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Research article

Infinity norm upper bounds for the inverse of SDD_1 matrices

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Abstract: In this paper, a new proof that SDD_1 matrices is a subclass of H-matrices is presented, and some properties of SDD_1 matrices are obtained. Based on the new proof, some upper bounds of the infinity norm of inverse of SDD_1 matrices and SDD matrices are given. Moreover, we show that these new bounds of SDD matrices are better than the well-known Varah bound for SDD matrices in some cases. In addition, some numerical examples are given to illustrate the corresponding results.

Keywords: SDD_1 matrices; SDD matrices; upper bound; positive diagonal matrix; infinity norm
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1. Introduction

Let n be an integer number, $N = \{1, 2, \ldots, n\}$, and $\mathbb{C}^{n\times n}$ be the set of all complex matrices of order n. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is called a strictly diagonally dominant (SDD) matrix if

$$|a_{ii}| > r_i(A), i \in N,$$

where

$$r_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \ i \in N.$$

It was shown that SDD matrices is a subclass of H-matrices [1], where a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is an H-matrix if and only if there exists a positive diagonal matrix X such that AX is an SDD matrix [1].

In 2011, a new subclass of H-matrices was proposed by J. M. Peña, which is called SDD_1 matrices [2], and the definition of SDD_1 matrix is given as follows.

Definition 1. [2] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is called an SDD_1 matrix if

$$|a_{ii}| > p_i(A), i \in N_1(A),$$

where

$$p_i(A) = \sum_{j \in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j \in N_2(A)\backslash\{i\}} \frac{r_j(A)}{|a_{jj}|} |a_{ij}|,$$

$N_1(A) = \{i|\ |a_{ii}| \leq r_i(A)\}$ *and* $N_2(A) = \{i|\ |a_{ii}| > r_i(A)\}$.

In [2], J. M. Peña "proved" the following result:

Theorem 1. ([2 Theorem 2.3]) If a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is an SDD_1 matrix by rows, then it is an H-matrix.

From the definition of H-matrix and Theorem 1, given an SDD_1 matrix A, there exists a correspondingly positive diagonal matrix D, such that AD is an SDD matrix. The great interest of the constitution of positive diagonal matrix D was commented in the introduction in [2], and divided it into two cases, that is, the given SDD_1 matrix has a unique row i strictly diagonally dominant and at least two rows i and j strictly diagonally dominant, to constitute positive diagonal matrix. However, Dai in [3] found that the proof of Theorem 1 is incorrect, and a counter example was given as follows.

Example 1. [3] Let us consider SDD_1 matrices

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

From the proof of Theorem 1 in [2], it is easy to obtain that $D = diag\{\frac{3}{4}, \frac{1}{3}, \frac{1}{4}, 1\}$, however, AD is not an SDD matrix by rows.

Dai found that the proof of the case that the given SDD_1 matrix has at least two rows i and j strictly diagonally dominant is incorrect, and a correct proof of Theorem 1 was presented in [3]. The correct proof of Theorem 1 divides the case that SDD_1 matrices have at least two rows i and j strictly diagonally dominant into $S = \emptyset$ and $S \neq \emptyset$, where S is given as follows:

$$S = \{i|a_{ij} = 0,\ for\ some\ i \in N_2(A),\ all\ j \in N_2(A)\backslash\{i\}\}.$$

However, when we use the correct proof to give the upper bound for the infinity norm of the inverse of SDD_1 matrices, the upper bound needs to be considered in different cases. Therefore, in order to avoid the difficult, we need to improve the proof of Theorem 1.

In addition, it was shown that upper bound of the infinity norm of inverse of a given nonsingular matrix has many potential applications in computational mathematics, such as for bounding the condition number and for proving the convergence of iteration methods. Moreover, upper bounds of the infinity norm of inverse for different classes of matrix have been widely studied, such as Nekrasov matrices [4–6], S-Nekrasov matrices [7, 8], QN-Nekrasov matrices [8], $\{p_1, p_2\}$-Nekrasov matrices [9, 10], DZT matrices [11, 12], S-SDD matrices [13], S-$SDDS$ matrices [14] and so on. However, the estimation of upper bounds of the infinity norm of inverse for SDD_1 matrices has never been reported.

