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Research article

Fixed point approach for solving a system of Volterra integral equations and Lebesgue integral concept in $\mathrm{F_{CM}}$-spaces

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Abstract: The goal of this manuscript is to obtain some tripled fixed point results under a new contractive condition and triangular property in the context of fuzzy cone metric spaces ($\mathrm{F_{CM}}$-spaces). Moreover, two examples and corollaries are given to validate our work. Ultimately, as applications, the notion of Lebesgue integral is represented by the fuzzy method to discuss the existence of fixed points. Also, the existence and uniqueness solution for a system of Volterra integral equations are studied by the theoretical results.

Keywords: tripled fixed point; fuzzy cone metric spaces; Lebesgue-integrable mapping; Volterra integral equation
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1. Introduction

Let X be a nonempty set. In 2011, Berinde and Borcut [1] initiated the concept of a tripled fixed point of a mapping $H : X \times X \times X \to X$. Some tripled fixed (coincidence) point theorems are obtained via the mixed monotone (g-monotone) property, see [2]. Their obtained results are generalizations and extensions of the work due to Bhaskar and Lakshmikantham [3]. As an application, they studied the existence of solutions of a periodic boundary value problem whose coupled fixed point technique cannot solve such a problem.

The concept of cone metric spaces was reintroduced in 2007 by Huang and Zhang [4] by replacing the set of real numbers with an ordered Banach space. This concept was first initiated in literature by showing its importance via a numerical approach by Kantorovich [5]. Note that cone (normed) metric spaces have interesting applications in fixed point theory and the numerical analysis. For instance, see [6–10].

The notion of a fuzzy set was appeared in 1965 by Zadeh [11]. This notion has hardly been studied and extended. Its application on variant fields became fruitful and needful. Data analysis, computational intelligence and artificial intelligence are intensively developed. This theory is also generalized and extended in many directions by means of the theories of aggregation operators and triangular norms and co-norms, see [12–15]. Kramosil and Michalek [16] initiated the notion of a fuzzy metric space. Defining a fuzzy metric is one of the essential problems in fuzzy mathematics, which was frequently used in pattern recognition and fuzzy optimization. In 1994, George and Veeramani [17] presented some fixed point results in fuzzy metric spaces.

On the other hand, by combining the concepts of a cone metric space setting and a fuzzy set, Oner et al. [18] presented in 2015 the notion of a fuzzy cone metric space (as an abbreviation, (F_{CM}-spaces) and the fuzzy cone Banach contraction result was established. Further results in this direction have been investigated, see [19–23]. Very recently, Waheed et al. [24] established some coupled fixed point results in F_{CM}-spaces. Going in the same direction, the aim of this work is to present some tripled fixed point results in this setting for new contractive mappings via a triangular property. We also give nontrivial examples and two illustrated applications making effective the presented results. To our knowledge, this work is the first time to deal with tripled fixed point notion in F_{CM}-spaces.

In mathematics, the Volterra integral equations are a special type of integral equations. They are divided into two groups referred to as the first and the second kind. It is known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Finding solutions of linear or nonlinear Volterra integral equations is highly interesting for researchers and scientists in this field and there are available studies to find analytical or numerical solutions for Volterra integral equations (see [25–28]). In fact, Volterra integral equations are usually solved analytically or numerically by finding approximate solutions to the problems using numerical or analytical approximation methods. For instance, well-posedness and regularity of backward (doubly) stochastic Volterra integral equations have been studied in [29, 30]. Among the methods used to solve Volterra integral equations are Sinc-collocation method [31], Barycentric Lagrange interpolation and the equidistance Chebyshev interpolation nodes [32], relaxed Monte Carlo method [33], etc. We will be concerned in this work to use a fixed point technique via a tripled fixed point approach to solve a system of Volterra integral equations. We will also present an application on Lebesgue integral type mappings using a tripled fixed point result.

This paper is organized as follows: Section 2 giving the essential definitions and known results in the literature that help us in the rest of the paper. In Section 3, we prove our main tripled fixed point results and we provide some concrete examples. The aim of Section 4 is to apply our obtained results by ensuring the existence of a tripled fixed point for a Lebesgue integral mapping and a unique solution for a system of Volterra integral equations.

2. Necessary facts

Definition 2.1. [34] An operation $\star : [0,1]\times[0,1] \to [0,1]$ is described as a continuous $\nu-$norm if it fulfills the following:

- $\star$ is associative and commutative,
- $\star$ is continuous,
- for each $\lambda \in [0,1]$, $1 \star \lambda = \lambda$,
- for each $\lambda_1, \lambda_2, \beta_1, \beta_2 \in [0,1]$, if $\lambda_1 \leq \lambda_2$ and $\beta_1 \leq \beta_2$, then $\lambda_1 \star \beta_1 \leq \lambda_2 \star \beta_2$.

Here $\mathbb{N}$, $\beth$, ϑ and $\nu-$norm, represent the set of natural numbers, the real Banach space, a zero element in $\beth$ and a continuous $\nu-$norm, respectively.

Definition 2.2. [4] A subset $\Upsilon \in \beth$ is called a cone if the following hold:

(1) $\Upsilon \neq \emptyset$ is closed and $\Upsilon \neq \{\vartheta\}$;
(2) If $\lambda_1, \beta_1 \in (0,\infty)$ and $\theta, \rho \in \Upsilon$, then $\lambda_1\theta + \beta_1\rho \in \Upsilon$;
(3) If both $\theta \in \Upsilon$ and $-\theta \in \Upsilon$, then $\theta = \vartheta$.

A partial ordering on a cone Υ is described as $\theta \leq \rho \Longleftrightarrow \rho - \theta \in \Upsilon$. $\theta < \rho$ stands for $\theta \leq \rho$ and $\theta \neq \rho$, while $\theta \ll \rho$ stands for $\rho - \theta \in int(\Upsilon)$. Here each cone has non-empty interior.

Definition 2.3. [18] A 3−tuple $(\Omega, \Theta_{\varpi}, \star)$ is called an $F_{CM}-$space if Υ is a cone in $\beth$, Ω is an arbitrary set, $\star$ is a $\nu-$norm, and Θ_{ϖ} is a fuzzy set on $\Omega^2 \times int(\Upsilon)$ such that the following are satisfied, for all $\theta, \rho, \delta \in \Omega$ and $\nu, \mu \in int(\Upsilon)$,

($\heartsuit_1$) $\Theta_{\varpi}(\theta, \rho, \nu) > \vartheta$ and $\Theta_{\varpi}(\theta, \rho, \nu) = 1 \Leftrightarrow \theta = \rho$;
($\heartsuit_2$) $\Theta_{\varpi}(\theta, \rho, \nu) = \Theta_{\varpi}(\rho, \theta, \nu)$;
($\heartsuit_3$) $\Theta_{\varpi}(\theta, \rho, \nu) \star \Theta_{\varpi}(\rho, \delta, \mu) \leq \Theta_{\varpi}(\theta, \delta, \nu + \mu)$;
($\heartsuit_4$) $\Theta_{\varpi}(\theta, \rho, \cdot) : int(\Upsilon) \to [0,1]$ is continuous.

Definition 2.4. [18] Let $(\Omega, \Theta_{\varpi}, \star)$ be an $F_{CM}-$space, $\theta \in \Omega$ and (θ_i) be a sequence in Ω.

- (θ_i) is called convergent to some θ if for $\nu \gg \vartheta$ and $0 < u < 1$, there exists $i_1 \in \mathbb{N}$ such that $\Theta_{\varpi}(\theta_i, \theta, \nu) > 1 - u$, $\forall i > i_1$, and we can write $\lim_{i\to\infty} \theta_i = \theta$.
- (θ_i) is called a Cauchy sequence if for $\nu \gg \vartheta$ and $0 < u < 1$, there exists $i_1 \in \mathbb{N}$ such that

$$\Theta_{\varpi}(\theta_k, \theta_i, \nu) > 1 - u, \ \forall k, i > i_1.$$

- If every Cauchy sequence is convergent in Ω, then the triple $(\Omega, \Theta_{\varpi}, \star)$ is called complete.
- (θ_i) is called a fuzzy cone contraction (F_{cc}) if there exists $\beta \in (0,1)$ such that

$$\frac{1}{\Theta_{\varpi}(\theta_i, \theta_{i+1}, \nu)} - 1 \leq \beta\left(\frac{1}{\Theta_{\varpi}(\theta_{i-1}, \theta_i, \nu)} - 1\right), \ \forall \nu \gg \vartheta, \ i \geq 1.$$

Definition 2.5. [35] Assume that $(\Omega, \Theta_{\varpi}, \star)$ is an $F_{CM}-$space. Then the fuzzy cone metric Θ_{ϖ} is called *triangular* if

$$\frac{1}{\Theta_{\varpi}(\theta, \delta, \nu)} - 1 \leq \left(\frac{1}{\Theta_{\varpi}(\theta, \rho, \nu)} - 1\right) + \left(\frac{1}{\Theta_{\varpi}(\rho, \delta, \nu)} - 1\right), \ \forall \theta, \rho, \delta \in \Omega, \ \nu \gg \vartheta.$$

Lemma 2.6. *[18] Suppose that* $(\Omega, \Theta_{\varpi}, \star)$ *is an* $F_{CM}-$*space,* $\theta \in \Omega$ *and* (θ_i) *is a sequence in* Ω*, then*

$$\theta_i \to \theta \Leftrightarrow \lim_{i\to\infty} \Theta_{\varpi}(\theta_i, \theta, \nu) = 1, \text{ for } \nu \gg \vartheta.$$

Definition 2.7. [18] Let $(\Omega, \Theta_{\varpi}, \star)$ be an $F_{CM}-$space and $\Xi : \Omega \to \Omega$. Then Ξ is called an $\mathrm{F_{cc}}$ if there exists $g \in (0, 1)$ such that

$$\left(\frac{1}{\Theta_{\varpi}(\Xi\theta, \Xi\rho, \nu)} - 1\right) \le g\left(\frac{1}{\Theta_{\varpi}(\theta, \rho, \nu)} - 1\right), \ \forall \theta, \rho \in \Omega, \ \nu \gg \vartheta.$$

Definition 2.8. [3] A pair (θ, ρ) is called a coupled FP of the mapping $\Xi : \Omega \times \Omega \to \Omega$ if

$$\Xi(\theta, \rho) = \theta \text{ and } \Xi(\rho, \theta) = \rho.$$

Definition 2.9. [1] Let $\Omega \neq \emptyset$. Then a triple $(\theta, \rho, \delta) \in \Omega^3$ is called a TFP of the mapping $\Xi : \Omega^3 \to \Omega$ if $\theta = \Xi(\theta, \rho, \delta)$, $\rho = \Xi(\rho, \delta, \theta)$ and $\delta = \Xi(\delta, \theta, \rho)$.

