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Research article

Least energy sign-changing solutions of Kirchhoff equation on bounded domains

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Abstract: We deal with sign-changing solutions for the Kirchhoff equation

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = \lambda u + \mu |u|^2 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $a, b > 0$ and $\lambda, \mu \in \mathbb{R}$ being parameters, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. Combining Nehari manifold method with the quantitative deformation lemma, we prove that there exists $\mu^* > 0$ such that above problem has at least a least energy sign-changing (or nodal) solution if $\lambda < a\lambda_1$ and $\mu > \mu^*$, where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta u, H_0^1(\Omega))$. It is noticed that the nonlinearity $\lambda u + \mu|u|^2u$ fails to satisfy super-linear near zero and super-three-linear near infinity, respectively.

Keywords: Kirchhoff equation; nonlocal term; variation methods; sign-changing solutions
Mathematics Subject Classification: 35J60, 35J20

1. Introduction and main results

In this article, we are concerned with sign-changing solutions for the Kirchhoff equation

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = \lambda u + \mu |u|^2 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b > 0$ and $\lambda, \mu \in \mathbb{R}$ being parameters, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. Problem (1.1) comes from the following general Kirchhoff equation

$$-(a+b\int_\Omega |\nabla u|^2dx)\Delta u = f(x,u), \tag{1.2}$$

which is related to the following stationary analogue of the equation of Kirchhoff type

$$u_{tt} - (a + b\int_{\Omega} |\nabla u|^2 dx)\Delta u = f(x, u). \tag{1.3}$$

It is noticed that, as a generalization of the well-known D'Alembert wave equation for free vibration of elastic strings, Kirchhoff firstly introduced Eq (1.3) in paper [13]. More backgrounds about Kirchhoff type problems, we refer the readers to [24]. Since the pioneer work of Lions [19], there are many results for Kirchhoff type problems. On the one hand, we shall recall some results about the perturbed problem. For example, He and Zou [10] considered the Kirchhoff-type problem

$$-(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(u) \ \text{ in } \mathbb{R}^3, \tag{1.4}$$

where $\varepsilon > 0$ is a parameter, $V > 0$ is a continuous function and $f(u) \sim |u|^{p-2}u$ $(4 < p < 6)$. Combining Ljusternik-Schnirelmann theory and minimax methods, they proved the multiplicity of positive solutions, which concentrate on the minima of $V(x)$ as $\varepsilon \to 0$ while vanishing elsewhere. Later, Wang et al. [32] extended the results obtained in [10] to the critical case, i.e., $f(u) \sim \lambda|u|^{p-2}u + |u|^4u$ $(4 < p < 6)$. In [6], Figueiredo et al. considered the following Kirchhoff-type equation

$$-\varepsilon^2 M(\varepsilon^{2-N}\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u + V(x)u = f(u) \ \text{ in } \mathbb{R}^N, \tag{1.5}$$

where $N \geq 1$, M and V are continuous functions, the authors studied the existence and concentration behaviors of positive solutions to Kirchhoff type Eq (1.5). In [15], Luo et al. considered the Kirchhoff problem

$$-(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = u^p, \ \ u > 0 \text{ in } \mathbb{R}^3, \tag{1.6}$$

where $1 < p < 5$. By Lyapunov-Schmidt reduction method, under some mild assumptions on the function V, the authors obtained multi-peak solutions for $\varepsilon > 0$ sufficiently small. On the other hand, we cit some results about the non perturbed problem. For example, Li et al. [18] considered the following Kirchhoff type equation

$$[a + \lambda\int_{\mathbb{R}^N} |\nabla u|^2 dx + \lambda b\int_{\mathbb{R}^N} u^2 dx][-\Delta u + bu] = f(u), \ \text{in } \mathbb{R}^N, \tag{1.7}$$

where $N \geq 3$, and a, b are positive constants, $\lambda \geq 0$ is a parameter. Without usual compactness conditions, they proved the existence of a positive solution to Kirchhoff type Eq (1.7). In [16], by using a monotonicity trick and a new version of global compactness lemma, Li and Ye had proved that (1.4) had a ground state solution in the case $\varepsilon = 1$ and $f(x, u) = |u|^{p-1}u$ with $2 < p < 5$. By using variational methods and Schwartz symmetric arrangement, Guo [9] generalized the result obtained in [16] to the Kirchhoff-type problem with general nonlinearity. Later, by introducing some new tricks, Tang and Chen [29] proved that the problem (1.4) with $\varepsilon = 1$ had a ground state solution of Nehari-Pohozaev type and a least energy solution under some mild assumptions on V and f. In [35], Xie et al. investigated bound state solutions for small linear perturbations of Kirchhoff type problems with critical exponent. In [11], Huang et al. studied the Brezis-Nirenberg problem to a class of Kirchhoff type problem with critical Sobolev exponent on bounded domain in $\mathbb{R}^4$. In [21], using variational methods, Maia proved the existence of a weak solution for a class of $p(x)$-Choquard equations with upper critical growth. Furthermore, in [21], the author also obtained a multiplicity of solutions for a class of $p(x)$-Choquard equations with a nonlocal and non-degenerate Kirchhoff term by using truncation arguments and Krasno-selskii's genus. Very recently, Vetro [30] considered the following nonlinear $p(x)$-Kirchhoff type problem

