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Research article

Results on controllability for Sobolev type fractional differential equations of order $1 < r < 2$ with finite delay

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Abstract: In this article, exact controllability results for Sobolev fractional delay differential system of $1 < r < 2$ are investigated. Fractional analysis, cosine and sine function operators, and Schauder's fixed point theorem are applied to verify the main results of this study. To begin, we use sufficient conditions to explore the controllability for fractional evolution differential system with finite delay. Lastly, an example is provided to illustrate the obtained theoretical results.

Keywords: fractional differential systems; controllability; Mainardi's Wright-type function; Sobolev type; mild solutions
Mathematics Subject Classification: 34A08, 26A33, 93B05, 46E36, 47D09

1. Introduction

In recent years, fractional calculus has been shown to be a terrific way to present the hereditary properties of various patterns, with a few repercussions. Fractional calculus ideas have dominated mathematics in recent decades. Numerous physical issues cannot be addressed with integer-order differential equations, and they should be addressed with fractional-order differential equations. As a result, numerous academics have recently made significant contributions to the fields of electromagnetics, fluid flow, signal, religion, porous media, control theory, viscoelasticity, biological, image processing, engineering difficulties, diffusion, and so forth. This combination has recently gained a lot of traction, owing to fractional differential equations' ability to reveal a few complicated wonders in a variety of diverse and limitless domains of research. For further details, refer to [1, 3, 11–13, 18–20, 31–35, 39, 44, 46, 53, 55, 56, 58, 59]. Very recently, in [48], the researchers presented the conditions for fractal stability, uniform boundedness, and asymptotic behaviors of second-order fractal differential equation solutions. Further, in [4], the authors discussed the Caputo fractional derivative for nonlinear Volterra integro-differential systems, multiple constant delays, and multiple kernels. The main goal of this article is to prove the qualitative properties of this equation's solutions, such as asymptotic stability, Mittag-Leffler stability of the zero solution, uniform stability, and also the boundedness of nonzero solutions. The qualitative aspects of Caputo fractional retarded Volterra integro-differential systems were investigated in [49].

The basic ideas underlying the design and analysis control systems are addressed with in mathematical control theory, which is a branch of application-oriented mathematics. Fractional derivatives with varying meanings can be used to address these types of difficulties. Controllability is amongst the most significant properties of a nonlinear model in control theory. The controllability problem's purpose is to show that a control function exists that directs the system's solution from its initial position to a final position, where the initial and final states may differ throughout space. Integro-differential equations, also known as dynamic systems or combined ordinary and partial dynamical systems, are used to simulate a wide range of scientific and engineering problems, including heat transport in memory materials, rheological properties, and a number of different physical processes. As a result, it's critical to investigate the controllability findings of such systems by utilizing existing approaches. Controllability is used in a multitude of industries, physics, power systems, chemical outgrowth control, electronics, engineering, including economics, biology, chemistry, space technology, transportation, robotics, and other fields. The researcher's papers can be found here [5, 9, 14, 15, 17, 22, 24, 35–38, 40–45, 51–53, 57, 59]. Furthermore, fractional evolution differential systems of the Sobolev type are frequently encountered in a variety of applications, thermodynamics, including fluid flow through fissured rocks, and shear in second order fluids, which can be referred to [6, 9, 18, 21, 50].

The authors [57] recently discussed the controllability problem as well as some unusual results for mild solutions to fractional differential equations of order $r \in (1,2)$. Further, the authors [16], investigate the presence of nonlocal conditions for fractional differential inclusions of order $r \in (1,2)$. [23] uses cosine families, measure of noncompactness, Laplace transform, and Mönch's fixed point theorem to prove the existence and controllability of a fractional delay integro-differential system of order $r \in (1,2)$. In [24], the researchers indicated the fractional derivatives of order $r \in (1,2)$ with control problems by referring to the fixed point theorem. In [26, 54], the authors established the existence, uniqueness, and approximate controllability results for fractional evolution equations of order $1 < r < 2$ by utilizing finite delay, nonlocal conditions, and integrodifferential systems. In [25], the authors signified the Caputo fractional differential evolution inclusions of order $r \in (1,2)$ by employing multivalued map, cosine and sine functions of operators, infinite delay, and Dhage's fixed point theorem. The Sobolev type, nonlocal conditions, fixed point theorems, and Volterra-Fredholm integro-differential system were applied to obtain the fractional differential inclusions of order $r \in (1,2)$ with control problems in [50]. In [27, 28, 30], the researchers developed the optimal control results for fractional differential systems of order $r \in (1,2)$ using different fixed point approaches, and integrodifferential systems, and hemivariational inequalities. Very recently, in [6, 7], the authors discussed approximate controllability results of fractional stochastic evolution systems of order $1 < r < 2$ with delay by referring to the integro-differential systems, Sobolev type, Wiener process, and fixed point theorems. Using the measure of noncompactness, integrodifferential systems, and Mönch fixed point theorem, the authors recently explored nonlocal controllability results for fractional evolution equations of order $r \in (1,2)$ in [29]. The exact controllability for Caputo fractional evolution equations of order $r \in (1,2)$ with finite delay utilizing the Sobolev type, mild solution, Schauder's fixed point theorem, cosine operators, and Mainardi's Wright-type function is the motivation for the current study.

The existing manuscript has inspired the laws mentioned above. Assume the following form for the Sobolev fractional evolution system of order $1 < r < 2$ with delay:

$$\begin{cases} {}^{C}D^{r}_{\vartheta}(Mz(\vartheta)) + Az(\vartheta) = g(\vartheta, z_{\vartheta}) + \mathcal{B}x(\vartheta), \ \vartheta \in V = [0, c], \\ z(\vartheta) = \hbar(\vartheta), \ \vartheta \in [-v, 0], \ z'(0) = z_1 \in \mathcal{Z}, \end{cases} \tag{1.1}$$

where ${}^{C}D^{r}_{\vartheta}$ represents Caputo fractional derivative of order $r \in (1,2)$; $A : D(A) \to \mathcal{Y}$ and $M : D(M) \to \mathcal{Y}$, where $D(A)$ and $D(M)$ are subsets of $\mathcal{Z}$; the control function $x \in \mathcal{H}$, where either $\mathcal{H} = L^2(V, \mathcal{X})$ for $\frac{3}{2} < r < 2$ or $\mathcal{H} = L^{\infty}(V, \mathcal{X})$ for $1 < r < 2$, $\mathcal{X}$ is also a Banach space; Moreover, the bounded linear operator $\mathcal{B}$ maps from $\mathcal{X}$ into $\mathcal{Z}$; $g : [0, c] \times C \to \mathcal{Z}$ with $C = C([-v, 0], \mathcal{Z})$ will be given later; $z : V^* = [-v, c] \to \mathcal{Z}$ is continuous, the element z_{ϑ} in C defined by $z_{\vartheta}(\varpi) = z(\vartheta + \varpi)$, $-v \le \varpi \le 0$; The domain $D(M)$ of M becomes a Banach space with $\|z\|_{D(M)} = \|Mz\|_{\mathcal{Y}}$, $z \in D(M)$ and $\hbar \in C(M) = C([-v, 0], D(M))$.