Example 2.10. Let $\Omega = [0, \infty)$ and $\Xi : \Omega^3 \to \Omega$ be a mapping given by

$$\Xi(\theta, \rho, \delta) = \frac{\theta + \rho + \delta}{3}, \ \forall \theta, \rho, \delta \in \Omega.$$

Then Ξ has a TFP when $\theta = \rho = \delta$.

3. Results and examples

This part is concerned with presenting the main theoretical results of our paper. In addition, some supporting examples are provided.

Theorem 3.1. *Let* $\Xi : \Omega^3 \to \Omega$ *be a mapping defined on a complete* $F_{CM}-$*space* $(\Omega, \Theta_{\varpi}, \star)$ *in which* Θ_{ϖ} *is triangular and fulfills*

$$\begin{aligned} &\frac{1}{\Theta_{\varpi}\left(\Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \Xi\left(\varrho, \widehat{\varrho}, \widetilde{\varrho}\right), \nu\right)} - 1 \\ &\quad \le u_1\left(\frac{1}{\Theta_{\varpi}\left(\varkappa, \varrho, \nu\right)} - 1\right) + u_2\left(\frac{1}{Z\left(\Xi, \left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \left(\varrho, \widehat{\varrho}, \widetilde{\varrho}\right), \nu\right)} - 1\right), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \left(\frac{1}{Z\left(\Xi, \left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \left(\varrho, \widehat{\varrho}, \widetilde{\varrho}\right), \nu\right)} - 1\right) = & \left(\frac{1}{\Theta_{\varpi}\left(\varkappa, \Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \nu\right)} - 1 + \frac{1}{\Theta_{\varpi}\left(\varrho, \Xi\left(\varrho, \widehat{\varrho}, \widetilde{\varrho}\right), \nu\right)} - 1\right. \\ & + \left.\frac{1}{\Theta_{\varpi}\left(\varkappa, \Xi\left(\varrho, \widehat{\varrho}, \widetilde{\varrho}\right), \nu\right)} - 1 + \frac{1}{\Theta_{\varpi}\left(\varrho, \Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \nu\right)} - 1\right), \end{aligned}$$

for all $\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}, \varrho, \widehat{\varrho}, \widetilde{\varrho} \in \Omega$, $\nu \gg \vartheta$ *and* $u_1 \in [0, 1)$ *and* $u_2 \ge 0$ *with* $u_1 + 4u_2 < 1$. *Then* Ξ *has a unique TFP in* Ω.

Proof. Consider $\varkappa_0, \widehat{\varkappa}_0, \widetilde{\varkappa}_0 \in \Omega$. Describe sequences $\{\varkappa_\alpha\}$, $\{\widehat{\varkappa}_\alpha\}$ and $\{\widetilde{\varkappa}_\alpha\}$ in Ω such that

$$\begin{cases} \Xi\left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right) = \varkappa_{\alpha+1}, \\ \Xi\left(\widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha, \varkappa_\alpha\right) = \widehat{\varkappa}_{\alpha+1}, \quad \text{for } \alpha \geq 0. \\ \Xi\left(\widetilde{\varkappa}_\alpha, \varkappa_\alpha, \widehat{\varkappa}_\alpha\right) = \widetilde{\varkappa}_{\alpha+1}, \end{cases} \tag{3.2}$$

By (3.1), for $\nu \gg \vartheta$, we have

$$\begin{aligned} \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1 &= \frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa_{\alpha-1}, \widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}\right), \Xi\left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right), \nu\right)} - 1 \\ &\leq u_1\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1\right) \\ &\quad + u_2\left(\frac{1}{Z\left(\Xi, \left(\varkappa_{\alpha-1}, \widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}\right), \left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right), \nu\right)} - 1\right), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &\frac{1}{Z\left(\Xi, \left(\varkappa_{\alpha-1}, \widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}\right), \left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right), \nu\right)} - 1 \\ &= \left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \Xi\left(\varkappa_{\alpha-1}, \widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}\right), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \Xi\left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right), \nu\right)} - 1\right. \\ &\left. + \frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \Xi\left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \Xi\left(\varkappa_{\alpha-1}, \widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}\right), \nu\right)} - 1\right) \\ &= \left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_{\alpha+1}, \nu\right)} - 1\right) \\ &\leq 2\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1\right). \end{aligned} \tag{3.4}$$

By (3.4) and (3.3), for $\nu \gg \vartheta$, we have

$$\begin{aligned} \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1 &\leq u_1\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1\right) \\ &\quad + 2u_2\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1\right). \end{aligned}$$

By simple calculations, we obtain that

$$\frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1 \leq \nabla\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1\right), \text{ for } \nu \gg \vartheta, \tag{3.5}$$

where $\nabla = \frac{u_1 + 2u_2}{1 - 2u_2} < 1$. Analogously, one can write

$$\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1 \leq \nabla\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-2}, \varkappa_{\alpha-1}, \nu\right)} - 1\right), \text{ for } \nu \gg \vartheta. \tag{3.6}$$

By induction and from (3.5) and (3.6), for $\nu \gg \vartheta$, we conclude that

$$\frac{1}{\Theta_\varpi\left(\varkappa_\alpha, \varkappa_{\alpha+1}, \nu\right)} - 1 \leq \nabla\left(\frac{1}{\Theta_\varpi\left(\varkappa_{\alpha-1}, \varkappa_\alpha, \nu\right)} - 1\right)$$

$$
\begin{aligned}
&\leq \nabla^{2}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-2}, \varkappa_{\alpha-1}, \nu\right)}-1\right) \\
&\quad\vdots \\
&\leq \nabla^{\alpha}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{0}, \varkappa_{1}, \nu\right)}-1\right) \rightarrow 0 \text{ as } \alpha \rightarrow \infty.
\end{aligned}
$$

This implies that $\{\varkappa_{\alpha}\}$ is an $\mathrm{F_{cc}}$ and hence

$$
\lim_{\alpha \rightarrow \infty} \Theta_{\varpi}\left(\varkappa_{\alpha}, \varkappa_{\alpha+1}, \nu\right)=1.
$$

Now, for $\ell>\alpha$ and $\nu \gg \vartheta$, we get

$$
\begin{aligned}
\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha}, \varkappa_{\ell}, \nu\right)}-1 &\leq \frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha}, \varkappa_{\alpha+1}, \nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha+1}, \varkappa_{\alpha+2}, \nu\right)}-1+\cdots+\frac{1}{\Theta_{\varpi}\left(\varkappa_{\ell-1}, \varkappa_{\ell}, \nu\right)}-1 \\
&\leq \nabla^{\alpha}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{0}, \varkappa_{1}, \nu\right)}-1\right)+\nabla^{\alpha+1}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{0}, \varkappa_{1}, \nu\right)}-1\right) \\
&\quad+\cdots+\nabla^{\ell-1}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{0}, \varkappa_{1}, \nu\right)}-1\right) \\
&=\left(\nabla^{\alpha}+\nabla^{\alpha+1}+\cdots+\nabla^{\ell-1}\right)\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{0}, \varkappa_{1}, \nu\right)}-1\right) \\
&=\frac{\nabla^{\alpha}}{1-\nabla}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{0}, \varkappa_{1}, \nu\right)}-1\right) \rightarrow 0 \text{ as } \alpha \rightarrow \infty.
\end{aligned}
$$