In this paper, a new proof of Theorem 1 is given firstly. Secondly, some properties of SDD_1 matrices are presented. Finally, based on the new proof, some upper bounds of the infinity norm of inverse of SDD_1 matrices and SDD matrices are obtained. Moreover, it is shown that these new bounds of SDD matrices works better than the well-known Varah bound in some cases, and numerical examples are given to illustrate the corresponding results.

2. Main results

Firstly, some notations and a lemma are listed.

$D = diag\{d_1, d_2, \ldots, d_n\}$ denotes a diagonal matrix.

$(AD)_{ij}$ denotes the entry (i, j) of matrix AD, and $(AD)_{ii}$ denotes the diagonal element of the ith row of matrix AD.

Lemma 1. If a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is an SDD_1 matrix if and only if $|a_{ii}| > p_i(A)$ for all $i \in N$.

Proof. From Definition 1, we get that $|a_{ii}| > p_i(A)$ for any $i \in N_1(A)$ and for any $i \in N_2(A)$,

$$|a_{ii}| > r_i(A) \geq \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A)}{|a_{jj}|} \left|a_{ij}\right| = p_i(A), \tag{2.1}$$

thus, we obtain that a matrix A is an SDD_1 matrix if and only if $|a_{ii}| > p_i(A)$ for all $i \in N$. □

Next, a new proof of Theorem 1 is given as follows.

Proof. It is sufficient to prove that each SDD_1 matrix A is an H-matrix. In order to do that, let us define the diagonal matrix as $D = diag\{d_1, d_2, \ldots, d_n\}$, where

$$d_j = \begin{cases} 1 & , j \in N_1(A), \\ \frac{p_j(A)}{|a_{jj}|} + \varepsilon & , j \in N_2(A), \end{cases} \tag{2.2}$$

and

$$0 < \varepsilon < \min_{i\in N} \frac{|a_{ii}| - p_i(A)}{\sum\limits_{j\in N_2(A)\backslash\{i\}} |a_{ij}|}, \tag{2.3}$$

if $\sum\limits_{j\in N_2(A)\backslash\{i\}} |a_{ij}| = 0$, then the corresponding fraction is defined to be $\propto$.

Since matrix A is an SDD_1 matrix, D is a positive diagonal matrix.

In the following, we prove that AD is an SDD matrix, and divided it into two cases.

Case 1: for any $i \in N_1(A)$, it is easy to obtain that $|(AD)_{ii}| = |a_{ii}|$, and

$$\begin{aligned} r_i(AD) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| d_j &= \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \left(\frac{p_j(A)}{|a_{jj}|} + \varepsilon\right) \left|a_{ij}\right| \\ &\leq \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A)}{|a_{jj}|} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \varepsilon |a_{ij}| \ (by\ inequality\ (2.1)) \\ &= p_i(A) + \varepsilon \sum_{j\in N_2(A)\backslash\{i\}} |a_{ij}| \ (by\ the\ expression\ of\ p_i(A)) \\ &< p_i(A) + |a_{ii}| - p_i(A) \ (by\ inequality\ (2.3)) \\ &= |a_{ii}| = |(AD)_{ii}|. \end{aligned}$$

Case 2: for any $i \in N_2(A)$, we get that $|(AD)_{ii}| = |a_{ii}|(\frac{p_i(A)}{|a_{ii}|} + \varepsilon) = p_i(A) + \varepsilon|a_{ii}|$, and

$$\begin{aligned} r_i(AD) = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|d_j &= \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \left(\frac{p_j(A)}{|a_{jj}|} + \varepsilon\right)|a_{ij}| \\ &\leq \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A)}{|a_{jj}|}|a_{ij}| + \varepsilon \sum_{j\in N_2(A)\backslash\{i\}} |a_{ij}| (by\ inequality\ (2.1)) \\ &= p_i(A) + \varepsilon \sum_{j\in N_2(A)\backslash\{i\}} |a_{ij}|\ (by\ the\ expression\ of\ p_i(A)) \\ &< p_i(A) + \varepsilon|a_{ii}| = |(AD)_{ii}|\ (by\ |a_{ii}| > r_i(A), for\ i \in N_2(A)). \end{aligned}$$