The following sections are included in this paper: Preliminaries, assumptions, and the primary finding on Sobolev type, remarks, mild solutions, and lemmas are presented in Section 2. Exact controllability results for system (1.1) by referring to the Schauder's point theorem in Section 3. In Section 4, an example is provided to illustrate the obtained theoretical findings.

2. Preliminaries

This part will go well significant information, basic definitions, lemmas, and outcomes. Denote $D(A)$ and $R(A)$ are the domain and range of the A. We assume the resolvent set A by $\rho(A)$ and the resolvent of A by $R(\Lambda, A) = \frac{1}{(\Lambda I - A)} \in L_c(\mathcal{Z})$.

Definition 2.1. *[19] The Riemann-Liouville fractional integral of order β with the lower limit zero for $g : [0, \infty) \to \mathbb{R}$ is given by*

$$I^{\beta} g(\vartheta) = \frac{1}{\Gamma(\beta)} \int_0^{\vartheta} \frac{g(\varpi)}{(\vartheta - \varpi)^{1-\beta}} d\varpi, \quad \vartheta > 0, \ \beta \in \mathbb{R}^+,$$

if the right side is point-wise defined on $[0, \infty)$.

Definition 2.2. *[19] The Riemann-Liouville derivative of order β with the lower limit zero for $g : [0, \infty) \to \mathbb{R}$ is defined by*

$${}^{L}D^{\beta} g(\vartheta) = \frac{1}{\Gamma(l - \beta)} \frac{d^l}{d\vartheta^l} \int_0^{\vartheta} g(\varpi)(\vartheta - \varpi)^{l-\beta-1} d\varpi, \quad \vartheta > 0, \ l - 1 < \beta < l.$$

Definition 2.3. *[19] The Caputo derivative of order β with the lower limit 0 for g is given by*

$$ {}^{C}D^{\beta}g(\vartheta) = {}^{L}D^{\beta}\left(g(\vartheta) - \sum_{i=0}^{l-1}\frac{g^{(i)}(0)}{i!}\vartheta^{i}\right), \quad \vartheta > 0,\ l-1 < \beta < l,\ \beta \in \mathbb{R}^{+}. $$

Remark 2.4. *[19]*

(1) Caputo derivative of a constant function is equal to zero.
(2) If $g \in C^{l}[0,\infty)$, then

$$ {}^{C}D^{\beta}g(\vartheta) = \frac{1}{\Gamma(l-\beta)}\int_0^{\vartheta}(\vartheta-\varpi)^{l-\beta-1}g^{(l)}(\varpi)d\varpi = I^{l-\beta}g^{(l)}(\vartheta),\ \vartheta > 0,\ l-1<\beta<l. $$

(3) If g is an abstract function with values in $\mathcal{Z}$, then the integrals appear in the Definitions (2.2) *and* (2.3) *are taken in Bochner's sense.*

Definition 2.5. *[47] A one parameter family $\{\mathcal{P}(\vartheta)\}_{\vartheta\in\mathbb{R}}$ of bounded linear operators mapping $\mathcal{Z}$ into itself is said to be a strongly continuous cosine family if and only if*

(a) $\mathcal{P}(0) = I$;
(b) $\mathcal{P}(\vartheta)z$ is strongly continuous in ϑ on $\mathbb{R}$ for all fixed point $z \in \mathcal{Z}$;
(c) $\mathcal{P}(\varpi+\vartheta) + \mathcal{P}(\varpi-\vartheta) = 2\mathcal{P}(\varpi)\mathcal{P}(\vartheta)$ for all $\varpi, \vartheta \in \mathbb{R}$.

Consider the sine family $\{\mathcal{S}(\vartheta)\}_{\vartheta\in\mathbb{R}}$ associated with the strongly continuous cosine family $\{\mathcal{P}(\vartheta)\}_{\vartheta\in\mathbb{R}}$, then

$$ \mathcal{S}(\vartheta)z = \int_0^{\vartheta}\mathcal{P}(\varpi)zd\varpi, \quad z \in \mathcal{Z}, \quad \vartheta \in \mathbb{R}. \tag{2.1} $$

Moreover, if

$$ Az = \frac{d^2}{d\vartheta^2}\mathcal{P}(\vartheta)z\Big|_{\vartheta=0}, \text{ for all } z \in D(A). $$

In the above $D(A)$ determined by $D(A) = \{z \in \mathcal{Z} : \mathcal{P}(\vartheta)z \in C^2(\mathbb{R},\mathcal{Z})\}$, where, A denotes a closed, densely-determined operator in $\mathcal{Z}$.

We now present the following assumptions on operators A and M. More details refer in [21]:

($\mathbf{Q_1}$) $D(M) \subset D(A)$ and M is bijective.
($\mathbf{Q_2}$) The operators A and M are linear operators, and A is closed.
($\mathbf{Q_3}$) The linear operator $M^{-1} : \mathcal{Y} \to D(M) \subset \mathcal{Z}$ is compact ($\Rightarrow M^{-1}$ is bounded).

In the above assumption ($\mathbf{Q_3}$) $\Rightarrow$ M is closed in view of the fact: M^{-1} is closed and injective, then its inverse is also closed. By referring to ($\mathbf{Q_1}$)–($\mathbf{Q_3}$) and the closed graph theorem, we have the boundedness of the linear operator $-AM^{-1}$ mapping from $\mathcal{Y}$ into itself. Denote $\|M^{-1}\| = \widetilde{M}_1$ and $\|M\| = \widetilde{M}_2$. We will assume $P = \sup_{\vartheta\geq 0}\|\mathcal{P}(\vartheta)\| < \infty$.

Let us assume the following fractional evolution system:

$$ \begin{cases} {}^{C}D^{r}_{\vartheta}(Mz(\vartheta)) + Az(\vartheta) = g(\vartheta, z_{\vartheta}),\ \vartheta \in V = [0,c], \\ z(\vartheta) = \hbar(\vartheta),\ \vartheta \in [-\nu, 0],\ z'(0) = z_1 \in \mathcal{Z}. \end{cases} \tag{2.2} $$

With reference to Definitions 2.1–2.3, it is easier to rewrite system (2.2) in the similar fractional integral equation:

$$\begin{cases} Mz(\vartheta) = M\hbar(0) + Mz_1\vartheta + \frac{1}{\Gamma(r)}\int_0^{\vartheta}(\vartheta-\varpi)^{r-1}[-Az(\varpi) + g(\vartheta, z_\vartheta)]d\varpi, \ \vartheta \in V = [0, c], \\ z(\vartheta) = \hbar(\vartheta), \ \vartheta \in [-\nu, 0], \ z'(0) = z_1 \in \mathcal{Z}. \end{cases} \tag{2.3}$$

If the integral in (2.3) exists. Let $\eta = \frac{r}{2}$ for $r \in (1, 2)$, which is discussed in [16, 55].