This proves that the sequence $\{\varkappa_{\alpha}\}$ is Cauchy. Again, regarding to the sequence $\widehat{\varkappa}_{\alpha}$, by (3.1), for $\nu \gg \vartheta$, we get

$$
\begin{aligned}
\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha}, \widehat{\varkappa}_{\alpha+1}, \nu\right)}-1 &= \frac{1}{\Theta_{\varpi}\left(\Xi\left(\widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}, \varkappa_{\alpha-1}\right), \Xi\left(\widehat{\varkappa}_{\alpha}, \widetilde{\varkappa}_{\alpha}, \varkappa_{\alpha}\right), \nu\right)}-1 \\
&\leq u_{1}\left(\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha-1}, \widehat{\varkappa}_{\alpha}, \nu\right)}-1\right) \\
&\quad+u_{2}\left(\frac{1}{Z\left(\Xi,\left(\widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}, \varkappa_{\alpha-1}\right),\left(\widehat{\varkappa}_{\alpha}, \widetilde{\varkappa}_{\alpha}, \varkappa_{\alpha}\right), \nu\right)}-1\right),
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
&\frac{1}{Z\left(\Xi,\left(\widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}, \varkappa_{\alpha-1}\right),\left(\widehat{\varkappa}_{\alpha}, \widetilde{\varkappa}_{\alpha}, \varkappa_{\alpha}\right), \nu\right)}-1 \\
&=\left(\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha-1}, \Xi\left(\widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}, \varkappa_{\alpha-1}\right), \nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha}, \Xi\left(\widehat{\varkappa}_{\alpha}, \widetilde{\varkappa}_{\alpha}, \varkappa_{\alpha}\right), \nu\right)}-1\right. \\
&\quad\left.+\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha-1}, \Xi\left(\widehat{\varkappa}_{\alpha}, \widetilde{\varkappa}_{\alpha}, \varkappa_{\alpha}\right), \nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha}, \Xi\left(\widehat{\varkappa}_{\alpha-1}, \widetilde{\varkappa}_{\alpha-1}, \varkappa_{\alpha-1}\right), \nu\right)}-1\right) \\
&=\left(\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha-1}, \widehat{\varkappa}_{\alpha}, \nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha}, \widehat{\varkappa}_{\alpha+1}, \nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha-1}, \widehat{\varkappa}_{\alpha+1}, \nu\right)}-1\right) \\
&\leq 2\left(\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha-1}, \widehat{\varkappa}_{\alpha}, \nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\widehat{\varkappa}_{\alpha}, \widehat{\varkappa}_{\alpha+1}, \nu\right)}-1\right).
\end{aligned} \tag{3.8}
$$

It follows from (3.7) and (3.8) that, for $\nu \gg \vartheta$,

$$\begin{aligned}\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\alpha+1},\nu)}-1 &\leq u_1\left(\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-1},\varkappa_{\alpha},\nu)}-1\right)\\ &\quad +2u_2\left(\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-1},\varkappa_{\alpha},\nu)}-1+\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\alpha+1},\nu)}-1\right).\end{aligned}$$

Again, by simple calculations, we obtain that

$$\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa}_{\alpha+1},\nu)}-1 \leq \nabla\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha-1},\widehat{\varkappa}_{\alpha},\nu)}-1\right),\ \text{for } \nu \gg \vartheta, \tag{3.9}$$

where $\nabla = \frac{u_1+2u_2}{1-2u_2} < 1$. Similarly, we have

$$\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha-1},\widehat{\varkappa}_{\alpha},\nu)}-1 \leq \nabla\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha-2},\widehat{\varkappa}_{\alpha-1},\nu)}-1\right),\ \text{for } \nu \gg \vartheta. \tag{3.10}$$

From (3.9), (3.10) and by induction for $\nu \gg \vartheta$, we get

$$\begin{aligned}\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa}_{\alpha+1},\nu)}-1 &\leq \nabla\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha-1},\widehat{\varkappa}_{\alpha},\nu)}-1\right)\\ &\leq \nabla^{2}\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha-2},\widehat{\varkappa}_{\alpha-1},\nu)}-1\right)\\ &\ \ \vdots\\ &\leq \nabla^{\alpha}\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{0},\widehat{\varkappa}_{1},\nu)}-1\right)\to 0 \text{ as } \alpha\to\infty.\end{aligned}$$

This yields that the sequence $\{\widehat{\varkappa}_{\alpha}\}$ is an $\mathrm{F_{cc}}$ and hence

$$\lim_{\alpha\to\infty}\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa}_{\alpha+1},\nu)=1.$$

Now, for $\ell > \alpha$ and $\nu \gg \vartheta$, we get

$$\begin{aligned}\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa}_{\ell},\nu)}-1 &\leq \frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa}_{\alpha+1},\nu)}-1+\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\alpha+1},\widehat{\varkappa}_{\alpha+2},\nu)}-1+\cdots+\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{\ell-1},\widehat{\varkappa}_{\ell},\nu)}-1\\ &\leq \nabla^{\alpha}\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{0},\widehat{\varkappa}_{1},\nu)}-1\right)+\nabla^{\alpha+1}\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{0},\widehat{\varkappa}_{1},\nu)}-1\right)\\ &\quad +\cdots+\nabla^{\ell-1}\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{0},\widehat{\varkappa}_{1},\nu)}-1\right)\\ &= \left(\nabla^{\alpha}+\nabla^{\alpha+1}+\cdots+\nabla^{\ell-1}\right)\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{0},\widehat{\varkappa}_{1},\nu)}-1\right)\\ &= \frac{\nabla^{\alpha}}{1-\nabla}\left(\frac{1}{\Theta_{\varpi}(\widehat{\varkappa}_{0},\widehat{\varkappa}_{1},\nu)}-1\right)\to 0 \text{ as } \alpha\to\infty.\end{aligned}$$

Hence, $\{\widehat{\varkappa}_{\alpha}\}$ is a Cauchy sequence. With the same approach, one can prove that the sequence $\{\widetilde{\varkappa}_{\alpha}\}$ is also Cauchy in Ω. Since Ω is complete, there are $\varkappa, \widehat{\varkappa}$ and $\widetilde{\varkappa}$ in Ω such that $\varkappa_{\alpha}\to\varkappa$, $\widehat{\varkappa}_{\alpha}\to\widehat{\varkappa}$ and $\widetilde{\varkappa}_{\alpha}\to\widetilde{\varkappa}$ as $\alpha\to\infty$. So, we can write

$$\lim_{\alpha\to\infty}\Theta_{\varpi}(\varkappa_{\alpha},\varkappa,\nu)=1,\ \lim_{\alpha\to\infty}\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa},\nu)=1,\ \lim_{\alpha\to\infty}\Theta_{\varpi}(\widetilde{\varkappa}_{\alpha},\widetilde{\varkappa},\nu)=1 \text{ for } \nu\gg\vartheta.$$

Hence

$$\begin{aligned}
\lim_{\alpha\to\infty} \varkappa_{\alpha+1} &= \lim_{\alpha\to\infty} \Xi\left(\varkappa_\alpha, \widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha\right) = \Xi\left(\lim_{\alpha\to\infty} \varkappa_\alpha, \lim_{\alpha\to\infty} \widehat{\varkappa}_\alpha, \lim_{\alpha\to\infty} \widetilde{\varkappa}_\alpha\right) = \Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right) = \varkappa,\\
\lim_{\alpha\to\infty} \widehat{\varkappa}_{\alpha+1} &= \lim_{\alpha\to\infty} \Xi\left(\widehat{\varkappa}_\alpha, \widetilde{\varkappa}_\alpha, \varkappa_\alpha\right) = \Xi\left(\lim_{\alpha\to\infty} \widehat{\varkappa}_\alpha, \lim_{\alpha\to\infty} \widetilde{\varkappa}_\alpha, \lim_{\alpha\to\infty} \varkappa_\alpha\right) = \Xi\left(\widehat{\varkappa}, \widetilde{\varkappa}, \varkappa\right) = \widehat{\varkappa},\\
\lim_{\alpha\to\infty} \widetilde{\varkappa}_{\alpha+1} &= \lim_{\alpha\to\infty} \Xi\left(\widetilde{\varkappa}_\alpha, \varkappa_\alpha, \widehat{\varkappa}_\alpha\right) = \Xi\left(\lim_{\alpha\to\infty} \widetilde{\varkappa}_\alpha, \lim_{\alpha\to\infty} \varkappa_\alpha, \lim_{\alpha\to\infty} \widehat{\varkappa}_\alpha\right) = \Xi\left(\widetilde{\varkappa}, \varkappa, \widehat{\varkappa}\right) = \widetilde{\varkappa}.
\end{aligned}$$

Therefore, a mapping Ξ has a TFP $(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa})$ in Ω^3.

For uniqueness, assume that $(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1)$ is another TFP of Ξ such that $(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1) \neq (\varkappa, \widehat{\varkappa}, \widetilde{\varkappa})$. From (3.1), for $\nu \gg \vartheta$, we can write

$$\begin{aligned}
\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1 &= \frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \Xi\left(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1\right), \nu\right)} - 1\\
&\leq u_1\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right)\\
&\quad + u_2\left(\frac{1}{Z\left(\Xi, \left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \left(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1\right), \nu\right)} - 1\right),
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&\frac{1}{Z\left(\Xi, \left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \left(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1\right), \nu\right)} - 1\\
&= \left(\frac{1}{\Theta_\varpi\left(\varkappa, \Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_1, \Xi\left(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1\right), \nu\right)} - 1\right.\\
&\quad \left. + \frac{1}{\Theta_\varpi\left(\varkappa, \Xi\left(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1\right), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_1, \Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \nu\right)} - 1\right)\\
&= \left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa_1, \varkappa_1, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right)\\
&= 2\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right).
\end{aligned} \tag{3.12}$$

From (3.12) in (3.11), we obtain that

$$\begin{aligned}
\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1 &\leq u_1\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right) + 2u_2\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right)\\
&= \left(u_1 + 2u_2\right)\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right)\\
&= \left(u_1 + 2u_2\right)\left(\frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}\right), \Xi\left(\varkappa_1, \widehat{\varkappa}_1, \widetilde{\varkappa}_1\right), \nu\right)} - 1\right)\\
&\leq \left(u_1 + 2u_2\right)^2\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right)\\
&\ \vdots\\
&\leq \left(u_1 + 2u_2\right)^\alpha\left(\frac{1}{\Theta_\varpi\left(\varkappa, \varkappa_1, \nu\right)} - 1\right) \to 0 \text{ as } \alpha \to \infty,
\end{aligned}$$

where $u_1 + 2u_2 < 1$. This implies that $\Theta_\varpi(\varkappa, \varkappa_1, \nu) = 1$, for $\nu \gg \vartheta$. Thus, $\varkappa = \varkappa_1$. By the same manner, one can obtain $\widehat{\varkappa} = \widehat{\varkappa}_1$ and $\widetilde{\varkappa} = \widetilde{\varkappa}_1$. This completes the proof. □

Corollary 3.2. *Theorem 3.1 is also true if we replace the condition (3.1) with one of the following:*

($\clubsuit_1$) *For* $\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}, \varrho, \widehat{\varrho}, \widetilde{\varrho} \in \Omega$, $\nu \gg \vartheta$,

$$\begin{aligned}&\frac{1}{\Theta_\varpi\left(\Xi(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \Xi(\varrho, \widehat{\varrho}, \widetilde{\varrho}), \nu\right)} - 1\\ &\leq u_1\left(\frac{1}{\Theta_\varpi(\varkappa, \varrho, \nu)} - 1\right) + u_2\left(\frac{1}{\Theta_\varpi\left(\varkappa, \Xi(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varrho, \Xi(\varrho, \widehat{\varrho}, \widetilde{\varrho}), \nu\right)} - 1\right),\end{aligned}$$

for any $u_1 \in [0, 1)$ *and* $u_2 \geq 0$ *with* $(u_1 + 2u_2) < 1$.