From Cases 1 and 2, we obtain that $|(AD)_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|d_j = r_i(AD)$ for any $i \in N$, that is, AD is an SDD matrix, then according to the definition of H-matrix, A is an H-matrix. □

Since the definition of SDD_1 matrix was proposed, some properties of SDD_1 matrices were obtained, such as Schur complements of SDD_1 matrices [2], subdirect sums of SDD_1 matrices [15]. Next, some new properties of SDD_1 matrices are listed as follows.

Theorem 2. If a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is an SDD_1 matrix by rows, and $N_1(A) \neq \emptyset$, then for each $i \in N_1(A)$, there is at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$.

Proof. Suppose on the contrary that for each $i \in N_1(A)$, $a_{ij} = 0$, where $j \in N_2(A)$ and $j \neq i$, then it is easy to obtain that $p_i(A) = r_i(A)$ for any $i \in N_1(A)$ from Definition 1, thus we obtain that $|a_{ii}| > p_i(A) = r_i(A) \leq |a_{ii}|$, which does not hold, hence for each $i \in N_1(A)$, there is at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$. □

Theorem 3. If a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ is an SDD_1 matrix by rows, and for each $i \in N_2(A)$, there is at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, then $|a_{ii}| > p_i(A) > 0$ for any $i \in N$ and $|a_{ii}| > r_i(A) > p_i(A) > 0$ for any $i \in N_2(A)$.

Proof. From the Lemma 1, we get that $|a_{ii}| > p_i(A)$ for any $i \in N$ and $|a_{ii}| > r_i(A) \geq p_i(A)$ for all $i \in N_2(A)$.

Since A is an SDD_1 matrix, and from the condition that for each $i \in N_2(A)$, A has at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, it is easy to obtain that $|a_{ii}| > r_i(A) > p_i(A) > 0$ for any $i \in N_2(A)$.

We next prove that $|a_{ii}| > p_i(A) > 0$ for any $i \in N$, and consider the following two cases separately.

Case 1: if $N_1(A) = \emptyset$, then A is an SDD matrix, and from the condition that for each $i \in N_2(A)$, A has at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, thus it is easy to get $|a_{ii}| > p_i(A) > 0$ for any $i \in N = N_2(A)$.

Case 2: if $N_1(A) \neq \emptyset$, then from Theorem 2 and the condition that for each $i \in N_2(A)$, A has at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, we obtain that $|a_{ii}| > p_i(A) > 0$ for all $i \in N$.

From Cases 1 and 2, we obtain that $|a_{ii}| > p_i(A) > 0$ for any $i \in N$. □

Theorem 4. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD_1 matrix by rows, and for each $i \in N_2(A)$, A has at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, then there exists a diagonal matrix $D = diag\{d_1, d_2, \ldots, d_n\}$, where $d_i = \frac{p_i(A)}{|a_{ii}|}, i = 1, 2, \ldots, n$, such that AD is an SDD matrix.

Proof. In order to prove that matrix AD is an SDD matrix, we need to prove that matrix AD satisfies the following inequalities:

$$|(AD)_{ii}| > r_i(AD)\ for\ any\ i \in N.$$

Since for each $i \in N_2(A)$, there is at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, from Theorems 2 and 3, we obtain that $|a_{ii}| > p_i(A) > 0$ for any $i \in N$ and $|a_{ii}| > r_i(A) > p_i(A) > 0$ for all $i \in N_2(A)$.