Definition 2.6. *[16] For every $x \in \mathcal{H}$ and $C(M)$ is said to be a mild solution of system* (1.1)*, we mean a function $z \in C(V^*, \mathcal{Z})$ which satisfies*

$$\begin{aligned} z(\vartheta) =& M^{-1}\mathcal{P}_\eta(\vartheta)M\hbar(0) + M^{-1}\mathcal{Q}_\eta(\vartheta)Mz_1 + \int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}M^{-1}\mathcal{G}_\eta(\vartheta-\varpi)g(\varpi, z_\varpi)d\varpi \\ & \int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}M^{-1}\mathcal{G}_\eta(\vartheta-\varpi)\mathcal{B}x(\varpi)d\varpi, \ \vartheta \in V, \end{aligned} \tag{2.4}$$

where $\mathcal{P}_\eta(\cdot), \mathcal{Q}_\eta(\cdot)$ and $\mathcal{G}_\eta(\cdot)$ are called the characteristic solution operator and presented as

$$\mathcal{P}_\eta(\vartheta) = \int_0^{\infty} S_\eta(\xi)\mathcal{P}(\vartheta^\eta\xi)d\xi, \quad \mathcal{Q}_\eta(\vartheta) = \int_0^{\vartheta}\mathcal{P}_\eta(\varpi)d\varpi, \quad \mathcal{G}_\eta(\vartheta) = \int_0^{\infty}\eta\xi S_\eta(\xi)\mathcal{S}(\vartheta^\eta\xi)d\xi,$$
$$S_\eta(\xi) = \frac{1}{\eta}\xi^{-1-\frac{1}{\eta}}\zeta_\eta(\xi^{-\frac{1}{\eta}}) \geq 0, \quad \zeta_\eta(\xi) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\xi^{-k\eta-1}\frac{\Gamma(k\eta+1)}{k!}\sin(k\pi\eta), \quad \xi \in (0, \infty),$$

and $S_\eta(\cdot)$ is the Mainardi's Wright-type function of defined on $(0, \infty)$ *such that*

$$S_\eta(\xi) \geq 0 \text{ for } \xi \in (0, \infty) \text{ and } \int_0^{\infty} S_\eta(\xi)d\xi = 1.$$

Lemma 2.7. *[16] The operators $\mathcal{P}_\eta(\vartheta)$, $\mathcal{Q}_\eta(\vartheta)$ and $\mathcal{G}_\eta(\vartheta)$ have the following characteristics:*

(a) *For $\vartheta \geq 0$, the operators $\mathcal{P}_\eta(\vartheta)$, $\mathcal{Q}_\eta(\vartheta)$ and $\mathcal{G}_\eta(\vartheta)$ are compact;*
(b) *For any fixed $\vartheta \geq 0$, the operators $\mathcal{P}_\eta(\vartheta)$, $\mathcal{Q}_\eta(\vartheta)$ and $\mathcal{G}_\eta(\vartheta)$ are linear and bounded, i.e., for all $z \in \mathcal{Z}$, the subsequent*

$$\|\mathcal{P}_\eta(\vartheta)z\| \leq P\|z\|, \quad \|\mathcal{Q}_\eta(\vartheta)z\| \leq P\|z\|\vartheta, \quad \|\mathcal{G}_\eta(\vartheta)z\| \leq \frac{P}{\Gamma(2\eta)}\|z\|\vartheta^\eta;$$

(c) $\{\mathcal{P}_\eta(\vartheta), \vartheta \geq 0\}$, $\{\mathcal{Q}_\eta(\vartheta), \vartheta \geq 0\}$ *and* $\{\vartheta^{\eta-1}\mathcal{G}_\eta(\vartheta), \vartheta \geq 0\}$ *are strongly continuous.*

3. Main results

Before beginning and analyzing the main results, we make the following assumptions to arrive at the principal result:

Definition 3.1. *(Controllability) The system* (1.1) *is called controllable on V if and only if for all continuous initial function $\hbar \in C(M)$ and for every $z_1, y \in D(M)$, there exists $x \in L^2(V, \mathcal{X})$ such that a mild solution z of system* (1.1) *satisfies $z(c) = y$.*

($\mathbf{H_1}$) The linear operator $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ is bounded, and the operator $W : \mathcal{H} \to D(M)$ determined by

$$Wx = \int_0^c (c-\varpi)^{\eta-1} M^{-1} \mathcal{G}_\eta(c-\varpi)\mathcal{B}x(\varpi)d\varpi,$$

has an inverse operator $W^{-1} : D(M) \to \mathcal{H}$, that is, $WW^{-1} = I_{D(M)}$, and there exists $P_1, P_2 > 0$ such that

$$\|\mathcal{B}\| \le P_1, \ \|W^{-1}\| \le P_2,$$

where we consider the norm $\|\cdot\|_{D(M)}$ on $D(M)$ for determining P_2.

So suffices to say that Wx in $D(M)$ and W is clearly determined. In fact, it holds

$$\begin{aligned}\|MWx\| &= \left\| \int_0^c (c-\varpi)^{\eta-1} \mathcal{G}_\eta(c-\varpi)\mathcal{B}x(\varpi)d\varpi \right\| \\ &\le \int_0^c (c-\varpi)^{\eta-1} \|\mathcal{G}_\eta(c-\varpi)\|\|\mathcal{B}x(\varpi)\|d\varpi \\ &\le \frac{P\|\mathcal{B}\|}{\Gamma(2\eta)} \int_0^c (c-\varpi)^{2\eta-1}\|x(\varpi)\|d\varpi \\ &= \frac{P\|\mathcal{B}\|}{\Gamma(2\eta)} \sqrt{\frac{c^{4\eta-1}}{4\eta-1}}\|x\|_{\mathcal{H}},\end{aligned}$$

in order to get $\eta \in (\frac{3}{2}, 2)$ and $x \in L^2(V, \mathcal{X})$, meanwhile

$$\begin{aligned}\|MWx\| &\le \frac{P\|\mathcal{B}\|c^{2\eta}}{\Gamma(2\eta+1)}\|x\|_{L^\infty(V,\mathcal{X})} \\ &= \frac{P\|\mathcal{B}\|c^{2\eta}}{\Gamma(2\eta+1)}\|x\|_{\mathcal{H}},\end{aligned}$$

in order to get $\eta \in (1, 2)$ and $x \in L^\infty(V, \mathcal{X})$.

We also see that

$$\int_0^\vartheta (\vartheta-\varpi)^{2\eta-1}\|x(\varpi)\|d\varpi \le N_\eta\|x\|_{\mathcal{H}}, \text{ for all } \vartheta \in V, \tag{3.1}$$

where $N_\eta = \sqrt{\frac{c^{4\eta-1}}{4\eta-1}}$, for $\eta \in (\frac{3}{2}, 2)$ and $x \in L^2(V, \mathcal{X})$, meanwhile, $N_\eta = \frac{c^{2\eta}}{2\eta}$ for $\eta = (1, 2)$ and $x \in L^\infty(V, \mathcal{X})$.