($\clubsuit_2$) *For* $\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}, \varrho, \widehat{\varrho}, \widetilde{\varrho} \in \Omega$, $\nu \gg \vartheta$,

$$\begin{aligned}&\frac{1}{\Theta_\varpi\left(\Xi(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \Xi(\varrho, \widehat{\varrho}, \widetilde{\varrho}), \nu\right)} - 1\\ &\leq u_1\left(\frac{1}{\Theta_\varpi(\varkappa, \varrho, \nu)} - 1\right) + u_2\left(\frac{1}{\Theta_\varpi\left(\varkappa, \Xi(\varrho, \widehat{\varrho}, \widetilde{\varrho}), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varrho, \Xi(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \nu\right)} - 1\right),\end{aligned}$$

for any $u_1 \in [0, 1)$ *and* $u_2 \geq 0$ *with* $u_1 + 2u_2 < 1$.

Example 3.3. Let $\Omega = (0, \infty)$, $\star$ be a $\nu-$norm and $\Theta_\varpi : \Omega^2 \times (0, \infty) \to [0, 1]$ be described as

$$\Theta_\varpi(\varkappa, \varrho, \nu) = \frac{\nu}{d(\varkappa, \varrho) + \nu},\ d(\varkappa, \varrho) = |\varkappa - \varrho|,$$

for all $\varkappa, \varrho \in \Omega$, for $\nu > 0$. Clearly, $(\Omega, \Theta_\varpi, \star)$ is a complete $F_{CM}-$space. Define the mapping $\Xi : \Omega^3 \to \Omega$ by

$$\Xi(\varkappa, \varrho, \ell) = \begin{cases} \frac{\varkappa - \varrho - \ell}{21}, & \text{if } \varkappa, \varrho, \ell \in [0, 1),\\ \frac{3\varkappa + 3\varrho + 3\ell - 6}{7} & \text{if } \varkappa, \varrho, \ell \in [1, \infty).\end{cases}$$

Then, for $\nu \gg \vartheta$,

$$\begin{aligned}&\frac{1}{\Theta_\varpi\left(\Xi(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \Xi(\varrho, \widehat{\varrho}, \widetilde{\varrho}), \nu\right)} - 1\\ &= \frac{1}{\Theta_\varpi\left(\frac{\varkappa - \widehat{\varkappa} - \widetilde{\varkappa}}{21}, \frac{\varrho - \widehat{\varrho} - \widetilde{\varrho}}{21}, \nu\right)} - 1\\ &= \frac{1}{\nu}\left(d\left(\frac{\varkappa - \widehat{\varkappa} - \widetilde{\varkappa}}{21}, \frac{\varrho - \widehat{\varrho} - \widetilde{\varrho}}{21}\right)\right)\\ &= \frac{1}{21\nu}\left|\varkappa - \widehat{\varkappa} - \widetilde{\varkappa} - \varrho + \widehat{\varrho} + \widetilde{\varrho}\right|\\ &\leq \frac{1}{21\nu}\left[\left|(\varkappa - \varrho) + (\varkappa - (\varkappa - \widehat{\varkappa} - \widetilde{\varkappa})) + (\varrho - (\varrho - \widehat{\varrho} - \widetilde{\varrho})) + (\varkappa - (\varrho - \widehat{\varrho} - \widetilde{\varrho})) + (\varrho - (\varkappa - \widehat{\varkappa} - \widetilde{\varkappa}))\right|\right]\\ &\leq \frac{1}{21\nu}|\varkappa - \varrho| + \frac{1}{21\nu}\left(\left|\varkappa - (\varkappa - \widehat{\varkappa} - \widetilde{\varkappa})\right| + \left|\varrho - (\varrho - \widehat{\varrho} - \widetilde{\varrho})\right|\left|\varkappa - (\varrho - \widehat{\varrho} - \widetilde{\varrho})\right| + \left|\varrho - (\varkappa - \widehat{\varkappa} - \widetilde{\varkappa})\right|\right)\\ &= \frac{1}{21}\left(\frac{1}{\Theta_\varpi(\varkappa, \varrho, \nu)} - 1\right) + \frac{1}{12}\left(\frac{1}{\Theta_\varpi\left(\varkappa, \Xi(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \nu\right)} - 1 + \frac{1}{\Theta_\varpi\left(\varrho, \Xi(\varrho, \widehat{\varrho}, \widetilde{\varrho}), \nu\right)} - 1\right.\end{aligned}$$

$$+\frac{1}{\Theta_{\varpi}\left(\varkappa,\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\varrho,\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\nu\right)}-1\Bigg)$$
$$=\frac{1}{21}\left(\frac{1}{\Theta_{\varpi}\left(\varkappa,\varrho,\nu\right)}-1\right)+\frac{1}{12}\left(\frac{1}{Z\left(\Xi,\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1\right).$$

Hence, all the conditions of Theorem 3.1 are fulfilled with $u_1=u_2=\frac{1}{21}$. Therefore, Ξ possesses a point $(3,3,3)$ as unique TFP, that is,

$$\Xi(3,3,3)=\frac{3(3)+3(3)+3(3)-6}{7}=3.$$

Theorem 3.4. *Let $(\Omega,\Theta_{\varpi},\star)$ be a complete F_{CM}−space, $\Xi:\Omega^3\to\Omega$ be a given mapping and Θ_{ϖ} be triangular satisfying*

$$\frac{1}{\Theta_{\varpi}\left(\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1$$
$$\leq u_1\left(\frac{1}{\Theta_{\varpi}\left(\varkappa,\varrho,\nu\right)}-1\right)+u_2\left(\frac{1}{\Theta_{\varpi}\left(\varkappa,\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\varrho,\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1\right)$$
$$+u_3\left(\frac{1}{\Theta_{\varpi}\left(\varrho,\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\nu\right)\star\Theta_{\varpi}\left(\varrho,\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1\right)\tag{3.13}$$

for all $\varkappa,\widehat{\varkappa},\widetilde{\varkappa},\varrho,\widehat{\varrho},\widetilde{\varrho}\in\Omega$, $\nu\gg\vartheta$ and $u_1\in[0,1)$ and $u_2,u_3\geq 0$ with $u_1+2u_2+u_3<1$. Then Ξ has a unique TFP in Ω.

Proof. Assume that the sequences given in (3.2) are valid. Then from (3.13), for $\nu\gg\vartheta$, we get

$$\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\varkappa_{\alpha+1},\nu\right)}-1$$
$$=\frac{1}{\Theta_{\varpi}\left(\Xi\left(\varkappa_{\alpha-1},\widehat{\varkappa}_{\alpha-1},\widetilde{\varkappa}_{\alpha-1}\right),\Xi\left(\varkappa_\alpha,\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha\right),\nu\right)}-1$$
$$\leq u_1\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-1},\varkappa_\alpha,\nu\right)}-1\right)$$
$$+u_2\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-1},\Xi\left(\varkappa_{\alpha-1},\widehat{\varkappa}_{\alpha-1},\widetilde{\varkappa}_{\alpha-1}\right),\nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\Xi\left(\varkappa_\alpha,\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha\right),\nu\right)}-1\right)$$
$$+u_3\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\Xi\left(\varkappa_{\alpha-1},\widehat{\varkappa}_{\alpha-1},\widetilde{\varkappa}_{\alpha-1}\right),\nu\right)\star\Theta_{\varpi}\left(\varkappa_\alpha,\Xi\left(\varkappa_\alpha,\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha\right),\nu\right)}-1\right)$$
$$=u_1\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-1},\varkappa_\alpha,\nu\right)}-1\right)+u_2\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-1},\varkappa_\alpha,\nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\varkappa_{\alpha+1},\nu\right)}-1\right)$$
$$+u_3\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\varkappa_\alpha,\nu\right)\star\Theta_{\varpi}\left(\varkappa_\alpha,\varkappa_{\alpha+1},\nu\right)}-1\right)$$
$$=u_1\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-1},\varkappa_\alpha,\nu\right)}-1\right)+u_2\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_{\alpha-1},\varkappa_\alpha,\nu\right)}-1+\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\varkappa_{\alpha+1},\nu\right)}-1\right)$$
$$+u_3\left(\frac{1}{\Theta_{\varpi}\left(\varkappa_\alpha,\varkappa_{\alpha+1},\nu\right)}-1\right).$$

After a routine calculation, we have

$$\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\alpha+1},\nu)}-1\leq \gimel\left(\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-1},\varkappa_{\alpha},\nu)}-1\right),\ \text{for } \nu\gg\vartheta, \tag{3.14}$$

where $\gimel=\frac{u_1+u_2}{1-u_2-u_3}<1$. Similarly,

$$\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-1},\varkappa_{\alpha},\nu)}-1\leq \gimel\left(\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-2},\varkappa_{\alpha-1},\nu)}-1\right),\ \text{for } \nu\gg\vartheta. \tag{3.15}$$