$$-\triangle^K_{p(x)} u(x) = f(x, u(x), \nabla u(x)) \text{ in } \Omega,\ u|_{\partial\Omega} = 0, \tag{1.8}$$

where $\triangle^K_{p(x)} u(x) = (a_p - b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. The existence of two different notions of solutions is discussed in [30] with respect to a Galerkin approximation method, jointly with the theory of pseudomonotone operators.

In recent years, many authors also pay their attention to find sign-changing solutions to Kirchhoff type equations. For example, in [23, 38], Zhang, Perera and Mao obtained the existence of sign-changing solution of problem (1.2) by using the method of invariant sets of descent flow. Via variational methods and invariant sets of descent flow, Mao and Luan [22] obtained existence of signed and sign-changing solutions for problem (1.2) with asymptotically 3-linear bounded nonlinearity. In [7], Figueiredo and Nascimento considered the following Kirchhoff equation

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 dx)\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.9}$$

where Ω is a bounded domain in $\mathbb{R}^3$ and $M \in C^1$. Using minimization argument together with quantitative deformation lemma, authors studied the existence of sign-changing solution for Eq (1.9). Later, Figueiredo and Santos Júnior [8] extended the results obtained in [7] to the unbounded domains. In [26], together with constraint variational methods and quantitative deformation lemma, Shuai studied the existence and asymptotic behavior of least energy sign-changing solution to problem (1.2). It is noticed that Ye [36] also obtained some results similar to paper [26]. Based on variational methods, Lu [20] obtained the ground states and least energy sign-changing solutions for problem (1.2). In [28], without the usual Nehari-type monotonicity condition on f, Tang and Cheng improved and generalized results obtained in [26].

In [4], Deng, Peng and Shuai considered the following Kirchhoff problem

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(x, u), \text{ in } \mathbb{R}^3. \tag{1.10}$$

When $V(x) = V(|x|)$ and $f(x, u) = f(|x|, u)$, by using a Nehari manifold and gluing solution pieces together, they obtained the existence of a sign-changing solution which changes signs exactly k times for any $k \in \mathbb{N}$. In [14], Li et al. investigated the existence and the concentration of sign-changing solutions to a class of Kirchhoff-type systems with Hartree-type nonlinearity in $\mathbb{R}^3$. When $f(x, u) = f(u)$, with the help of variational methods in association with the deformation lemma and Miranda's theorem, Wang et al. [33] investigated sign-changing solution to problem (1.10) where the potential V is not necessarily radially symmetric. When the potential $V(x)$ is a nonnegative continuous function with a potential well which possesses k disjoint bounded components, Deng and Shuai [5] obtained multiple sign-changing multi-bump solutions for problem (1.10). When the nonlinearity involved a combination of concave and convex terms, Shao and Mao [25] got the existence of infinitely many high-energy solutions for a class of Kirchhoff problem by using Fountain theorem. In [17], Li et al.

also considered the existence of sign-changing solution to problem (1.10) when $f(x,u) = f(u)$. When $f(x,u) = f(u)$ and f is odd, combining with Ljusternik-Schnirelmann theory and minimax methods, Sun et al. [27] obtained infinitely many sign-changing solutions for Kirchhoff problem (1.10). In [2], Cassani et al. considered the following Kirchhoff type equation

$$[1 + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx][-\Delta u + V(x)u] = f(u) \text{ in } \mathbb{R}^3. \tag{1.11}$$

The authors obtained that, for any $n \in \mathbb{N}$ there exists $\lambda_n > 0$ such that for any $\lambda \in (0, \lambda_n)$, problem (1.11) has at least n pairs of radially symmetric sign-changing solutions with positive energy. In [31], the last author of this paper investigated the existence and the energy property of least energy sign-changing solution to the following Kirchhoff problem with critical growth

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx)\Delta u = |u|^4 u + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.12}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. Subsequently, Zhang [37] generalized results obtained in [31] to a class of general Kirchhoff problem.