Now we introduce the assumption:

($\mathbf{H_2}$) g satisfies the accompanying two conditions:

- **(i)** For any $z \in C$, $g(\cdot, z) : V \to \mathcal{Y}$ is strongly measurable and for all $\vartheta \in V$, the continuous function $g(\vartheta, \cdot)$ maps from C into $\mathcal{Y}$.
- **(ii)** For any $\wp > 0$, there is a measurable function $h_\wp$ such that

$$\sup_{\|z\|\le\wp} \|g(\vartheta, z)\| \le h_\wp(\vartheta), \text{ with } \|h_\wp\|_\infty = \sup_{\varpi\in V} h_\wp(\varpi) < \infty,$$

$$\sup_{\vartheta\in V} \int_0^\vartheta (\vartheta-\varpi)^{2\eta-1} h_\wp(\varpi)d\varpi \le \delta\wp,$$

for all $\wp > 0$ sufficiently large and some $\delta > 0$.

It's important to note that the following:

$$\delta > \limsup_{\wp\to\infty} \frac{c^{2\eta}\|h_{\wp}\|_{\infty}}{2\eta^{\wp}}.$$

We shall state the conventional method for dealing with controllability problems as follows for the purpose of simplicity. According to our assumptions, the following control formula for $x(\cdot)$ is suitable:

$$x(\vartheta) = W^{-1}\Big[y - M^{-1}\mathcal{P}_{\eta}(c)M\hbar(0) - M^{-1}Q_{\eta}(c)Mz_1 - \int_0^c (c-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(c-\varpi)g(\varpi, z_{\varpi})d\varpi\Big]. \quad (3.2)$$

Now we introduce the operator Φ such that

$$\begin{aligned}
(\Phi z)(\vartheta) &= M^{-1}\mathcal{P}_{\eta}(\vartheta)M\hbar(0) + M^{-1}Q_{\eta}(\vartheta)Mz_1 + \int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(\vartheta-\varpi)g(\varpi, z_{\varpi})d\varpi \\
&\quad + \int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(\vartheta-\varpi)\mathcal{B}x(\varpi)d\varpi, \text{ for all } \vartheta \in V, \\
(\Phi z)(\vartheta) &= \hbar(\vartheta), \ -v \le \vartheta \le 0,
\end{aligned}$$

$\Phi : C(V^*, \mathcal{Z}) \to C(V^*, \mathcal{Z})$, has a fixed point. Obviously, this fixed point is just a solution of system (1.1). Moreover, we have

$$\begin{aligned}
(\Phi z)(c) &= M^{-1}\mathcal{P}_{\eta}(c)M\hbar(0) + M^{-1}Q_{\eta}(c)Mz_1 + \int_0^c (c-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(c-\varpi)g(\varpi, z_{\varpi})d\varpi \\
&\quad + \int_0^c (c-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(c-\varpi)\mathcal{B}x(\varpi)d\varpi \\
&= M^{-1}\mathcal{P}_{\eta}(c)M\hbar(0) + M^{-1}Q_{\eta}(c)Mz_1 + \int_0^c (c-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(c-\varpi)g(\varpi, z_{\varpi})d\varpi \\
&\quad + \int_0^c (c-\varpi)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(c-\varpi)\mathcal{B}W^{-1}\Big[y - M^{-1}\mathcal{P}_{\eta}(c)M\hbar(0) - M^{-1}Q_{\eta}(c)Mz_1 \\
&\quad - \int_0^c (c-u)^{\eta-1}M^{-1}\mathcal{G}_{\eta}(c-u)g(u, z_u)du\Big]d\varpi = y,
\end{aligned}$$

which implies x moves the system (1.1) from $\hbar(0)$ to y in finite time c. Hence, we claim (1.1) is controllable on $[0, c]$.

For all $\wp > 0$, determine $\mathrm{B}_{\wp}$ is bounded, closed and convex subset in $C(V^*, \mathcal{Z})$, then $\mathrm{B}_{\wp} = \{z \in C(V^*, \mathcal{Z}) : \|z(\vartheta)\| \le \wp, \ \vartheta \in V^*\}$.

By referring the hypotheses $(\mathbf{H_1})$–$(\mathbf{H_2})$, we provide following results for proving the primary results.

Lemma 3.2. *There exists* $\mu \ge \max\left\{\max_{\vartheta\in[-v,0]}\|\hbar(\vartheta)\|, \frac{P^*}{1-\kappa}\right\}$, *where*

$$\kappa = \begin{cases} \frac{P\widetilde{M}_1\delta}{\Gamma(2\eta)}\left(1 + \frac{\sqrt{c}P\|\mathcal{B}\|\|W^{-1}\|N_{\eta}}{\Gamma(2\eta)}\right) < 1, \ \mathcal{H} = L^2(V, \mathcal{X}), \\ \frac{P\widetilde{M}_1\delta}{\Gamma(2\eta)}\left(1 + \frac{P\|\mathcal{B}\|\|W^{-1}\|N_{\eta}}{\Gamma(2\eta)}\right) < 1, \quad \mathcal{H} = L^{\infty}(V, \mathcal{X}), \end{cases} \quad (3.3)$$

$$P^* = \begin{cases} \widetilde{M}_1 P\|M\hbar(0)\| + \widetilde{M}_1 Pc\|Mz_1\| \\ \quad + \frac{\sqrt{c}P\|\mathcal{B}\|\|W^{-1}\|\widetilde{M}_1}{\Gamma(2\eta)} N_\eta(\|My\| + P\|M\hbar(0)\| + Pc\|Mz_1\|), \ \mathcal{H} = L^2(V, \mathcal{X}), \\ \widetilde{M}_1 P\|M\hbar(0)\| + \widetilde{M}_1 Pc\|Mz_1\| \\ \quad + \frac{P\|\mathcal{B}\|\|W^{-1}\|\widetilde{M}_1}{\Gamma(2\eta)} N_\eta(\|My\| + P\|M\hbar(0)\| + Pc\|Mz_1\|), \quad \mathcal{H} = L^\infty(V, \mathcal{X}), \end{cases}$$

such that $\Phi B_\mu \subset B_\mu$.

Proof. Let the control function x determined in (3.2) satisfies

$$\begin{aligned} \|x(\vartheta)\| &\le \|W^{-1}\| \Big\| y - M^{-1}\mathcal{P}_\eta(c)M\hbar(0) - M^{-1}\mathcal{Q}_\eta(c)Mz_1 \\ &\quad - \int_0^c (c-\varpi)^{\eta-1} M^{-1}\mathcal{G}_\eta(c-\varpi)g(\varpi, z_\varpi)d\varpi \Big\|_{D(M)} \\ &\le \|W^{-1}\| \Big\| M\Big(y - M^{-1}\mathcal{P}_\eta(c)M\hbar(0) - M^{-1}\mathcal{Q}_\eta(c)Mz_1 \\ &\quad - \int_0^c (c-\varpi)^{\eta-1} M^{-1}\mathcal{G}_\eta(c-\varpi)g(\varpi, z_\varpi)d\varpi \Big) \Big\| \\ &\le \|W^{-1}\| \Big(\|My\| + \|\mathcal{P}_\eta(c)M\hbar(0)\| + \|\mathcal{Q}_\eta(c)Mz_1\| \\ &\quad - \int_0^c (c-\varpi)^{\eta-1} \|\mathcal{G}_\eta(c-\varpi)g(\varpi, z_\varpi)\| d\varpi \Big) \\ &\le \|W^{-1}\| \Big(\|My\| + P\|M\hbar(0)\| + Pc\|Mz_1\| + \frac{P}{\Gamma(2\eta)} \int_0^c (c-\varpi)^{2\eta-1} h_\mu(\varpi)d\varpi \Big) \\ &\le \|W^{-1}\| \Big(\|My\| + P\|M\hbar(0)\| + Pc\|Mz_1\| + \frac{P}{\Gamma(2\eta)}\delta\mu \Big), \end{aligned}$$