It follows from (3.14), (3.15) and induction that, for $\nu\gg\vartheta$,

$$\begin{aligned}\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\alpha+1},\nu)}-1 &\leq \gimel\left(\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-1},\varkappa_{\alpha},\nu)}-1\right)\\ &\leq \gimel^{2}\left(\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha-2},\varkappa_{\alpha-1},\nu)}-1\right)\\ &\ \ \vdots\\ &\leq \gimel^{\alpha}\left(\frac{1}{\Theta_{\varpi}(\varkappa_{0},\varkappa_{1},\nu)}-1\right)\to 0 \text{ as } \alpha\to\infty.\end{aligned}$$

This implies that $\{\varkappa_{\alpha}\}$ is an $\mathrm{F_{cc}}$ and so we get

$$\lim_{\alpha\to\infty}\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\alpha+1},\nu)=1.$$

Now, for $\ell>\alpha$ and $\nu\gg\vartheta$, we obtain

$$\begin{aligned}\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\ell},\nu)}-1 &\leq \frac{1}{\Theta_{\varpi}(\varkappa_{\alpha},\varkappa_{\alpha+1},\nu)}-1+\frac{1}{\Theta_{\varpi}(\varkappa_{\alpha+1},\varkappa_{\alpha+2},\nu)}-1+\cdots+\frac{1}{\Theta_{\varpi}(\varkappa_{\ell-1},\varkappa_{\ell},\nu)}-1\\ &\leq \gimel^{\alpha}\left(\frac{1}{\Theta_{\varpi}(\varkappa_{0},\varkappa_{1},\nu)}-1\right)+\gimel^{\alpha+1}\left(\frac{1}{\Theta_{\varpi}(\varkappa_{0},\varkappa_{1},\nu)}-1\right)\\ &\quad+\cdots+\gimel^{\ell-1}\left(\frac{1}{\Theta_{\varpi}(\varkappa_{0},\varkappa_{1},\nu)}-1\right)\\ &= \left(\gimel^{\alpha}+\gimel^{\alpha+1}+\cdots+\gimel^{\ell-1}\right)\left(\frac{1}{\Theta_{\varpi}(\varkappa_{0},\varkappa_{1},\nu)}-1\right)\\ &= \frac{\gimel^{\alpha}}{1-\gimel}\left(\frac{1}{\Theta_{\varpi}(\varkappa_{0},\varkappa_{1},\nu)}-1\right)\to 0 \text{ as } \alpha\to\infty.\end{aligned}$$

This proves that the sequence $\{\varkappa_{\alpha}\}$ is Cauchy. In the same scenario, it can be shown that the sequences $\{\widehat{\varkappa}_{\alpha}\}$ and $\{\widetilde{\varkappa}_{\alpha}\}$ are Cauchy. Since Ω is complete, there are $\varkappa,\widehat{\varkappa}$ and $\widetilde{\varkappa}$ in Ω such that $\varkappa_{\alpha}\to\varkappa,\widehat{\varkappa}_{\alpha}\to\widehat{\varkappa}$ and $\widetilde{\varkappa}_{\alpha}\to\widetilde{\varkappa}$ as $\alpha\to\infty$. Hence, one can write

$$\lim_{\alpha\to\infty}\Theta_{\varpi}(\varkappa_{\alpha},\varkappa,\nu)=1,\ \lim_{\alpha\to\infty}\Theta_{\varpi}(\widehat{\varkappa}_{\alpha},\widehat{\varkappa},\nu)=1,\ \lim_{\alpha\to\infty}\Theta_{\varpi}(\widetilde{\varkappa}_{\alpha},\widetilde{\varkappa},\nu)=1 \text{ for } \nu\gg\vartheta.$$

Thus,

$$\lim_{\alpha\to\infty}\varkappa_{\alpha+1} = \lim_{\alpha\to\infty}\Xi(\varkappa_{\alpha},\widehat{\varkappa}_{\alpha},\widetilde{\varkappa}_{\alpha})=\Xi\left(\lim_{\alpha\to\infty}\varkappa_{\alpha},\lim_{\alpha\to\infty}\widehat{\varkappa}_{\alpha},\lim_{\alpha\to\infty}\widetilde{\varkappa}_{\alpha}\right)=\Xi(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})=\varkappa,$$

$$\lim_{\alpha\to\infty}\widehat{\varkappa}_{\alpha+1} = \lim_{\alpha\to\infty}\Xi\left(\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha,\varkappa_\alpha\right) = \Xi\left(\lim_{\alpha\to\infty}\widehat{\varkappa}_\alpha,\lim_{\alpha\to\infty}\widetilde{\varkappa}_\alpha,\lim_{\alpha\to\infty}\varkappa_\alpha\right) = \Xi\left(\widehat{\varkappa},\widetilde{\varkappa},\varkappa\right) = \widehat{\varkappa},$$
$$\lim_{\alpha\to\infty}\widetilde{\varkappa}_{\alpha+1} = \lim_{\alpha\to\infty}\Xi\left(\widetilde{\varkappa}_\alpha,\varkappa_\alpha,\widehat{\varkappa}_\alpha\right) = \Xi\left(\lim_{\alpha\to\infty}\widetilde{\varkappa}_\alpha,\lim_{\alpha\to\infty}\varkappa_\alpha,\lim_{\alpha\to\infty}\widehat{\varkappa}_\alpha\right) = \Xi\left(\widetilde{\varkappa},\varkappa,\widehat{\varkappa}\right) = \widetilde{\varkappa}.$$

Therefore, a mapping Ξ possesses a TFP $(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})$ in Ω^3.

Now, let $(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)$ be a TFP of Ξ such that $(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)\neq(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})$. From (3.1), for $\nu\gg\vartheta$, we have

$$\begin{aligned}
\frac{1}{\Theta_\varpi(\varkappa,\varkappa_1,\nu)}-1 &= \frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\Xi\left(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1\right),\nu\right)}-1\\
&\leq u_1\left(\frac{1}{\Theta_\varpi(\varkappa,\varkappa_1,\nu)}-1\right)\\
&\quad +u_2\left(\frac{1}{\Theta_\varpi\left(\varkappa,\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\nu\right)}-1+\frac{1}{\Theta_\varpi\left(\varkappa_1,\Xi\left(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1\right),\nu\right)}-1\right)\\
&\quad +u_3\left(\frac{1}{\Theta_\varpi\left(\varkappa_1,\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\nu\right)\star\Theta_\varpi\left(\varkappa_1,\Xi\left(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1\right),\nu\right)}-1\right)\\
&= (u_1+u_3)\left(\frac{1}{\Theta_\varpi(\varkappa,\varkappa_1,\nu)}-1\right)\\
&= (u_1+u_3)\left(\frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\Xi\left(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1\right),\nu\right)}-1\right)\\
&\leq (u_1+u_3)^2\left(\frac{1}{\Theta_\varpi(\varkappa,\varkappa_1,\nu)}-1\right)\\
&\vdots\\
&\leq (u_1+u_3)^\alpha\left(\frac{1}{\Theta_\varpi(\varkappa,\varkappa_1,\nu)}-1\right)\to 0 \text{ as } \alpha\to\infty.
\end{aligned}$$

Therefore, $\Theta_\varpi(\varkappa,\varkappa_1,\nu)=1$. Thus it follows that $\varkappa=\varkappa_1$. Analogously, one can obtain $\widehat{\varkappa}=\widehat{\varkappa}_1$ and $\widetilde{\varkappa}=\widetilde{\varkappa}_1$. This finishes the proof. □

Corollary 3.5. *Theorem 3.4 is valid if we replace the condition (3.13) with the following condition: For* $\varkappa,\widehat{\varkappa},\widetilde{\varkappa},\varrho,\widehat{\varrho},\widetilde{\varrho}\in\Omega$, $\nu\gg\vartheta$,

$$\begin{aligned}
&\frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1\\
&\leq u_1\left(\frac{1}{\Theta_\varpi(\varkappa,\varrho,\nu)}-1\right)+u_3\left(\frac{1}{\Theta_\varpi\left(\varrho,\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\nu\right)\star\Theta_\varpi\left(\varrho,\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1\right),
\end{aligned}$$

for any $u_1\in[0,1)$ *and* $u_3\geq 0$ *with* $u_1+2u_3<1$.

In order to support Theorem 3.4, the example below is considered.

Example 3.6. Assume that all the requirements of Example 3.3 hold. Then, from (3.13), for $\nu\gg\vartheta$, we get

$$\frac{1}{\Theta_\varpi\left(\Xi\left(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}\right),\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)}-1$$

$$
\begin{aligned}
&= \frac{1}{\Theta_{\varpi}\left(\frac{\varkappa-\widehat{\varkappa}-\widetilde{\varkappa}}{21}, \frac{\varrho-\widehat{\varrho}-\widetilde{\varrho}}{21}, \nu\right)} - 1 \\
&= \frac{1}{\nu}\left(d\left(\left(\frac{\varkappa-\widehat{\varkappa}-\widetilde{\varkappa}}{21}, \frac{\varrho-\widehat{\varrho}-\widetilde{\varrho}}{21}\right)\right)\right) \\
&= \frac{1}{21\nu}\left|\varkappa-\widehat{\varkappa}-\widetilde{\varkappa}-\varrho+\widehat{\varrho}+\widetilde{\varrho}\right| \\
&\leq \frac{1}{21\nu}\left|(\varkappa-\varrho)+(\varkappa-(\varkappa-\widehat{\varkappa}-\widetilde{\varkappa}))+(\varrho-(\varrho-\widehat{\varrho}-\widetilde{\varrho}))\right| \\
&\leq \frac{1}{21\nu}\left|\varkappa-\varrho\right| + \frac{1}{21\nu}\left|(\varkappa-(\varkappa-\widehat{\varkappa}-\widetilde{\varkappa}))+(\varrho-(\varrho-\widehat{\varrho}-\widetilde{\varrho}))\right| \\
&= \frac{1}{21}\left(\frac{1}{\Theta_{\varpi}(\varkappa,\varrho,\nu)} - 1\right) + \frac{1}{21}\left(\frac{1}{\Theta_{\varpi}(\varkappa,\Xi(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\nu)} - 1 + \frac{1}{\Theta_{\varpi}(\varrho,\Xi(\varrho,\widehat{\varrho},\widetilde{\varrho}),\nu)} - 1\right).
\end{aligned}
$$

It is easy to prove that all the requirements of Theorem 3.4 are satisfied with $u_1 = u_2 = \frac{1}{21}$ and $u_3 = 0$. Hence, Ξ has a unique strong TFP in Ω, which is $(3, 3, 3)$.