Therefore, for any $i \in N$, it is easy to get $|(AD)_{ii}| = p_i(A)$, and from $0 < \frac{p_j(A)}{|a_{jj}|} < \frac{r_j(A)}{|a_{jj}|} < 1$ for any $j \in N_2(A)$, Theorems 2 and 3, we get that

$$\begin{aligned} r_i(AD) = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|d_j &= \sum_{j\in N_1(A)\backslash\{i\}} \frac{p_j(A)}{|a_{jj}|}|a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{p_j(A)}{|a_{jj}|}|a_{ij}| \\ &< \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A)}{|a_{jj}|}|a_{ij}| \\ &= p_i(A) = |(AD)_{ii}|. \end{aligned}$$

Obviously, for any $i \in N$, we get $|(AD)_{ii}| > r_i(AD)$, that is, AD is an SDD matrix. □

Finally, some upper bounds of the infinity norm of inverse of SDD_1 matrices and SDD matrices are established. Before that, a theorem which will be used later is listed.

Theorem 5. (Varah bound) [4] Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD matrix, then

$$||A^{-1}||_\infty \leq \frac{1}{\min\limits_{1\leq i\leq n} (|a_{ii}| - r_i(A))}. \tag{2.4}$$

Theorem 6. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD_1 matrix, then

$$||A^{-1}||_\infty \leq \frac{\max\{1, \max\limits_{i\in N_2(A)} \frac{p_i(A)}{|a_{ii}|} + \varepsilon\}}{\min\{\min\limits_{i\in N_1(A)} H_i, \min\limits_{i\in N_2(A)} Q_i\}}, \tag{2.5}$$

where

$$H_i = |a_{ii}| - \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| - \sum_{j\in N_2(A)\backslash\{i\}} \left(\frac{p_j(A)}{|a_{jj}|} + \varepsilon\right)\left|a_{ij}\right|,\ i \in N_1(A),$$

$$Q_i = \varepsilon(|a_{ii}| - \sum_{j\in N_2(A)\backslash\{i\}} |a_{ij}|) + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|}\left|a_{ij}\right|,\ i \in N_2(A),$$

and ε satisfy inequality (2.3).

Proof. From the new proof of Theorem 1, we obtain that there exists a positive diagonal matrix D such that AD is an SDD matrix, where D is defined as Eq (2.2). Therefore, we have the following inequality:

$$||A^{-1}||_\infty = ||D(D^{-1}A^{-1})||_\infty = ||D(AD)^{-1}||_\infty \leq ||D||_\infty||(AD)^{-1}||_\infty.$$

Since the matrix D is positive diagonal, it is easy to obtain that

$$\|D\|_\infty = \max_{1\le i\le n} d_i = \max\{1, \max_{i\in N_2(A)} \frac{p_i(A)}{|a_{ii}|} + \varepsilon\},$$

where ε satisfy inequality (2.3).

Since AD is an SDD matrix, by Theorem 5, we obtain

$$\begin{aligned}\|(AD)^{-1}\|_\infty &\le \frac{1}{\min\limits_{1\le i\le n}(|(AD)_{ii}| - r_i(AD))} \\ &= \frac{1}{\min\limits_{1\le i\le n}(|a_{ii}|d_i - r_i(AD))} \\ &= \frac{1}{\min\{\min\limits_{i\in N_1(A)} H_i, \min\limits_{i\in N_2(A)} Q_i\}}.\end{aligned}$$

Thus, we get

$$\|A^{-1}\|_\infty \le \frac{\max\{1, \max_{i\in N_2(A)} \frac{p_i(A)}{|a_{ii}|} + \varepsilon\}}{\min\{\min\limits_{i\in N_1(A)} H_i, \min\limits_{i\in N_2(A)} Q_i\}}.$$

□

Based the new proof, the upper bound of the infinity norm of inverse of SDD_1 matrix is presented, and since SDD matrices is a subclass of SDD_1 matrices, from Theorem 6, it is easy to obtain the following Corollary 1.