However, as far as we know, when studying least energy sign-changing solution to Kirchhoff equation, the nonlinearity always satisfies the growth conditions of super-linear near zero or super-three-linear near infinity except [3, 39]. In [3], when f satisfies asymptotically linear growth at infinity about u, Cheng and Tang obtained the existence and asymptotic behavior of least energy sign-changing solution for Eq (1.2) with bounded domain. It is notice that the results obtained in [3] still depends on the fact that f is super-linear near zero about u. Recently, Zhong and Tang [39] considered the Kirchhoff-type problem

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u + |u|^2 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.13}$$

where $a, b > 0$, $\lambda < a\lambda_1$, λ_1 is the principal eigenvalue of $(-\Delta, H_0^1(\Omega))$, and Ω is a smooth bounded domain in $\mathbb{R}^3, N = 1, 2, 3$. By Nehari manifold argument, the authors proved that there exists $\wedge > 0$ such that the Eq (1.13) has at least one least energy sign-changing solution u_b for all $0 < b < \wedge$ and $\lambda < a\lambda_1$ and obtained that its energy is strictly larger than twice that of ground state solutions. Furthermore, they also studied the asymptotic behavior of u_b as $b \to 0$ and the nonexistence of sign-changing solution for Eq (1.13). Obviously, the nonlinearity $\lambda u + |u|^2 u$ fails to satisfy super-linear near zero and super-three-linear near infinity, respectively. However, since their results strongly depends on the condition $0 < b < \wedge$, the methods used in [39] seem not valid for all $b > 0$.

In this paper, inspired by above works, we consider the existence of least energy sign-changing solution to Kirchhoff Eq (1.1) for all $b > 0$. Our method is closely related to the works in [1, 12], where authors dealt with p-Laplacian equation and Schrödinger-Poisson system respectively.

Let $L^p(\Omega)$ be a Lebesgue space with the norm $|u|_p := (\int_\Omega |u|^p dx)^{\frac{1}{p}}$, $1 \le p < \infty$ and $H_0^1(\Omega)$ be Sobolev space equipped with the inner product and norm

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx, \|u\| = (u, u)^{\frac{1}{2}}.$$

Associated with Eq (1.1), the energy functional is defined by

$$\Phi(u) = \frac{a}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 dx + \frac{b}{4}\|u\|^4 - \frac{\mu}{4}\int_\Omega |u|^4 dx = \frac{1}{2}\Phi_\lambda(u) + \frac{1}{4}\Phi_\mu(u), u \in H_0^1(\Omega),$$

where $\Phi_\lambda(u) = a\|u\|^2 - \lambda \int_\Omega |u|^2 dx, \Phi_\mu(u) = b\|u\|^4 - \mu \int_\Omega |u|^4 dx$. Moreover, $\Phi(u)$ belongs to C^1, and

$$\langle \Phi'(u), v\rangle = a\int_\Omega \nabla u \cdot \nabla v dx - \lambda \int_\Omega uv dx + b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v dx - \mu \int_\Omega |u|^2 uv dx$$

for any $u, v \in H_0^1(\Omega)$.

Let $u^+ = \max\{u(x), 0\}, u^- = \min\{u(x), 0\}$, if $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$ is a solution of problem (1.1), then we said that u is a sign-changing solution of Eq (1.1).

Let $\mathcal{M} = \{u \in H_0^1(\Omega), u^\pm \neq 0 \text{ and } \langle \Phi'(u), u^\pm\rangle = 0\}, m = \inf_{u\in\mathcal{M}} \Phi(u)$ and λ_1 be the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

In fact, $\lambda_1 = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{|u|_2^2}$. We remark that if $u \in H_0^1(\Omega)$ is a sign-changing solution of Eq (1.1), then

$$\langle \Phi'(u), u^\pm\rangle = \Phi_\lambda(u^\pm) + b\|u\|^2\|u^\pm\|^2 - \mu \int_\Omega |u^\pm|^4 dx = 0.$$

Hence, it follows from $\lambda < a\lambda_1$ that $b\|u\|^2\|u^\pm\|^2 - \mu \int_\Omega |u^\pm|^4 dx = -\Phi_\lambda(u^\pm) < 0$, that is

$$\mu > \max\left\{\frac{b\|u\|^2\|u^+\|^2}{\int_\Omega |u^+|^4 dx}, \frac{b\|u\|^2\|u^-\|^2}{\int_\Omega |u^-|^4 dx}\right\}, \text{ for any } u \in \mathcal{M}.$$