4. Applications

4.1. Lebesgue integral type mappings

This part is devoted to discuss an application on Lebesgue integral type mappings to strengthen our theoretical results.

In 2002, Branciari [36] presented the following theorem:

Theorem 4.1. *Let Ξ be a mapping defined in a complete metric space (Ω, d) satisfying*

$$
\int_0^{d(\Xi\varkappa,\Xi\varrho)} \omega(\eta)d\eta \leq \rho \int_0^{d(\varkappa,\varrho)} \omega(\eta)d\eta,
$$

where $\rho \in [0, 1)$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable function which is summable, positive and such that $\int_0^{\epsilon} \omega(\eta)d\eta > 0$ for each $\epsilon > 0$. Then Ξ has a unique fixed point $q \in \Omega$. Moreover, for all $\varkappa \in \Omega$, $\lim_{\alpha\to\infty} \Xi^{\alpha}\varkappa = q$.

According to the above idea, we obtain a unique TFP result in $F_{CM}-$space.

Theorem 4.2. *Let $\Xi : \Omega^3 \to \Omega$ be a mapping defined on a complete $F_{CM}-$space $(\Omega, \Theta_{\varpi}, \star)$ in which Θ_{ϖ} is triangular and fulfills*

$$
\int_0^{\left(\frac{1}{\Theta_{\varpi}(\Xi(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\Xi(\varrho,\widehat{\varrho},\widetilde{\varrho}),\nu)}-1\right)} \omega(\eta)d\eta \leq \quad u_1 \int_0^{\left(\frac{1}{\Theta_{\varpi}(\varkappa,\varrho,\nu)}-1\right)} \omega(\eta)d\eta + u_2 \int^{\left(\frac{1}{Z(\Xi,(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),(\varrho,\widehat{\varrho},\widetilde{\varrho}),\nu)}-1\right)} \omega(\eta)d\eta, \tag{4.1}
$$

where $\left(\frac{1}{Z(\Xi,(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),(\varrho,\widehat{\varrho},\widetilde{\varrho}),\nu)} - 1\right)$ is defined in Theorem 3.1, for all $\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}, \varrho, \widehat{\varrho}, \widetilde{\varrho} \in \Omega$, $\nu \gg \vartheta$ and $u_1 \in [0, 1)$, $u_2 \geq 0$ with $u_1 + 4u_2 < 1$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable function which is summable, positive and such that $e^{\epsilon} > 0$ for each $\epsilon > 0$. Then Ξ possesses a unique TFP in Ω.

Proof. Define a sequences $\{\varkappa_\alpha\}$, $\{\widehat{\varkappa}_\alpha\}$ and $\{\widetilde{\varkappa}_\alpha\}$ as (3.2). Then by (4.1) and from some statements of the proof of Theorem 3.1, for $\nu \gg \vartheta$, we get

$$\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_\alpha,\varkappa_{\alpha+1},\nu)}-1\right)}\omega(\eta)d\eta=\int_0^{\left(\frac{1}{\Theta_\varpi(\Xi(\varkappa_{\alpha-1},\widehat{\varkappa}_{\alpha-1},\widetilde{\varkappa}_{\alpha-1}),\Xi(\varkappa_\alpha,\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha),\nu)}-1\right)}\omega(\eta)d\eta\le\nabla\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_{\alpha-1},\varkappa_\alpha,\nu)}-1\right)}\omega(\eta)d\eta, \quad (4.2)$$

where $\nabla=\frac{u_1+2u_2}{1-2u_2}<1$. Similarly, by using the same arguments, we obtain that

$$\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_{\alpha-1},\varkappa_\alpha,\nu)}-1\right)}\omega(\eta)d\eta\le\nabla\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_{\alpha-2},\varkappa_{\alpha-1},\nu)}-1\right)}\omega(\eta)d\eta,\ \text{for } \nu\gg\vartheta. \quad (4.3)$$

From (4.2), (4.3) and by induction for $\nu \gg \vartheta$, we can write

$$\begin{aligned}\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_\alpha,\varkappa_{\alpha+1},\nu)}-1\right)}\omega(\eta)d\eta &\le \nabla\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_{\alpha-1},\varkappa_\alpha,\nu)}-1\right)}\omega(\eta)d\eta\\ &\le \nabla^2\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_{\alpha-2},\varkappa_{\alpha-1},\nu)}-1\right)}\omega(\eta)d\eta\\ &\vdots\\ &\le \nabla^\alpha\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_0,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta\to 0 \text{ as } \alpha\to\infty.\end{aligned}$$

This proves that the sequence $\{\varkappa_\alpha\}$ is an $\mathrm{F_{cc}}$ and hence

$$\lim_{\alpha\to\infty}\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_\alpha,\varkappa_{\alpha+1},\nu)}-1\right)}\omega(\eta)d\eta=0\Rightarrow\lim_{\alpha\to\infty}\left(\frac{1}{\Theta_\varpi(\varkappa_\alpha,\varkappa_{\alpha+1},\nu)}-1\right)=0,\ \text{for } \nu\gg\vartheta.$$

Thus

$$\lim_{\alpha\to\infty}\Theta_\varpi(\varkappa_\alpha,\varkappa_{\alpha+1},\nu)=1,\ \text{for } \nu\gg\vartheta.$$

Now, for $\ell>\alpha$ and $\nu\gg\vartheta$, we get

$$\begin{aligned}\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_\alpha,\varkappa_\ell,\nu)}-1\right)}\omega(\eta)d\eta &\le \int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_\alpha,\varkappa_{\alpha+1},\nu)}-1\right)}\omega(\eta)d\eta+\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_{\alpha+1},\varkappa_{\alpha+2},\nu)}-1\right)}\omega(\eta)d\eta\\ &\quad+\cdots+\int_0^{\frac{1}{\Theta_\varpi(\varkappa_{\ell-1},\varkappa_\ell,\nu)}-1}\omega(\eta)d\eta\\ &\le \nabla^\alpha\int_0^{\frac{1}{\Theta_\varpi(\varkappa_0,\varkappa_1,\nu)}-1}\omega(\eta)d\eta+\nabla^{\alpha+1}\int_0^{\frac{1}{\Theta_\varpi(\varkappa_0,\varkappa_1,\nu)}-1}\omega(\eta)d\eta\\ &\quad+\cdots+\nabla^{\ell-1}\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_0,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta\\ &= \left(\nabla^\alpha+\nabla^{\alpha+1}+\cdots+\nabla^{\ell-1}\right)\int_0^{\left(\frac{1}{\Theta_\varpi(\varkappa_0,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta\end{aligned}$$

$$= \frac{\nabla^{\alpha}}{1-\nabla}\int^{\left(\frac{1}{\Theta_{\varpi}(\varkappa_0,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta \to 0 \text{ as } \alpha\to\infty.$$

It follows that

$$\lim_{\alpha\to\infty}\int_0^{\left(\frac{1}{\Theta_{\varpi}(\varkappa_\alpha,\varkappa_\ell,\nu)}-1\right)}\omega(\eta)d\eta = 0 \Rightarrow \lim_{\alpha\to\infty}\left(\frac{1}{\Theta_{\varpi}(\varkappa_\alpha,\varkappa_\ell,\nu)}-1\right)=0.$$

This proves that $\{\varkappa_\alpha\}$ is a Cauchy sequence in Ω. By the same approach, we can show that $\{\widehat{\varkappa}_\alpha\}$ and $\{\widetilde{\varkappa}_\alpha\}$ are Cauchy sequences in Ω. Since Ω is complete, there are $\varkappa$, $\widehat{\varkappa}$ and $\widetilde{\varkappa}$ in Ω such that $\varkappa_\alpha \to \varkappa$, $\widehat{\varkappa}_\alpha \to \widehat{\varkappa}$ and $\widetilde{\varkappa}_\alpha \to \widetilde{\varkappa}$ as $\alpha\to\infty$. Therefore,

$$\lim_{\alpha\to\infty}\Theta_{\varpi}(\varkappa_\alpha,\varkappa,\nu)=1, \lim_{\alpha\to\infty}\Theta_{\varpi}(\widehat{\varkappa}_\alpha,\widehat{\varkappa},\nu)=1, \lim_{\alpha\to\infty}\Theta_{\varpi}(\widetilde{\varkappa}_\alpha,\widetilde{\varkappa},\nu)=1 \text{ for } \nu\gg\vartheta.$$