Corollary 1. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \ge 2)$ be an SDD matrix, then

$$\|A^{-1}\|_\infty \le \frac{\max_{i\in N} \frac{p_i(A)}{|a_{ii}|} + \varepsilon}{\min\limits_{i\in N} M_i}, \tag{2.6}$$

where

$$M_i = \varepsilon(|a_{ii}| - r_i(A)) + \sum_{j\in N\setminus\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right|,\ i \in N \tag{2.7}$$

and

$$0 < \varepsilon < \min_{i\in N} \frac{|a_{ii}| - p_i(A)}{r_i(A)}. \tag{2.8}$$

Example 2. Considering the following SDD_1 matrices

$$A_1 = \begin{bmatrix} 4 & 1 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 0 & 8 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 4 & 1 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}.$$

Obviously, A_1 and A_2 are also SDD matrices. By calculation, we have

$$p_1(A_1) = 1,\ p_2(A_1) = 1,\ p_3(A_1) = 1.5\ p_4(A_1) = 1.5 \textit{ and } 0 < \varepsilon_1 < 1,$$

and

$$p_1(A_2) = 1,\ p_2(A_2) = 1,\ p_3(A_2) = 1,\ p_4(A_2) = 0 \textit{ and } 0 < \varepsilon_2 < 1.5.$$

By the Varah bound (2.4) of Theorem 5, we obtain that $\|A_1^{-1}\|_\infty \le 1$ and $\|A_2^{-1}\|_\infty \le 5$. By the bound (2.6) of Corollary 1, we obtain that $\|A_1^{-1}\|_\infty \le \frac{0.25+\varepsilon_1}{0.375+\varepsilon_1}$ (where $0 < \varepsilon_1 < 1$) and $\|A_2^{-1}\|_\infty \le 5 + \frac{5}{4\varepsilon_2}$(where $0 < \varepsilon_2 < 1.5$). In fact, $\|A_1^{-1}\|_\infty \approx 0.4434$ and $\|A_2^{-1}\|_\infty = 5$. Obviously, for the matrix A_1, it is easy to obtain that $\|A_1^{-1}\|_\infty \approx 0.4434 < \frac{0.25+\varepsilon_1}{0.375+\varepsilon_1} < 1$ for any the number $0 < \varepsilon_1 < 1$. However, for the matrix A_2, we have that $\|A_2^{-1}\|_\infty = 5 < 5 + \frac{5}{4\varepsilon_2}$ for any the number $0 < \varepsilon_2 < 1.5$, which means that the bound in Corollary 1 is better than the Varah bound in Theorem 5 in some cases. Then, a meaningful discussion is concerned: under what conditions, the bound in Corollary 1 is better than the Varah bound in Theorem 5.

The following Theorem 7 shows that the bound in Corollary 1 is better than in Theorem 5 in some conditions.

Theorem 7. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \ge 2)$ be an SDD matrix, if

$$\max_{i\in N} \frac{p_i(A)}{|a_{ii}|} \min_{i\in N}(|a_{ii}| - r_i(A)) \le \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right|,$$

then

$$\|A^{-1}\|_\infty \le \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|} + \varepsilon}{\min\limits_{i\in N} M_i} \le \frac{1}{\min\limits_{1\le i\le n} (|a_{ii}| - r_i(A))},$$

where M_i is given as in Eq (2.7) and ε satisfy inequality (2.8).

Proof. From the condition

$$\max_{i\in N} \frac{p_i(A)}{|a_{ii}|} \min_{i\in N}(|a_{ii}| - r_i(A)) \le \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right|,$$

it is easy to obtain that

$$\max_{i\in N} \frac{p_i(A)}{|a_{ii}|} \min_{i\in N}(|a_{ii}| - r_i(A)) + \varepsilon \min_{i\in N}(|a_{ii}| - r_i(A)) \le \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right| + \varepsilon \min_{i\in N}(|a_{ii}| - r_i(A)),$$

thus, from combining similar terms at the left end of the above inequality, we obtain the following inequality