On the other hand, if $\mu < \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \max\left\{\frac{b\|u\|^2\|u^+\|^2}{\int_\Omega |u^+|^4 dx}, \frac{b\|u\|^2\|u^-\|^2}{\int_\Omega |u^-|^4 dx}\right\}$, then, for any $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, we get

$$b\|u\|^2\|u^+\|^2 - \mu \int_\Omega |u^+|^4 dx \geq 0 \text{ or } b\|u\|^2\|u^-\|^2 - \mu \int_\Omega |u^-|^4 dx \geq 0.$$

So, it follows from $\Phi_\lambda(u^\pm) > 0$ that $\langle \Phi'(u), u^+\rangle \neq 0$ or $\langle \Phi'(u), u^+\rangle \neq 0$, that is $\mathcal{M} = \emptyset$. Hence, the Eq (1.1) has no sign-changing solution.

Let

$$\mu^* = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \left\{\max\left\{\frac{b\|u\|^2\|u^+\|^2}{\int_\Omega |u^+|^4 dx}, \frac{b\|u\|^2\|u^-\|^2}{\int_\Omega |u^-|^4 dx}\right\}\right\}.$$

Our result is the following theorem.

Theorem 1.1. *If $\lambda < a\lambda_1$ and $\mu > \mu^*$, then the Eq* (1.1) *has a least energy sign-changing solution.*

Remark 1.1. *Although we obtain the existence of least energy sign-changing solution to Eq* (1.1) *for all $b > 0$, the parameter μ needs to be larger than one positive constant to achieve our goal. So, for all $b > 0$ and $\mu > 0$, we do not know whether Eq* (1.1) *has a least energy sign-changing solution or not.*

2. Technical lemmas

For any $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, it is easy to see that

$$\Phi_\lambda(u) = \Phi_\lambda(u^+) + \Phi_\lambda(u^-), \tag{2.1}$$
$$\Phi_\mu(u) = \Phi_\mu(u^+) + \Phi_\mu(u^-) + 2b\|u^+\|^2\|u^-\|^2, \tag{2.2}$$
$$\Phi(u) = \Phi(u^+) + \Phi(u^-) + \frac{b}{2}\|u^+\|^2\|u^-\|^2, \tag{2.3}$$
$$\langle \Phi'(u), u^\pm \rangle = \langle \Phi'(u^\pm), u^\pm \rangle + b\|u^+\|^2\|u^-\|^2. \tag{2.4}$$

Denote

$$\begin{aligned}\mathcal{N} &= \left\{u \in H_0^1(\Omega), u^\pm \neq 0 : \Phi_\mu(u^\pm) + b\|u^+\|^2\|u^-\|^2 < 0\right\} \\ &= \left\{u \in H_0^1(\Omega), u^\pm \neq 0 : b\|u\|^2\|u^\pm\|^2 - \mu \int_\Omega |u^\pm|^4 dx < 0\right\}.\end{aligned} \tag{2.5}$$

Lemma 2.1. *If $\lambda < a\lambda_1$ and $\mu > \mu^*$, then $\mathcal{N} \neq \emptyset$ and $\mathcal{M} \subset \mathcal{N}$.*

Proof. Suppose that $\mu > \mu^*$, it follows from the definition of μ^* that there exists $v \in H_0^1(\Omega)$ with $v^\pm \neq 0$ such that

$$\mu > \max\left\{\frac{b\|v\|^2\|v^+\|^2}{\int_\Omega |v^+|^4 dx}, \frac{b\|v\|^2\|v^-\|^2}{\int_\Omega |v^-|^4 dx}\right\} \geq \mu^*.$$

So, $b\|v\|^2\|v^\pm\|^2 - \mu\int_\Omega |v^\pm|^4 dx < 0$, that is, $v \in \mathcal{N}$. Hence we obtain $\mathcal{N} \neq \emptyset$.

In the following, we prove $\mathcal{M} \subset \mathcal{N}$. For any $u \in \mathcal{M}$, then we have that

$$b\|u\|^2\|u^\pm\|^2 - \mu \int_\Omega |u^\pm|^4 dx = -\Phi_\lambda(u^\pm). \tag{2.6}$$

Thanks to $\lambda < a\lambda_1$, we get $\Phi_\lambda(u^\pm) > 0$. Then, from (2.6), we conclude that $u \in \mathcal{N}$, that is $\mathcal{M} \subset \mathcal{N}$. □