Hence

$$\begin{aligned}
\lim_{\alpha\to\infty}\varkappa_{\alpha+1} &= \lim_{\alpha\to\infty}\Xi(\varkappa_\alpha,\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha) = \Xi\left(\lim_{\alpha\to\infty}\varkappa_\alpha, \lim_{\alpha\to\infty}\widehat{\varkappa}_\alpha, \lim_{\alpha\to\infty}\widetilde{\varkappa}_\alpha\right) = \Xi(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}) = \varkappa,\\
\lim_{\alpha\to\infty}\widehat{\varkappa}_{\alpha+1} &= \lim_{\alpha\to\infty}\Xi(\widehat{\varkappa}_\alpha,\widetilde{\varkappa}_\alpha,\varkappa_\alpha) = \Xi\left(\lim_{\alpha\to\infty}\widehat{\varkappa}_\alpha, \lim_{\alpha\to\infty}\widetilde{\varkappa}_\alpha, \lim_{\alpha\to\infty}\varkappa_\alpha\right) = \Xi(\widehat{\varkappa},\widetilde{\varkappa},\varkappa) = \widehat{\varkappa},\\
\lim_{\alpha\to\infty}\widetilde{\varkappa}_{\alpha+1} &= \lim_{\alpha\to\infty}\Xi(\widetilde{\varkappa}_\alpha,\varkappa_\alpha,\widehat{\varkappa}_\alpha) = \Xi\left(\lim_{\alpha\to\infty}\widetilde{\varkappa}_\alpha, \lim_{\alpha\to\infty}\varkappa_\alpha, \lim_{\alpha\to\infty}\widehat{\varkappa}_\alpha\right) = \Xi(\widetilde{\varkappa},\varkappa,\widehat{\varkappa}) = \widetilde{\varkappa}.
\end{aligned}$$

For uniqueness, suppose that $(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)$ is another TFP of Ξ such that $(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)\neq(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})$. By a similar method to the proof of Theorem 3.1 and using (4.1), for $\nu\gg\vartheta$, we have

$$\begin{aligned}
\int_0^{\left(\frac{1}{\Theta_{\varpi}(\varkappa,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta &= \int_0^{\left(\frac{1}{\Theta_{\varpi}(\Xi(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\Xi(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\nu)}-1\right)}\omega(\eta)d\eta\\
&\leq (u_1+2u_2)\int_0^{\left(\frac{1}{\Theta_{\varpi}(\varkappa,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta\\
&= (u_1+2u_2)\int_0^{\left(\frac{1}{\Theta_{\varpi}(\Xi(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\Xi(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\nu)}-1\right)}\omega(\eta)d\eta\\
&\leq (u_1+2u_2)^2\int_0^{\left(\frac{1}{\Theta_{\varpi}(\varkappa,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta\\
&\vdots\\
&\leq (u_1+2u_2)^{\alpha}\int_0^{\left(\frac{1}{\Theta_{\varpi}(\varkappa,\varkappa_1,\nu)}-1\right)}\omega(\eta)d\eta \to 0 \text{ as } \alpha\to\infty.
\end{aligned}$$

Hence, we obtain $\Theta_{\varpi}(\varkappa,\varkappa_1,\nu)=1$, for $\nu\gg\vartheta$. This implies that $\varkappa=\varkappa_1$. Similarly, $\widehat{\varkappa}=\widehat{\varkappa}_1$ and $\widetilde{\varkappa}=\widetilde{\varkappa}_1$. This finishes the required. □

4.2. Solving a system of Volterra integral equations

In this part, we apply Theorem 3.1 to ensure the existence of a solution to the system of Volterra integral equations (see also [37,38]). Solving this system is equivalent to find a TFP of the mapping Ξ.

Let $B=(C[0,1],\mathbb{R})$ be the space of all real continuous functions and describe a supremum norm on B by

$$\|\varkappa\| = \sup_{l\in[0,1]} |\varkappa(l)|, \text{ for all } \varkappa \in B.$$

Define a distance $d: B\times B \to \mathbb{R}$ by

$$d(\varkappa,\varrho) = \sup_{l\in[0,1]} |\varkappa(l)-\varrho(l)| = \|\varkappa-\varrho\|, \ \forall \varkappa,\varrho \in B.$$

Since $\star$ is a ν−norm, we get $\star(r,s)=rs$, $\forall r,s\in[0,1]$. Define a fuzzy metric $\Theta_{\varpi}: B\times B\times(0,\infty)\to[0,1]$ by

$$Q_c(\varkappa,\varrho,\nu) = \frac{\nu}{\nu+d(\varkappa,\varrho)}, \ d(\varkappa,\varrho) = \|\varkappa-\varrho\|, \tag{4.4}$$

for $\varkappa,\varrho\in B$ and $\nu \gg \vartheta$. Obviously, Θ_{ϖ} is triangular and $(B,\Theta_{\varpi},\star)$ is a complete F_{CM}−space. Consider the following system:

$$\begin{cases} \varrho(\eta) = \xi_1(\eta) + \int\limits_0^1 \mho_1(\eta,\zeta,\varrho(\zeta))d\zeta, \\ \widehat{\varrho}(\eta) = \xi_2(\eta) + \int\limits_0^1 \mho_2(\eta,\zeta,\widehat{\varrho}(\zeta))d\zeta, \\ \widetilde{\varrho}(\eta) = \xi_3(\eta) + \int\limits_0^1 \mho_3(\eta,\zeta,\widetilde{\varrho}(\zeta))d\zeta, \end{cases} \tag{4.5}$$

where $\eta\in\mathbb{R}$, and $\xi_1,\xi_2,\xi_3\in B$.

To study the existence of the solution to system (4.5), we consider the following:

(h_1) The functions $\xi_i:[0,1]\to\mathbb{R}$ and $\mho_i:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$, $(i=1,2,3)$ are continuous;

(h_2) For $\varrho,\widehat{\varphi},\widetilde{\kappa}\in A$, $\widehat{\varrho},\widetilde{\varphi},\kappa\in G$ and $\widetilde{\varrho},\varphi,\widehat{\kappa}\in H$, where $A,G,H\subset B$ such that

$$\begin{cases} A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}(\eta) = \int\limits_0^1 \mho_1(\eta,\zeta,(\varrho,\widehat{\varrho},\widetilde{\varrho})(\zeta))d\zeta \\ G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}(\eta) = \int\limits_0^1 \mho_2(\eta,\zeta,(\varphi,\widehat{\varphi},\widetilde{\varphi})(\zeta))d\zeta \quad , \ \eta\in[0,1]; \\ H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}(\eta) = \int\limits_0^1 \mho_3(\eta,\zeta,(\kappa,\widehat{\kappa},\widetilde{\kappa})(\zeta))d\zeta \end{cases}$$

(h_3) There exists $\gamma\in[0,1]$ such that

$$\begin{cases} \left\|\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}+\xi_1\right)-\left(G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}+\xi_2\right)\right\| \le \gamma\mho\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})},G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}\right) \\ \left\|\left(G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}+\xi_2\right)-\left(H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}+\xi_3\right)\right\| \le \gamma\mho\left(G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})},H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right) \\ \left\|\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}+\xi_1\right)-\left(H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}+\xi_3\right)\right\| \le \gamma\mho\left(A_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})},H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right) \end{cases},$$

where

$$\mho\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})},G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}\right) = \max\left\{ \begin{array}{c} \|A_{\varkappa}-G_{\ell}\|, \\ \left\|A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}-A_{\varrho}\right\| + \left\|G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}-G_{\varphi}\right\| \\ +\left\|G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}-A_{\varrho}\right\| + \left\|A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}-G_{\varphi}\right\| \end{array} \right\}.$$

Analogously, $\mho\left(G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}, H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)$ and $\mho\left(A_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)$, where $A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}, G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}, H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}, A_{\xi_1}, G_{\xi_2}, H_{\xi_3}, A_{\varrho}, G_{\varphi}, H_{\kappa} \in B$.

Theorem 4.3. *Problem (4.5) has a unique solution, provided that the conditions* $(h_1)-(h_3)$ *hold.*

Proof. Define an operator $\Xi : B^3 \to B$ by

$$\begin{cases} \Xi(\varrho,\widehat{\varrho},\widetilde{\varrho}) = A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})} + \xi_1, \\ \Xi(\varphi,\widehat{\varphi},\widetilde{\varphi}) = G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})} + \xi_2, \\ \Xi(\kappa,\widehat{\kappa},\widetilde{\kappa}) = H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})} + \xi_3. \end{cases} \tag{4.6}$$

Now, we need to assume the the following cases:

($\heartsuit_1$) If $\mho\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}, G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}\right) = \|A_{\varkappa} - G_{\ell}\|$, then by (4.4) and (4.5), we get

$$\begin{aligned} \frac{1}{\Theta_{\varpi}\left(\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\Xi\left(\varphi,\widehat{\varphi},\widetilde{\varphi}\right),\nu\right)} - 1 &= \frac{1}{\nu}\left\|\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right) - \Xi\left(\varphi,\widehat{\varphi},\widetilde{\varphi}\right)\right\| \\ &= \frac{\gamma}{\nu}\mho\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}, G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}\right) \\ &= \frac{\gamma}{\nu}\left\|A_{\varrho} - G_{\varphi}\right\| \\ &= \gamma\left(\frac{1}{\Theta_{\varpi}\left(\varrho,\varphi,\nu\right)} - 1\right), \end{aligned}$$

for $\nu \gg \vartheta$, and for $\varrho, \widehat{\varphi}, \in A$, $\widehat{\varrho}, \widetilde{\varphi} \in G$ and $\widetilde{\varrho}, \varphi \in H$. Therefore Ξ fulfills all the conditions of Theorem 3.1 with $u_1 = \gamma$ and $u_2 = 0$. Thus, the problem (4.5) has a unique solution in B.