$$\begin{aligned}\left(\max_{i\in N}\frac{p_i(A)}{|a_{ii}|}+\varepsilon\right)\min_{i\in N}(|a_{ii}|-r_i(A)) &\le \min_{i\in N}\sum_{j\in N\backslash\{i\}}\frac{r_j(A)-p_j(A)}{|a_{jj}|}\left|a_{ij}\right|+\varepsilon\min_{i\in N}(|a_{ii}|-r_i(A))\\ &=\min_{i\in N}\sum_{j\in N\backslash\{i\}}\frac{r_j(A)-p_j(A)}{|a_{jj}|}\left|a_{ij}\right|+\min_{i\in N}\left(\varepsilon(|a_{ii}|-r_i(A))\right)\\ &\le\min_{i\in N}\left(\varepsilon(|a_{ii}|-r_i(A))+\sum_{j\in N\backslash\{i\}}\frac{r_j(A)-p_j(A)}{|a_{jj}|}\left|a_{ij}\right|\right)\\ &=\min_{i\in N}M_i.\end{aligned}\tag{2.9}$$

Since A is an SDD matrix, we have

$$|a_{ii}|>r_i(A)\ and\ M_i>0\ for\ any\ i\in N.$$

Therefore, from inequality (2.9), it is easy to have

$$\frac{\max\limits_{i\in N}\frac{p_i(A)}{|a_{ii}|}+\varepsilon}{\min\limits_{i\in N}M_i}\le\frac{1}{\min\limits_{1\le i\le n}(|a_{ii}|-r_i(A))},$$

and thus from Corollary 1, we have

$$\|A^{-1}\|_\infty\le\frac{\max\limits_{i\in N}\frac{p_i(A)}{|a_{ii}|}+\varepsilon}{\min\limits_{i\in N}M_i}\le\frac{1}{\min\limits_{1\le i\le n}(|a_{ii}|-r_i(A))}.$$

□

The following Example 3 also illustrates the Theorem 7.

Example 3. This is the previous Example 2. For the matrix A_1, by a simple calculation, we obtain

$$\frac{p_1(A_1)}{|a_{11}|}=0.25,\ \frac{p_2(A_1)}{|a_{22}|}=0.25,\ \frac{p_3(A_1)}{|a_{33}|}=0.1875\ and\ \frac{p_4(A_1)}{|a_{44}|}=0.1875,$$

thus,

$$\sum_{j\in N\backslash\{1\}}\frac{r_j(A_1)-p_j(A_1)}{|a_{jj}|}\left|a_{1j}\right|=0.375,\sum_{j\in N\backslash\{2\}}\frac{r_j(A_1)-p_j(A_1)}{|a_{jj}|}\left|a_{2j}\right|=0.5625,$$

$$\sum_{j\in N\backslash\{3\}}\frac{r_j(A_1)-p_j(A_1)}{|a_{jj}|}\left|a_{3j}\right|=1\ and\ \sum_{j\in N\backslash\{4\}}\frac{r_j(A_1)-p_j(A_1)}{|a_{jj}|}\left|a_{4j}\right|=1.$$

It is easy to verify that

$$\max_{i\in N}\frac{p_i(A_1)}{|a_{ii}|}\min_{i\in N}(|a_{ii}|-r_i(A_1))=0.25<0.375=\min_{i\in N}\sum_{j\in N\backslash\{i\}}\frac{r_j(A_1)-p_j(A_1)}{|a_{jj}|}\left|a_{ij}\right|,$$

that is, the matrix A_1 satisfies the conditions of Theorem 7. Therefore, from Theorem 7, we obtain that for any $0 < \varepsilon_1 < 1$,

$$||A_1^{-1}||_\infty \leq \frac{0.25+\varepsilon_1}{0.375+\varepsilon_1} < 1 = \frac{1}{\min\limits_{1\leq i\leq n}(|a_{ii}| - r_i(A_1))}.$$

However, the upper bound (2.5) contains the parameter ε. Next, based on the Theorem 4, a upper bound of the infinity norm of inverse of SDD_1 matrices is presented as follows, and this upper bound only depends on the elements of given matrices.