Lemma 2.2. *If $\lambda < a\lambda_1$ and $\mu > \mu^*$ hold, then for any $u \in \mathcal{N}$, there is a unique pair (s_u, t_u) with $s_u, t_u > 0$ such that $s_u u^+ + t_u u^- \in \mathcal{M}$ and $\Phi(s_u u^+ + t_u u^-) = \max_{s,t>0} \Phi(su^+ + tu^-)$. Furthermore, if $\langle \Phi'(u), u^\pm \rangle \leq 0$, then we have $0 < s_u, t_u \leq 1$.*

Proof. For any $u \in \mathcal{N}$, it follows from (2.4) that $su^+ + tu^- \in \mathcal{M}$ if and only if the positive pair (s, t) satisfies

$$\begin{cases} \langle \Phi'(su^+ + tu^-), su^+ \rangle = s^2\Phi_\lambda(u^+) + s^4\Phi_\mu(u^+) + bs^2t^2\|u^+\|^2\|u^-\|^2 = 0, \\ \langle \Phi'(tu^+ + tu^-), tu^- \rangle = t^2\Phi_\lambda(u^-) + t^4\Phi_\mu(u^-) + bs^2t^2\|u^+\|^2\|u^-\|^2 = 0. \end{cases}$$

That is

$$\begin{cases} s^2\Phi_\mu(u^+) + bt^2\|u^+\|^2\|u^-\|^2 = -\Phi_\lambda(u^+), \\ t^2\Phi_\mu(u^-) + bs^2\|u^+\|^2\|u^-\|^2 = -\Phi_\lambda(u^-), \end{cases} \tag{2.7}$$

which is equivalent to

$$\begin{bmatrix} \Phi_\mu(u^+) & b\|u^+\|^2\|u^-\|^2 \\ b\|u^+\|^2\|u^-\|^2 & \Phi_\mu(u^-) \end{bmatrix} \begin{bmatrix} s^2 \\ t^2 \end{bmatrix} = \begin{bmatrix} -\Phi_\lambda(u^+) \\ -\Phi_\lambda(u^-) \end{bmatrix}.$$

Then, since $u \in \mathcal{N}$, one has

$$\begin{vmatrix} \Phi_\mu(u^+) & b\|u^+\|^2\|u^-\|^2 \\ b\|u^+\|^2\|u^-\|^2 & \Phi_\mu(u^-) \end{vmatrix} = \Phi_\mu(u^+)\Phi_\mu(u^-) - b^2\|u^+\|^4\|u^-\|^4 > 0.$$

Hence, thanks to $\Phi_\lambda(u^\pm) > 0$, Eq (2.7) has a unique solution (s_u, t_u) such that $s_u, t_u > 0$.

Fixed $u \in \mathcal{N}$, define $\phi_u : (0, \infty) \times (0, \infty) \to \mathbb{R}$ by $\phi_u(s, t) = \Phi(su^+ + tu^-)$. Now we prove $\phi_u(s_u, t_u) = \Phi(s_u u^+ + t_u u^-) = \max_{s,t>0} \phi_u(s, t)$, where (s_u, t_u) is the unique solution of Eq (2.7). It follows that

$$\begin{aligned} \nabla\phi_u(s, t) &= \left(\frac{\partial\phi_u}{\partial s}(s, t), \frac{\partial\phi_u}{\partial t}(s, t)\right) \\ &= (\langle\Phi'(su^+ + tu^-), u^+\rangle, \langle\Phi'(su^+ + tu^-), u^-\rangle) \\ &= \left(\frac{1}{s}\langle\Phi'(su^+ + tu^-), su^+\rangle, \frac{1}{t}\langle\Phi'(su^+ + tu^-), tu^-\rangle\right), \end{aligned}$$

which shows that a positive pair (s, t) is a critical point of ϕ_u if and only if $su^+ + tu^- \in \mathcal{M}$. So, since (s_u, t_u) is the unique solution of Eq (2.7), we deduce that (s_u, t_u) is a unique critical point of the function ϕ_u.

Since $u \in \mathcal{N}$, we have

$$\begin{cases} \frac{\partial^2\phi_u}{\partial s^2}(s_u, t_u) = \Phi_\lambda(u^+) + 3s_u^2\Phi_\mu(u^+) + bt_u^2\|u^+\|^2\|u^-\|^2 = 2s_u^2\Phi_\mu(u^+) < 0, \\ \frac{\partial^2\phi_u}{\partial t^2}(s_u, t_u) = \Phi_\lambda(u^-) + 3t_u^2\Phi_\mu(u^-) + bs_u^2\|u^+\|^2\|u^-\|^2 = 2t_u^2\Phi_\mu(u^-) < 0, \\ \frac{\partial^2\phi_u}{\partial s\partial t}(s_u, t_u) = \frac{\partial^2\phi_u}{\partial t\partial s}(s_u, t_u) = 2bs_u t_u\|u^+\|^2\|u^-\|^2. \end{cases} \tag{2.8}$$