which implies

$$\|x\|_{\mathcal{H}} \le \begin{cases} \sqrt{c}\|W^{-1}\| \Big(\|My\| + P\|M\hbar(0)\| + Pc\|Mz_1\| + \frac{P}{\Gamma(2\eta)}\delta\mu \Big), \ \mathcal{H} = L^2(V, \mathcal{X}), \\ \|W^{-1}\| \Big(\|My\| + P\|M\hbar(0)\| + Pc\|Mz_1\| + \frac{P}{\Gamma(2\eta)}\delta\mu \Big), \quad \mathcal{H} = L^\infty(V, \mathcal{X}). \end{cases} \tag{3.4}$$

Assume that $z \in \mathrm{B}_\mu$. If $\vartheta \in [-v, 0]$ then

$$\|(\Phi z)(\vartheta)\| \le \max_{\vartheta \in [-v,0]} \|\hbar(\vartheta)\|.$$

If $\vartheta \in [0, c]$ then

$$\begin{aligned} \|(\Phi z)(\vartheta)\| &\le \|M^{-1}\mathcal{P}_\eta(\vartheta)M\hbar(0)\| + \|M^{-1}\mathcal{Q}_\eta(\vartheta)Mz_1\| \\ &\quad + \int_0^\vartheta (\vartheta-\varpi)^{\eta-1} \|M^{-1}\mathcal{G}_\eta(\vartheta-\varpi)g(\varpi, z_\varpi)\| d\varpi \\ &\quad + \int_0^\vartheta (\vartheta-\varpi)^{\eta-1} \|M^{-1}\mathcal{G}_\eta(\vartheta-\varpi)\mathcal{B}x(\varpi)\| d\varpi \\ &\le \widetilde{M}_1 P\|M\hbar(0)\| + \widetilde{M}_1 Pc\|Mz_1\| + \frac{P}{\Gamma(2\eta)} \int_0^\vartheta (\vartheta-\varpi)^{2\eta-1} \|M^{-1}\|\|g(\varpi, z_\varpi)\| d\varpi \end{aligned}$$

$$
\begin{aligned}
&+\frac{P}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}\|M^{-1}\|\|\mathcal{B}x(\varpi)\|d\varpi\\
&\le \widetilde{M}_1P\|M\hbar(0)\|+\widetilde{M}_1Pc\|Mz_1\|+\frac{P\widetilde{M}_1}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}h_\mu(\varpi)d\varpi\\
&+\frac{P\widetilde{M}_1}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}\|\mathcal{B}\|\|x(\varpi)\|d\varpi\\
&\le \widetilde{M}_1P\|M\hbar(0)\|+\widetilde{M}_1Pc\|Mz_1\|+\frac{P\widetilde{M}_1}{\Gamma(2\eta)}\delta\mu+\frac{P\widetilde{M}_1N_\eta}{\Gamma(2\eta)}\|\mathcal{B}\|\|x\|_{\mathcal{H}}\\
&=\kappa\mu+P^*\le\mu.
\end{aligned}
$$

Hence, $\Phi B_\mu \subset B_\mu$, for every $\mu \ge \max\left\{\max_{\vartheta\in[-\nu,0]}\|\hbar(\vartheta)\|, \frac{P^*}{1-\kappa}\right\}$ sufficiently large. The proof has been addressed. □

Lemma 3.3. *For any fixed $\vartheta \in V$ then $J_\mu(\vartheta) = \{(\Phi z)(\vartheta) : z \in B_\mu\}$ is precompact in $\mathcal{Z}$.*

Proof. This is trivial for all $\vartheta \in [-\nu, 0]$, hence $J_\mu = \{\hbar(\vartheta)\}$. So let $\vartheta \in (0, c)$ be fixed.

$$(\Phi z)(\vartheta) = M^{-1}(\Phi_0 z)(\vartheta),$$

where

$$
\begin{aligned}
(\Phi_0 z)(\vartheta) &= \mathcal{P}_\eta(\vartheta)M\hbar(0)+\mathcal{Q}_\eta(\vartheta)Mz_1+\int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}\mathcal{G}_\eta(\vartheta-\varpi)g(\varpi,z_\varpi)d\varpi\\
&+\int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}\mathcal{G}_\eta(\vartheta-\varpi)\mathcal{B}x(\varpi)d\varpi.
\end{aligned}
$$

Furthermore, for any $z \in B_\mu$, we find

$$
\begin{aligned}
\|(\Phi_0 z)(\vartheta)\| &= \|\mathcal{P}_\eta(\vartheta)M\hbar(0)\|+\|\mathcal{Q}_\eta(\vartheta)Mz_1\|+\int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}\|\mathcal{G}_\eta(\vartheta-\varpi)g(\varpi,z_\varpi)\|d\varpi\\
&+\int_0^{\vartheta}(\vartheta-\varpi)^{\eta-1}\|\mathcal{G}_\eta(\vartheta-\varpi)\mathcal{B}x(\varpi)\|d\varpi\\
&\le P\|M\hbar(0)\|+Pc\|Mz_1\|+\frac{P}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}\|g(\varpi,z_\varpi)\|d\varpi\\
&+\frac{P}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}\|\mathcal{B}x(\varpi)\|d\varpi\\
&\le P\|M\hbar(0)\|+Pc\|Mz_1\|+\frac{P}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}h_\mu(\varpi)d\varpi\\
&+\frac{P\|\mathcal{B}\|}{\Gamma(2\eta)}\int_0^{\vartheta}(\vartheta-\varpi)^{2\eta-1}\|x(\varpi)\|d\varpi\\
&\le P\|M\hbar(0)\|+Pc\|Mz_1\|+\frac{P}{\Gamma(2\eta)}\Big[\frac{c^{2\eta}}{2\eta}\|h_\mu\|_\infty+N_\eta\|\mathcal{B}\|\|x\|_{\mathcal{H}}\Big].
\end{aligned}
$$

Then $\{(\Phi_0 z)(\vartheta) : z \in B_\mu\}$ is bounded in $\mathcal{Y}$ referring (3.4). Hence, M^{-1} mapping from $\mathcal{Y}$ into $\mathcal{Z}$ is compact, then $(\Phi z)(\vartheta) = M^{-1}(\{(\Phi_0 z)(\vartheta) : z \in B_\mu\})$ is precompact in $\mathcal{Z}$. □