Theorem 8. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD_1 matrix, and for each $i \in N_2(A)$, there is at least one $a_{ij} \neq 0$, where $j \in N_2(A)$ and $j \neq i$, then

$$||A^{-1}||_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N}\left(p_i(A) - \sum\limits_{j\in N\setminus\{i\}} \frac{p_j(A)}{|a_{jj}|}|a_{ij}|\right)}.$$

Proof. By Theorem 4, we obtain that there exists a positive diagonal matrix D such that AD is an SDD matrix, where D is defined as Theorem 4. Therefore, we get the following inequality:

$$||A^{-1}||_\infty = ||D(D^{-1}A^{-1})||_\infty = ||D(AD)^{-1}||_\infty \leq ||D||_\infty ||(AD)^{-1}||_\infty.$$

Since the matrix D is positive diagonal, it is easy to obtain that

$$||D||_\infty = \max_{1\leq i\leq n} d_i = \max_{i\in N} \frac{p_i(A)}{|a_{ii}|}.$$

Since AD is an SDD matrix, by Theorem 5, we obtain

$$\begin{aligned} ||(AD)^{-1}||_\infty &\leq \frac{1}{\min\limits_{1\leq i\leq n}(|(AD)_{ii}| - r_i(AD))} \\ &= \frac{1}{\min\limits_{1\leq i\leq n}(|a_{ii}|d_i - r_i(AD))} \\ &= \frac{1}{\min\limits_{i\in N}\left(p_i(A) - \sum\limits_{j\in N\setminus\{i\}} \frac{p_j(A)}{|a_{jj}|}|a_{ij}|\right)}. \end{aligned}$$

Thus, we get that

$$||A^{-1}||_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N}\left(p_i(A) - \sum\limits_{j\in N\setminus\{i\}} \frac{p_j(A)}{|a_{jj}|}|a_{ij}|\right)}.$$

□

Since SDD matrices is a subclass of SDD_1 matrices, from Theorem 8, it is easy to obtain the following corollary.

Corollary 2. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD matrix, if $r_i(A) \neq 0$ for all $i \in N$, then

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{r_j(A)-p_j(A)}{|a_{jj}|}|a_{ij}|}.$$

The following Theorems 9 and 10 show that the bound in Corollary 2 is better than in Theorem 5 in some conditions.

Theorem 9. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD matrix, if $r_i(A) \neq 0$ for all $i \in N$ and

$$\min_{i\in N} \sum_{j\in N\backslash\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right| \geq \min_{i\in N}(|a_{ii}| - r_i(A)),$$

then

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{r_j(A)-p_j(A)}{|a_{jj}|}|a_{ij}|} < \frac{1}{\min\limits_{1\leq i\leq n} (|a_{ii}| - r_i(A))}.$$

Proof. Since A is an SDD matrix, it is easy to get that

$$|a_{ii}| > p_i(A) \; for \; any \; i \in N,$$

thus

$$\max_{i\in N} \frac{p_i(A)}{|a_{ii}|} < 1,$$

and from the condition

$$\min_{i\in N} \sum_{j\in N\backslash\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right| \geq \min_{i\in N}(|a_{ii}| - r_i(A)),$$

we obtain

$$\frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{r_j(A)-p_j(A)}{|a_{jj}|}|a_{ij}|} < \frac{1}{\min\limits_{1\leq i\leq n} (|a_{ii}| - r_i(A))}.$$

Therefore, from Corollary 2, we get

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{r_j(A)-p_j(A)}{|a_{jj}|}|a_{ij}|} < \frac{1}{\min\limits_{1\leq i\leq n} (|a_{ii}| - r_i(A))}.$$

□

The following Example 4 also illustrates the Theorem 9.

Example 4. Considering the following SDD matrix

$$A_3 = \begin{bmatrix} 2.5 & 2 & 0.4 & 0 \\ 2 & 5.5 & 3 & 0 \\ 1 & 2 & 3.5 & 0 \\ 1 & 2 & 0 & 3.5 \end{bmatrix}.$$

By the Varah bound (2.4) of Theorem 5, we obtain that $\|A_3^{-1}\|_\infty \leq 10$.