Then we get

$$\begin{vmatrix} \frac{\partial^2\phi_u}{\partial s^2}(s_u, t_u) & \frac{\partial^2\phi_u}{\partial s\partial t}(s_u, t_u) \\ \frac{\partial^2\phi_u}{\partial t\partial s}(s_u, t_u) & \frac{\partial^2\phi_u}{\partial t^2}(s_u, t_u) \end{vmatrix} = 4s_u^2t_u^2[\Phi_\mu(u^+)\Phi_\mu(u^-) - b^2\|u^+\|^4\|u^-\|^4] > 0.$$

That is, the Hessian matrix of ϕ_u is negative definite at (s_u, t_u). So, we get $\phi_u(s_u, t_u) = \max_{s,t>0} \phi_u(s, t)$. Suppose $s_u \geq t_u > 0$. By $s_u u^+ + t_u u^- \in \mathcal{M}$, one has

$$\begin{cases} s_u^2\Phi_\mu(u^+) + bt_u^2\|u^+\|^2\|u^-\|^2 = -\Phi_\lambda(u^+), \\ t_u^2\Phi_\mu(u^-) + bs_u^2\|u^+\|^2\|u^-\|^2 = -\Phi_\lambda(u^-). \end{cases} \tag{2.9}$$

On the other hand, by $\langle\Phi'(u), u^\pm\rangle \leq 0$, one has

$$b\|u\|^2\|u^\pm\|^2 - \mu\int_\Omega |u^\pm|^4 dx \leq -\Phi_\lambda(u^\pm) < 0. \tag{2.10}$$

According to (2.9) and (2.10), we have that

$$\begin{aligned} -s_u^2\Phi_\lambda(u^+) &\geq s_u^2[b\|u\|^2\|u^+\|^2 - \mu\int_\Omega |u^+|^4 dx] = s_u^2[\Phi_\mu(u^+) + b\|u^+\|^2\|u^-\|^2] \\ &\geq s_u^2\Phi_\mu(u^+) + bt_u^2\|u^+\|^2\|u^-\|^2 = -\Phi_\lambda(u^+). \end{aligned}$$

Thanks to $\Phi_\lambda(u^+) > 0$, we conclude that $s_u \leq 1$. Thus, we have that $0 < s_u, t_u \leq 1$. □

Lemma 2.3. *If $\lambda < a\lambda_1$ and $\mu > \mu^*$, then there is $\tau > 0$ satisfying $\|u^\pm\| \geq \tau$ for all $u \in \mathcal{M}$.*

Proof. For any $u \in \mathcal{M}$, by Sobolev inequalities we have

$$(a - \frac{\lambda}{\lambda_1})\|u^\pm\|^2 \leq a\|u^\pm\|^2 - \lambda \int_\Omega |u^\pm|^2 dx \leq \Phi_\lambda(u^\pm) + b\|u\|^2\|u^+\|^2$$
$$= \mu \int_\Omega |u^\pm|^4 dx \leq \mu\alpha_4^4 \|u^\pm\|^4,$$

where α_4 is positive constant such that $|u|_4 \leq \alpha_4\|u\|$, $\forall u \in H_0^1(\Omega)$. So, $\|u^\pm\| \geq \sqrt{\frac{(a-\frac{\lambda}{\lambda_1})}{\mu\alpha_4^4}} := \tau > 0$. □

Lemma 2.4. *If $\lambda < a\lambda_1$ and $\mu > \mu^*$, then $m = \inf_{u\in\mathcal{M}} \Phi(u) > 0$ is achieved.*

Proof. Firstly, we assert that $m > 0$. In fact, for any $u \in \mathcal{M}$, by Lemma 2.3 we have

$$\Phi(u) - \frac{1}{4}\langle \Phi'(u), u\rangle = \frac{1}{4}\Phi_\lambda(u) \geq \frac{1}{4}(a - \frac{\lambda}{\lambda_1})\|u\|^2 \geq \frac{1}{4}(a - \frac{\lambda}{\lambda_1})\tau^2 > 0,$$

which implies $m > 0$.
In the following, we prove that m is achieved. Let $\{u_n\} \subset \mathcal{M}$ such that $\lim_{n\to\infty} \Phi(u_n) = m$. It is easy to see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then, there is $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$.
Furthermore, for all $p \in [1, 6)$, one has

$$u_n \to u \;\text{ in } L^p(\Omega), u_n(x) \to u(x) \;\text{ a.e. } x \in \Omega.$$

$$u_n^\pm \rightharpoonup u^\pm \;\text{ in } H_0^1(\Omega), u_n^\pm \to u^\pm \;\text{ in } L^p(\Omega), u_n^\pm(x) \to u^\pm(x) \;\text{ a.e. } x \in \Omega. \tag{2.11}$$