Lemma 3.4. $\Phi B_\mu = \{\Phi z : z \in B_\mu\}$ *is equicontinuous.*

Proof. Assume that $z \in \mathrm{B}_\mu$ and $0 < \vartheta_1 < \vartheta_2 \le c$, such that

$$\begin{aligned}
\|(\Phi z)(\vartheta_2) - (\Phi z)(\vartheta_1)\| \le\ & \|M^{-1}\mathcal{P}_\eta(\vartheta_2)M\hbar(0) - M^{-1}\mathcal{P}_\eta(\vartheta_1)M\hbar(0)\| \\
& + \|M^{-1}\mathcal{Q}_\eta(\vartheta_2)Mz_1 - M^{-1}\mathcal{Q}_\eta(\vartheta_1)Mz_1\| \\
& + \Big\| \int_0^{\vartheta_2} (\vartheta_2 - \varpi)^{\eta-1} M^{-1}\mathcal{G}_\eta(\vartheta_2 - \varpi) g(\varpi, z_\varpi) d\varpi \\
& - \int_0^{\vartheta_1} (\vartheta_1 - \varpi)^{\eta-1} M^{-1}\mathcal{G}_\eta(\vartheta_1 - \varpi) g(\varpi, z_\varpi) d\varpi \Big\| \\
& + \Big\| \int_0^{\vartheta_2} (\vartheta_2 - \varpi)^{\eta-1} M^{-1}\mathcal{G}_\eta(\vartheta_2 - \varpi) \mathcal{B}x(\varpi) d\varpi \\
& - \int_0^{\vartheta_1} (\vartheta_1 - \varpi)^{\eta-1} M^{-1}\mathcal{G}_\eta(\vartheta_1 - \varpi) \mathcal{B}x(\varpi) d\varpi \Big\| \\
\le\ & \|M^{-1}[\mathcal{P}_\eta(\vartheta_2) - \mathcal{P}_\eta(\vartheta_1)]M\hbar(0)\| + \|M^{-1}[\mathcal{Q}_\eta(\vartheta_2) - \mathcal{Q}_\eta(\vartheta_1)]Mz_1\| \\
& + \int_0^{\vartheta_1} \|M^{-1}[(\vartheta_2 - \varpi)^{\eta-1}\mathcal{G}_\eta(\vartheta_2 - \varpi) \\
& - (\vartheta_1 - \varpi)^{\eta-1}\mathcal{G}_\eta(\vartheta_1 - \varpi)] g(\varpi, z_\varpi)\| d\varpi \\
& + \int_0^{\vartheta_1} \|M^{-1}[(\vartheta_2 - \varpi)^{\eta-1}\mathcal{G}_\eta(\vartheta_2 - \varpi) \\
& - (\vartheta_1 - \varpi)^{\eta-1}\mathcal{G}_\eta(\vartheta_1 - \varpi)] \mathcal{B}x(\varpi)\| d\varpi \\
& + \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varpi)^{\eta-1} \|M^{-1}\mathcal{G}_\eta(\vartheta_2 - \varpi) g(\varpi, z_\varpi)\| d\varpi \\
& + \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varpi)^{\eta-1} \|M^{-1}\mathcal{G}_\eta(\vartheta_2 - \varpi) \mathcal{B}x(\varpi)\| d\varpi \\
\le\ & \sum_{i=1}^{6} \mathcal{O}_i.
\end{aligned}$$

Let $K_\eta(\vartheta) = \vartheta^{\eta-1}\mathcal{G}_\eta(\vartheta)$ for all $\vartheta \in V$, from Lemma 2.7(c) that $K_\eta(\vartheta)$ denotes the strongly continuous operator. Since choices $\epsilon > 0$, we get

$$\begin{aligned}
\mathcal{O}_3 \le\ & \int_0^{\vartheta_1 - \epsilon} \|K_\eta(\vartheta_2 - \varpi) - K_\eta(\vartheta_1 - \varpi)\| \|g(\varpi, z_\varpi)\| d\varpi \\
& + \int_{\vartheta_1 - \epsilon}^{\vartheta_1} \|K_\eta(\vartheta_2 - \varpi) - K_\eta(\vartheta_1 - \varpi)\| \|g(\varpi, z_\varpi)\| d\varpi \\
\le\ & \sup_{\varpi \in [0, \vartheta_1 - \epsilon]} \|K_\eta(\vartheta_2 - \varpi) - K_\eta(\vartheta_1 - \varpi)\| \int_0^{\vartheta_1} h_\mu(\varpi) d\varpi \\
& + \frac{2P}{\Gamma(2\eta)} \int_{\vartheta_1 - \epsilon}^{\vartheta_1} h_\mu(\varpi) d\varpi (\vartheta_2 - \vartheta_1 - \epsilon)^{2\eta - 1},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{O}_4 &\le \int_0^{\vartheta_1-\epsilon} \|K_\eta(\vartheta_2-\varpi)-K_\eta(\vartheta_1-\varpi)\|\|\mathcal{B}x(\varpi)\|d\varpi \\
&\quad + \int_{\vartheta_1-\epsilon}^{\vartheta_1} \|K_\eta(\vartheta_2-\varpi)-K_\eta(\vartheta_1-\varpi)\|\|\mathcal{B}x(\varpi)\|d\varpi \\
&\le \|\mathcal{B}\| \sup_{\varpi\in[0,\vartheta_1-\epsilon]} \|K_\eta(\vartheta_2-\varpi)-K_\eta(\vartheta_1-\varpi)\| \int_0^{\vartheta_1} \|x(\varpi)\|d\varpi \\
&\quad + \frac{2P\|\mathcal{B}\|}{\Gamma(2\eta)} \int_{\vartheta_1-\epsilon}^{\vartheta_1} \|x(\varpi)\|d\varpi(\vartheta_2-\vartheta_1-\epsilon)^{2\eta-1},
\end{aligned}
$$

$$
\mathcal{O}_5 \le \frac{P\|M^{-1}\|}{\Gamma(2\eta)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_1-\varpi)^{2\eta-1}\|g(\varpi,z_\varpi)\|d\varpi \le \frac{P\|M^{-1}\|}{\Gamma(2\eta)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_1-\varpi)^{2\eta-1}h_\mu(\varpi)d\varpi,
$$

$$
\mathcal{O}_6 \le \frac{P\|M^{-1}\|}{\Gamma(2\eta)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_1-\varpi)^{2\eta-1}\|\mathcal{B}x(\varpi)\|d\varpi \le \frac{P\|M^{-1}\|\|\mathcal{B}\|}{\Gamma(2\eta)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_1-\varpi)^{2\eta-1}\|x(\varpi)\|d\varpi.
$$

As a result, $\mathcal{O}_3$, and $\mathcal{O}_4$ tend to zero independently of $x \in \mathrm{B}_\mu$ as $\vartheta_2 \to \vartheta_1$, $\epsilon \to 0$.

Lemma 2.7(c), $\mathcal{P}_\eta(\vartheta)$, $\mathcal{Q}_\eta(\vartheta)$, and $\mathcal{G}_\eta(\vartheta)$ are continuous in the uniform operator topology for $\vartheta \ge 0$, and $\sup_{\varpi\in V} |h_\mu(\varpi)| < \infty$ and $x(\cdot)$ is bounded from (3.4). We easily seen that the terms $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_6 \to 0$ as $\vartheta_2 \to \vartheta_1$. Hence, $\Phi\mathrm{B}_\mu$ is equicontinuous and also bounded. □

Now we prove the main results of this paper.