($\heartsuit_2$) If

$$\mho\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}, G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}\right) = \begin{pmatrix} \left\|A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})} - A_{\varrho}\right\| + \left\|G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})} - G_{\varphi}\right\| \\ + \left\|G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})} - A_{\varrho}\right\| + \left\|A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})} - G_{\varphi}\right\| \end{pmatrix},$$

then from (4.4) and (4.5), we have

$$\begin{aligned} &\frac{1}{\Theta_{\varpi}\left(\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\Xi\left(\varphi,\widehat{\varphi},\widetilde{\varphi}\right),\nu\right)} - 1 \\ =\ & \frac{1}{\nu}\left\|\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right) - \Xi\left(\varphi,\widehat{\varphi},\widetilde{\varphi}\right)\right\| \\ =\ & \frac{\gamma}{\nu}\mho\left(A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})}, G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}\right) \\ =\ & \frac{\gamma}{\nu}\begin{pmatrix} \left\|A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})} - A_{\varrho}\right\| + \left\|G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})} - G_{\varphi}\right\| \\ + \left\|G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})} - A_{\varrho}\right\| + \left\|A_{(\varrho,\widehat{\varrho},\widetilde{\varrho})} - G_{\varphi}\right\| \end{pmatrix} \\ =\ & \gamma\left(\frac{1}{\Theta_{\varpi}\left(\varrho,\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)} - 1 + \frac{1}{\Theta_{\varpi}\left(\varphi,\Xi\left(\varphi,\widehat{\varphi},\widetilde{\varphi}\right),\nu\right)} - 1\right. \\ &\left. + \frac{1}{\Theta_{\varpi}\left(\varrho,\Xi\left(\varphi,\widehat{\varphi},\widetilde{\varphi}\right),\nu\right)} - 1 + \frac{1}{\Theta_{\varpi}\left(\varphi,\Xi\left(\varrho,\widehat{\varrho},\widetilde{\varrho}\right),\nu\right)} - 1\right) \end{aligned}$$

for $\nu \gg \vartheta$, and for $\varrho, \widehat{\varphi}, \in A$, $\widehat{\varrho}, \widetilde{\varphi} \in G$ and $\widetilde{\varrho}, \varphi \in H$. Therefore, Ξ satisfies all the conditions of Theorem 3.1 with $u_1 = 0$ and $u_2 = \gamma$. Thus the system (4.5) has a unique solution in B.

Similarly, we can finish the proof if we consider $\mho\left(G_{(\varphi,\widehat{\varphi},\widetilde{\varphi})}, H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)$ and $\mho\left(A_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, H_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)$ under the same cases ($\heartsuit_1$) and ($\heartsuit_2$). ∎

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

References

1. V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, *Nonlinear Anal. Theor.*, **74** (2011), 4889–4897. http://dx.doi.org/10.1016/j.na.2011.03.032
2. M. Borcut, V. Berinde, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, *Appl. Math. Comput.*, **218** (2012), 5929–5936. http://dx.doi.org/10.1016/j.amc.2011.11.049
3. T. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal. Theor.*, **65** (2006), 1379–1393. http://dx.doi.org/10.1016/j.na.2005.10.017
4. L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. http://dx.doi.org/10.1016/j.jmaa.2005.03.087
5. L. V. Kantorovich, The method of successive approximations for functional equations, *Acta Math.*, **71** (1939), 63–97. http://dx.doi.org/10.1007/BF02547750
6. J. Schroder, Das Iterationsverfahren bei allgemeinerem Abstandsbegriff, *Math. Z.*, **66** (1956), 111–116. http://dx.doi.org/10.1007/BF01186599
7. P. P. Zabrejko, K-metric and K-normed linear spaces: survey, *Collect. Math.*, **48** (1997), 825–859.
8. I. A. Rus, A. Petrusel, G. Petrusel, *Fixed point theory*, Cluj-Napoca: Cluj University Press, 2008.
9. S. Rezapour, R. Hamlbarani, Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **345** (2008), 719–724. http://dx.doi.org/10.1016/j.jmaa.2008.04.049
10. A. Amini-Harandi, M. Fakhar, Fixed point theory in cone metric spaces obtained via the scalarization method, *Comput. Math. Appl.*, **59** (2010), 3529–3534. http://dx.doi.org/10.1016/j.camwa.2010.03.046
11. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.
12. D. Dubois, H. Prade, *Fuzzy sets and systems*, New York: Academic Press, 1988.
13. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Dordrecht: Kluwer Acad. Press, 2000.
14. G. J. Klir, T. A. Folder, *Fuzzy sets, uncertainty and information*, Englewood Cliffs: Prentice Hall, 1988.
15. M. Mares, *Computations over fuzzy quantities*, Boca Raton: CRC–Press, 1994.

16. O. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika*, **11** (1975), 336–344.

17. A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Set. Syst.*, **64** (1994), 395–399. http://dx.doi.org/10.1016/0165-0114(94)90162-7

18. T. Oner, M. B. Kandemire, B. Tanay, Fuzzy cone metric spaces, *J. Nonlinear Sci. Appl.*, **8** (2015), 610–616. http://dx.doi.org/10.22436/jnsa.008.05.13

19. S. U. Rehman, H. Aydi, Rational fuzzy cone contractions on fuzzy cone metric spaces with an application to Fredholm integral equations, *J. Funct. Space.*, **2021** (2021), 5527864. http://dx.doi.org/10.1155/2021/5527864

20. T. Oner, On some results in fuzzy cone metric spaces, *International Journal of Advanced Computational Engineering and Networking*, **4** (2016), 37–39.

21. K. Javed, F. Uddin, H. Aydi, A. Mukheimer, M. Arshad, Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application, *J. Math.*, **2021** (2021), 6663707. http://dx.doi.org/10.1155/2021/6663707

22. S. Jabeen, S. U. Rehman, Z. Zheng, W. Wei, Weakly compatible and quasi-contraction results in fuzzy cone metric spaces with application to the Urysohn type integral equations, *Adv. Differ. Equ.*, **2020** (2020), 280. http://dx.doi.org/10.1186/s13662-020-02743-5

23. H. A. Hammad, M. De la Sen, Exciting fixed point results under a new control function with supportive application in fuzzy cone metric spaces, *Mathematics*, **9** (2021), 2267. http://dx.doi.org/10.3390/math9182267

24. M. T. Waheed, S. U. Rehman, N. Jan, A. Gumaei, M. Al-Rakhami, Some new coupled fixed-point findings depending on another function in fuzzy cone metric spaces with application, *Math. Probl. Eng.*, **2021** (2021), 4144966. http://dx.doi.org/10.1155/2021/4144966

25. I. Aziz, I. Siraj-Ul, F. Khan, A new method based on Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations, *J. Comput. Appl. Math.*, **272** (2014), 70–80. http://dx.doi.org/10.1016/j.cam.2014.04.027

26. A. Doucet, A. M. Johansen, V. B. Tadić, On solving integral equations using Markov chain Monte Carlo methods, *Appl. Math. Comput.*, **216** (2010), 2869–2880. http://dx.doi.org/10.1016/j.amc.2010.03.138

27. J. R. Loh, C. Phang, A new numerical scheme for solving system of Volterra integro-differential equation, *Alex. Eng. J.*, **57** (2018), 1117–1124. http://dx.doi.org/10.1016/j.aej.2017.01.021

28. M. Higazy, S. Aggarwal, T. A. Nofal, Sawi decomposition method for Volterra integral equation with application, *J. Math.*, **2020** (2020), 6687134. http://dx.doi.org/10.1155/2020/6687134

29. Y. Shi, J. Wen, J. Xiong, Backward doubly stochastic Volterra integral equations and their applications, *J. Differ. Equations*, **269** (2020), 6492–6528. http://dx.doi.org/10.1016/j.jde.2020.05.006

30. Y. Yong, Well-posedness and regularity of backward stochastic Volterra integral equations, *Probab. Theory Relat. Fields*, **142** (2008), 21–77. http://dx.doi.org/10.1007/s00440-007-0098-6

31. J. Rashidinia, M. Zarebnia, Solution of a Volterra integral equation by the Sinc-collocation method, *J. Comput. Appl. Math.*, **206** (2007), 801–813. http://dx.doi.org/10.1016/j.cam.2006.08.036

32. E. S. Shoukralla, B. M. Ahmed, Multi-techniques method for solving Volterra integral equations of the second kind, *14th International Conference on Computer Engineering and Systems (ICCES)*, 2019. http://dx.doi.org/10.1109/ICCES48960.2019.9068138
33. Z. Hong, X. Fang, Z. Yan, H. Hao, On solving a system of Volterra integral equations with relaxed Monte Carlo method, *J. Appl. Math. Phys.*, **4** (2016), 1315–1320. http://dx.doi.org/10.4236/jamp.2016.47140
34. B. Chweizer, A. Sklar, Statistical metric spaces, *Pac. J. Math.*, **10** (1960), 313–334. http://dx.doi.org/10.2140/pjm.1960.10.313
35. S. U. Rehman, H. X. Li, Fixed point theorems in fuzzy cone metric spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 5763–5769. http://dx.doi.org/10.22436/jnsa.010.11.14
36. A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, **29** (2002), 641824. http://dx.doi.org/10.1155/S0161171202007524
37. H. A. Hammad, H. Aydi, Y. U. Gaba, Exciting fixed point results on a novel space with supportive applications, *J. Funct. Space.*, **2021** (2021), 6613774. http://dx.doi.org/10.1155/2021/6613774
38. H. A. Hammad, M. De la Sen, H. Aydi, Analytical solution for differential and nonlinear integral equations via F_{w_e}-Suzuki contractions in modified w_e-metric-like spaces, *J. Funct. Space.*, **2021** (2021), 6128586. http://dx.doi.org/10.1155/2021/6128586

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