By $\{u_n\} \subset \mathcal{M}$, we have

$$a\|u_n^\pm\|^2 \leq a\|u_n^\pm\|^2 + b\|u_n\|^2\|u_n^+\|^2 = \lambda \int_\Omega |u_n^\pm|^2 dx + \mu \int_\Omega |u_n^\pm|^4 dx \leq \frac{\lambda}{\lambda_1}\|u_n^\pm\|^2 + \mu \int_\Omega |u_n^\pm|^4 dx.$$

So, $0 < (a - \frac{\lambda}{\lambda_1})\mu^{-1}\|u_n^\pm\|^2 \leq \int_\Omega |u_n^\pm|^4 dx$. It follows from (2.11) and Lemma 2.3 that $\int_\Omega |u^\pm|^4 dx \geq (a - \frac{\lambda}{\lambda_1})\mu^{-1}\tau^2 > 0$. Hence, we conclude that $u^\pm \neq 0$.
On the other hand, since $\{u_n\} \subset \mathcal{M}$, it follows from weakly lower semicontinuity of norm that

$$a\|u^\pm\|^2 + b\|u\|^2\|u^+\|^2 \leq \liminf_{n\to\infty}[a\|u_n^\pm\|^2 + b\|u_n\|^2\|u_n^+\|^2] = \lambda \int_\Omega |u^\pm|^2 dx + \mu \int_\Omega |u^\pm|^4 dx,$$

that is, $\langle \Phi'(u), u^\pm\rangle \leq 0$. So, it follows from (2.10) that $u \in \mathcal{N}$.

According to Lemma 2.2, there exists $0 < s_u, t_u \leq 1$ such that $\overline{u} := s_u u^+ + t_u u^- \in \mathcal{M}$. Thanks to (2.11) and the norm in $H_0^1(\Omega)$ is lower semicontinuous, we have that

$$m \leq \Phi(s_u u^+ + t_u u^-) = \Phi(s_u u^+ + t_u u^-) - \frac{1}{4}\langle \Phi(s_u u^+ + t_u u^-), s_u u^+ + t_u u^-\rangle$$
$$= \frac{s_u^2}{4}\Phi_\lambda(u^+) + \frac{t_u^2}{4}\Phi_\lambda(u^-) \leq \frac{1}{4}\Phi_\lambda(u^+) + \frac{1}{4}\Phi_\lambda(u^-) = \frac{a}{4}\|u\|^2 - \frac{\lambda}{4}|u|_2^2 \leq \liminf_{n\to\infty}[\frac{a}{4}\|u_n\|^2 - \frac{\lambda}{4}|u_n|_2^2]$$
$$= \liminf_{n\to\infty}[\Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle] = m.$$

Therefore, $s_u = t_u = 1$, and m is achieved by $\overline{u} = u \in \mathcal{M}$. □

3. The proof of main results

According to Lemma 2.4, we only prove that the minimizer u for m satisfies $\Phi'(u) = 0$.

Proof of Theorem 1.1.

Proof. Thanks to $u \in \mathcal{M}$, $\langle \Phi'(u), u^{\pm} \rangle = 0$. It follows from Lemma 2.2 that, for $(s,t) \in (\mathbb{R}_+ \times \mathbb{R}_+) \backslash (1,1)$,

$$\Phi(su^+ + tu^-) < \Phi(u^+ + u^-) = m. \tag{3.1}$$

Arguing indirectly, we suppose $\Phi'(u) \neq 0$. Then, it follows from $\Phi \in C^1$ that there are $\delta > 0$ and $\theta > 0$ such that $\|\Phi'(v)\| \geq \theta,$ for all $\|v - u\| \leq 3\delta$.