Theorem 3.5. *If* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *are satisfied. Then,* (1.1) *is controllable on* $[0, c]$ *if the condition* (3.3) *hold.*

Proof. By referring the Lemmas 3.2–3.4 and the Ascoli-Arzela theorem that $\Phi\mathrm{B}_\mu$ is precompact in $C(V^*, \mathcal{Z})$. As a result, Φ is a completely continuous operator on $C(V^*, \mathcal{Z})$. Referring the Schauder's fixed point theorem, Φ has a fixed point in B_μ. Any fixed point of Φ is a mild solution of (1.1) on V fulfilling $(\Phi z)(\vartheta) = z(\vartheta)$ in $\mathcal{Z}$. Therefore, the fractional evolution system (1.1) is controllable on V. □

Remark 3.6. *The primary discussion of this article, that is, Theorem 3.5 provides only the sufficient conditions for the controllability of the proposed system* (1.1)*. If the non-linearity of the function* $g(\vartheta, z)$ *does not satisfy the hypothesis* $(\mathbf{H_2})$*, then the proposed system* (1.1) *may or may not be controllable, one can check [38] with suitable modifications for fractional settings.*

4. Application

In this section, an example is given to illustrate our theory, we consider the following problem:

$$
\begin{cases}
{}^C D_\vartheta^{\frac{3}{2}}(z(\vartheta,\jmath)-\Delta z(\vartheta,\jmath)) = \Delta z(\vartheta,\jmath) + g(\vartheta, z(\vartheta-\tau,\jmath)) + \mathcal{B}x(\vartheta), \ \vartheta \in V_1 = [0,1], \ \jmath \in \mathcal{N}, \\
z(\vartheta,\jmath) = 0, \ \vartheta \in [0,1], \ \jmath \in \partial\mathcal{N}, \\
z(0,\jmath) = \hbar(0), \ z'(0,\jmath) = z_1(\jmath), \ \jmath \in \mathcal{N},
\end{cases}
\tag{4.1}
$$

where ${}^{C}D_{\vartheta}^{\frac{3}{2}}$ stands for Caputo fractional partial derivative. Assume that $\mathcal{N} \subset \mathbb{R}^N$ is a bounded domain and $\mathcal{X} = \mathcal{Z} = L^2([0,\pi]) = L^2(\mathcal{N})$. Let A be Laplace operator with Dirichlet boundary conditions presented as $Az = \Delta$, and $A : D(A) \to \mathcal{Y}$ by $Az = -\Delta$ and $M : D(M) \to \mathcal{Y}$ by $Mz = z - \Delta$, where $D(A)$ and $D(M)$ are subsets of $\mathcal{Z}$ and the domain of $D(A)$ and $D(M)$ is given by

$$D(A) = D(M) = \{g \in H_0^1(\mathcal{N}),\ Ag \in L^2(\mathcal{N})\}.$$

Obviously, we obtain $D(A) = H_0^1(\mathcal{N}) \cap H^2(\mathcal{N})$. A produces the uniformly bounded strongly continuous cosine family $\mathcal{P}(\vartheta)$ for $\vartheta \geq 0$, see [2]. Consider $\mathcal{V}_\ell = \ell^2\pi^2$ and $\wp_\ell(\jmath) = \sqrt{(2/\pi)}\sin(\ell\pi\jmath)$, for every $\ell \in \mathbb{N}$.

Assume that $\{-\mathcal{V}_\ell, \wp_\ell\}_{\ell=1}^{\infty}$ is the eigensystem of the operator A, since $0 < \mathcal{V}_1 \leq \mathcal{V}_2 \leq \cdots, \mathcal{V}_\ell \to \infty$ as $\ell \to \infty$, and $\{\wp_\ell\}_{\ell=1}^{\infty}$ form an orthonormal basis of $\mathcal{Z}$.

$$Az = \sum_{\ell=1}^{\infty} \mathcal{V}_\ell \langle z, \wp_\ell\rangle \wp_\ell,\ z \in D(A),$$

$$Mz = \sum_{\ell=1}^{\infty} (1 + \mathcal{V}_\ell)\langle z, \wp_\ell\rangle \wp_\ell,\ z \in D(M).$$

Further, for every $z \in \mathcal{Z}$ we obtain

$$M^{-1}z = \sum_{\ell=1}^{\infty} (1 + \mathcal{V}_\ell)^{-1}\langle z, \wp_\ell\rangle \wp_\ell,$$

$$-AM^{-1}z = \sum_{\ell=1}^{\infty} -\mathcal{V}_\ell(1 + \mathcal{V}_\ell)^{-1}\langle z, \wp_\ell\rangle \wp_\ell.$$

Consequently, cosine function $\mathcal{P}(\vartheta)$ defined by

$$\mathcal{P}(\vartheta)z = \sum_{\ell=1}^{\infty} \cos\left(\sqrt{\mathcal{V}_\ell}\vartheta\right)\langle z, \wp_\ell\rangle \wp_\ell,\ z \in \mathcal{Z},$$

and the sine function associated with cosine function given by

$$\mathcal{S}(\vartheta)z = \sum_{\ell=1}^{\infty} \frac{1}{\sqrt{\mathcal{V}_\ell}} \sin\left(\sqrt{\mathcal{V}_\ell}\vartheta\right)\langle z, \wp_\ell\rangle \wp_\ell,\ z \in \mathcal{Z}.$$

Accordingly, $\|M^{-1}\|$ is compact, bounded along with $\|M^{-1}\| \leq 1$ and $-AM^{-1}$ produces the above strongly continuous cosine family $\mathcal{P}(\vartheta)$ on $\mathcal{Z}$ along with $\|\mathcal{P}(\vartheta)\|_{L_c(\mathcal{Z})} \leq e^{-\vartheta} \leq 1$, for any $\vartheta \geq 0$.

Since $r = \frac{3}{2}$, we know that $\eta = \frac{3}{4}$. The control operator $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ is determined by

$$\mathcal{B}x = \sum_{\ell=1}^{\infty} a\mathcal{V}_\ell \langle \overline{x}, \wp_\ell\rangle \wp_\ell,\ a > 0.$$

In the above

$$\overline{x} = \begin{cases} x_\ell,\ \ell \in \mathbb{N}, \\ 0,\ \ \ell = J+1, J+2, \cdots, \end{cases}$$

for $J \in \mathbb{N}$. Now we denote $W : \mathcal{H} \to D(M)$ as follows:

$$Wx = \int_0^1 (1-\varpi)^{-\frac{1}{4}} M^{-1} \mathcal{G}_{\frac{3}{4}}(1-\varpi)\mathcal{B}x(\varpi)d\varpi.$$

Hence, $\|x\| = \left(\sum_{\ell=1}^{\infty}\langle x, \wp_\ell\rangle^2\right)^{\frac{1}{2}}$, for $x \in \mathcal{X}$, we obtain

$$\|\mathcal{B}x\| = \left(\sum_{\ell=1}^{\infty} a^2 \mathcal{V}_\ell^2 \langle \overline{x}, \wp_\ell\rangle^2\right)^{\frac{1}{2}} \leq aJ\mathcal{V}_J\|x\|.$$

Assume that $x(\varpi, \jmath) = z(\jmath) \in \mathcal{X}$ and $\overline{z}$ denotes z_ℓ if $\ell = 1, 2, \cdots, N$ or 0 if $\ell = J+1, \cdots$. Hence, we obtain

$$\begin{aligned}
Wx &= \int_0^1 (1-\varpi)^{-\frac{1}{4}} \frac{3}{4} \int_0^\infty M^{-1}\xi S_{\frac{3}{4}}(\xi)\mathcal{S}\left((1-\varpi)^{\frac{3}{4}}\xi\right)\mathcal{B}z d\xi d\varpi \\
&= a\int_0^1 (1-\varpi)^{-\frac{1}{4}} \frac{3}{4} \int_0^\infty M^{-1}\xi S_{\frac{3}{4}}(\xi) \sum_{\ell=1}^{J} \sqrt{\mathcal{V}_\ell}\sin\left(\sqrt{\mathcal{V}_\ell}(1-\varpi)^{\frac{3}{4}}\xi\right)\langle \overline{z}, \wp_\ell\rangle \wp_\ell d\xi d\varpi \\
&= a\int_0^\infty S_{\frac{3}{4}}(\xi) \sum_{\ell=1}^{J} \int_0^1 (1-\varpi)^{-\frac{1}{4}} \frac{3\sqrt{\mathcal{V}_\ell}}{4(1+\mathcal{V}_\ell)}\xi \sin\left(\sqrt{\mathcal{V}_\ell}(1-\varpi)^{\frac{3}{4}}\xi\right) d\varpi\langle \overline{z}, \wp_\ell\rangle \wp_\ell d\xi \\
&= a\int_0^\infty S_{\frac{3}{4}}(\xi) \sum_{\ell=1}^{J} \int_0^1 \frac{1}{(1+\mathcal{V}_\ell)} \frac{d}{d\varpi}\left[\cos\left(\sqrt{\mathcal{V}_\ell}(1-\varpi)^{\frac{3}{4}}\xi\right)\right] d\varpi\langle \overline{z}, \wp_\ell\rangle \wp_\ell d\xi \\
&= a\sum_{\ell=1}^{J} \int_0^\infty S_{\frac{3}{4}}(\xi) \frac{1}{(1+\mathcal{V}_\ell)}(1-\cos(\sqrt{\mathcal{V}_\ell}\xi))d\xi\langle \overline{z}, \wp_\ell\rangle \wp_\ell \\
&= a\sum_{\ell=1}^{\infty} \frac{1}{(1+\mathcal{V}_\ell)}\left(1-E_{\frac{3}{2},1}(-\mathcal{V}_\ell)\right)\langle z, \wp_\ell\rangle \wp_\ell.
\end{aligned}$$

In [8, 10], assume that $v = E_{\frac{3}{2},1}(-\frac{1}{10})$, then for any $\ell \in \mathbb{N}$, we obtain

$$-1 < E_{\frac{3}{2},1}(-\mathcal{V}_\ell) \leq v < 1,$$

it denotes

$$0 < 1 - v \leq 1 - E_{\frac{3}{2},1}(-\mathcal{V}_\ell) < 2.$$

Then, we classify W is surjective. We illustrate the W^{-1} mapping from $D(M)$ into $\mathcal{H}$ by

$$(W^{-1}z)(\vartheta, \jmath) = \frac{1}{a}\sum_{\ell=1}^{\infty} \frac{(1+\mathcal{V}_\ell)\langle z, \wp_\ell\rangle \wp_\ell}{1-E_{\frac{3}{2},1}(-\mathcal{V}_\ell)},$$

for any

$$z = \sum_{\ell=1}^{\infty}\langle z, \wp_\ell\rangle \wp_\ell \in \mathcal{Z}.$$

Thus

$$\|z\|_{D(M)} = \|Mz\| = \sqrt{\sum_{\ell=1}^{\infty}(1+\mathcal{V}_\ell)^2\langle z, \wp_\ell\rangle^2},$$

for $z \in D(M)$ in such a way

$$\|(W^{-1}z)(\vartheta, \cdot)\| \leq \frac{1}{a(1-v)}\|z\|_{D(M)}.$$

Note that $W^{-1}z$ is independent of $\vartheta \in V$. Moreover, we have

$$\|W^{-1}\| \leq \frac{1}{a(1-v)}.$$

Therefore assumption $(\mathbf{H_1})$ satisfied.

$(\mathbf{R_1})$ Determine $g : V_1 \times \mathbb{R} \to \mathbb{R}$. For any $z \in \mathbb{R}$, $g(\vartheta, z)$ is measurable and for every $\vartheta \in V_1$, $g(\vartheta, z)$ is continuous. Furthermore,

$$\limsup_{\iota\to\infty} \frac{1}{\iota} \sup_{\vartheta\in V_1,\ |z|\leq\iota} |g(\vartheta, z)| = \delta \leq \infty.$$

Consider $G : V_1 \times C([-1, 0], \mathcal{Z}) \to \mathcal{Y}$ by $G(\vartheta, y)(\jmath) = g(\vartheta, y(-v)(\jmath))$.

The system (4.1) can now be abstracted as follows:

$$\begin{cases} {}^{C}D^{r}_{\vartheta}z(\vartheta) = -Az(\vartheta) + G(\vartheta, z_\vartheta) + \mathcal{B}x(\vartheta),\ \vartheta \in V = [0, c],\ r \in (1, 2), \\ z(\vartheta) = \hbar(\vartheta), \vartheta \in [-1, 0],\ z'(0) = z_1 \in \mathcal{Z}. \end{cases} \tag{4.2}$$

As a result, all the assumptions of Theorem 3.5 are satisfied, if

$$\frac{2\delta}{\sqrt{\pi}}\Big[1 + \frac{\sqrt{2}}{\sqrt{\pi}a(1-v)}\Big] < 1.$$

Therefore, the system (4.1) is controllable on $[0, c]$.

Remark 4.1. *The above fractional partial differential system* (4.1) *provides only the sufficient conditions for the controllability. If the non-linearity of the function presented in the above fractional partial differential system* (4.1)*, that is, $g(\vartheta, z(\vartheta - \tau, \jmath))$ does not satisfy the hypothesis* $(\mathbf{H_2})$*, then the proposed system* (4.1) *may or may not be controllable, for more details, one can check the Example 1, Example 2 and Example 3, which are discussed in [38] with suitable modifications for fractional settings.*

5. Conclusions

This research explores the controllability of fractional delay differential equations of $r \in (1, 2)$. The main results of this work are evaluated using fractional computations, cosine and sine function operators, Sobolev type, and Schauder's fixed point theorem. At first, we used sufficient conditions to evaluate controllability results. Furthermore, an application is built to develop the theory of the main results. In the future, we shall concentrate our research on controllability results for Riemann-Liouville fractional neutral integrodifferential inclusions with infinite delay of order $r \in (1, 2)$.

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Conflict of interest

This work does not have any conflict of interest.

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