Since $\Phi_\lambda(u^{\pm}) > 0$, there exists $\sigma \in (0,1)$ small enough such that $\min_{t \in [1-\sigma, 1+\sigma]} \Phi_\lambda(tu^{\pm}) > 0$. Denote $\Pi := (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $\psi(s,t) = su^+ + tu^-$, $(s,t) \in \Pi$. It follows from (3.1) that

$$m_0 := \max_{\partial \Pi} \Phi \circ \psi < m. \tag{3.2}$$

Let $\varepsilon := \min\{(m - m_0)/3, \theta\delta/8\}$ and $S_\delta := B(u, \delta)$, it follows from Lemma 2.3 in [34] that there is $\gamma \in C([0,1] \times H_0^1(\Omega), H_0^1(\Omega))$ satisfying

(a) $\gamma(\alpha, v) = v$ if $\alpha = 0$ or $v \notin \Phi^{-1}([m - 2\varepsilon, m + 2\varepsilon] \cap S_{2\delta})$;

(b) $\Phi(\gamma(\alpha, v)) < m$ for all $v \in S_\delta$ with $\Phi(v) \leq m$ and $\alpha \in (0,1]$;

(c) $\Phi(\gamma(\alpha, v)) \leq \Phi(v)$ for all $v \in H_0^1(\Omega)$ and $\alpha \in [0,1]$.

We claim that

$$\max_{(s,t) \in \Pi} \Phi(\gamma(\alpha, \psi(s,t))) < m, \forall \alpha \in (0,1]. \tag{3.3}$$

Thanks to (b), we have that $\max_{\{(s,t) \in \Pi : \psi(s,t) \in S_\delta\}} \Phi(\gamma(\alpha, \psi(s,t))) < m, \forall \alpha \in (0,1]$. On the other hand, it follows from (c) that

$$\max_{\{(s,t) \in \Pi : \psi(s,t) \notin S_\delta\}} \Phi(\gamma(\alpha, \psi(s,t))) \leq \max_{\{(s,t) \in \Pi : \psi(s,t) \notin S_\delta\}} \Phi(\psi(s,t)) < m, \forall \alpha \in [0,1].$$

So, (3.3) can be concluded.

Since $\min_{t \in [1-\sigma, 1+\sigma]} \Phi_\lambda(tu^{\pm}) > 0$, it follows from the continuity of γ and Φ_λ that there is $\alpha_0 \in (0,1)$ such that

$$\Phi_\lambda(\gamma^{\pm}(\alpha_0, \psi(s,t))) > 0, \forall (s,t) \in \Pi. \tag{3.4}$$

Next, we prove that $\gamma(\alpha_0, \psi(\Pi)) \cap \mathcal{M} \neq \emptyset$. Let $\chi(s,t) := \gamma(\alpha_0, \psi(s,t))$ and

$$\Psi_0(s,t) := (\langle \Phi'(\psi(s,t)), su^+ \rangle, \langle \Phi'(\psi(s,t)), tu^- \rangle) := (\varphi_u^1(s,t), \varphi_u^2(s,t)),$$

$\Psi_1(s,t) := (\langle \Phi'(\chi(s,t)), (\chi(s,t))^+ \rangle, \langle \Phi'(\chi(s,t)), (\chi(s,t))^- \rangle)$.

Let

$$Q = \begin{bmatrix} \frac{\partial \varphi_u^1(s,t)}{\partial s}\Big|_{(1,1)} & \frac{\partial \varphi_u^2(s,t)}{\partial s}\Big|_{(1,1)} \\ \frac{\partial \varphi_u^1(s,t)}{\partial t}\Big|_{(1,1)} & \frac{\partial \varphi_u^2(s,t)}{\partial t}\Big|_{(1,1)} \end{bmatrix}.$$

From $u \in \mathcal{M}$, we can obtain that $\det Q > 0$. Since $\Psi_0(s,t)$ is a C^1 function and $(1,1)$ is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, \Pi, 0) = 1$. By $m_0 < m - 2\varepsilon$ and (a), we have that $\gamma(\alpha, \psi(s,t)) = \psi(s,t), \forall (s,t) \in \partial\Pi, \alpha \in [0,1]$. So, we conclude that $\Psi_0(s,t) = \Psi_1(s,t)$ on $\partial\Pi$. By degree theory, we have $\deg(\Psi_1, \Pi, 0) = 1$, which shows that $\Psi_1(s_0,t_0) = 0$ for some $(s_0,t_0) \in \Pi$. According to (3.4), we obtain $\chi(s_0,t_0) = \gamma(\alpha_0, \psi(s_0,t_0)) \in \mathcal{M}$, that is, we obtain that $\gamma(\alpha_0, \psi(\Pi)) \cap \mathcal{M} \neq \emptyset$. Thanks to (3.3), we conclude a contradiction. So, we obtain desire result. □

4. Conclusions

In this paper, by the minimization argument on the nodal Nehari manifold and the quantitative deformation lemma, we discussed the existence of least energy sign-changing solution for a class of Kirchhoff equation on bounded domains. Our result is complementary to the results by Zhong and Tang obtained in [39].

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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