By a simple calculation, we obtain

$$p_1(A_3) \approx 2.1610,\ p_2(A_3) \approx 4.4914,\ p_3(A_3) \approx 2.7782 \textit{ and } p_4(A_3) \approx 2.7782,$$

then,

$$\frac{p_1(A_3)}{|a_{11}|} \approx 0.8644,\ \frac{p_2(A_3)}{|a_{22}|} \approx 0.8166,\ \frac{p_3(A_3)}{|a_{33}|} \approx 0.7938 \textit{ and } \frac{p_4(A_3)}{|a_{44}|} \approx 0.7938$$

thus,

$$\sum_{j\in N\backslash\{1\}} \frac{r_j(A_3) - p_j(A_3)}{|a_{jj}|}|a_{1j}| \approx 0.2103, \sum_{j\in N\backslash\{2\}} \frac{r_j(A_3) - p_j(A_3)}{|a_{jj}|}|a_{2j}| \approx 0.3813,$$

$$\sum_{j\in N\backslash\{3\}} \frac{r_j(A_3) - p_j(A_3)}{|a_{jj}|}|a_{3j}| \approx 0.2806 \textit{ and } \sum_{j\in N\backslash\{4\}} \frac{r_j(A_3) - p_j(A_3)}{|a_{jj}|}|a_{4j}| \approx 0.2806.$$

It is easy to verify that

$$\min_{i\in N} \sum_{j\in N\backslash\{i\}} \frac{r_j(A_3) - p_j(A_3)}{|a_{jj}|}|a_{ij}| = 0.2103 > 0.1 = \min_{i\in N}(|a_{ii}| - r_i(A_3)).$$

Therefore, the matrix A_3 satisfies the conditions of Theorem 9, thus from the bound of Theorem 9, we obtain

$$\|A_3^{-1}\|_\infty \leq 4.1103.$$

In fact, $\|A_3^{-1}\|_\infty \approx 0.9480$. Obviously, $\|A_3^{-1}\|_\infty \approx 0.9480 < 4.1103 < 10$, which means that the bound in Corollary 2 is better than Varah bound of Theorem 5 in some conditions.

Theorem 10. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ $(n \geq 2)$ be an SDD matrix, if $r_i(A) \neq 0$ for all $i \in N$ and

$$\max_{i\in N} \frac{p_i(A)}{|a_{ii}|} \min_{i\in N}(|a_{ii}| - r_i(A)) \leq \min_{i\in N} \sum_{j\in N\backslash\{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right| < \min_{i\in N}(|a_{ii}| - r_i(A)),$$

then

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{r_j(A)-p_j(A)}{|a_{jj}|}|a_{ij}|} \leq \frac{1}{\min\limits_{1\leq i\leq n}(|a_{ii}| - r_i(A))}.$$

Proof. Since A is an SDD matrix, we have

$$|a_{ii}| > r_i(A) \ for \ any \ i \in N.$$

From the condition $r_i(A) \neq 0$ for all $i \in N$ and Theorem 4, it is easy to obtain that

$$\sum_{j \in N \setminus \{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right| > 0 \ for \ any \ i \in N.$$

Therefore, from the condition

$$\max_{i \in N} \frac{p_i(A)}{|a_{ii}|} \min_{i \in N}(|a_{ii}| - r_i(A)) \leq \min_{i \in N} \sum_{j \in N \setminus \{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} \left|a_{ij}\right|,$$

we obtain

$$\frac{\max\limits_{i \in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i \in N} \sum\limits_{j \in N \setminus \{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} |a_{ij}|} \leq \frac{1}{\min\limits_{1 \leq i \leq n} (|a_{ii}| - r_i(A))},$$

and thus from Corollary 2, we get

$$\|A^{-1}\|_\infty \leq \frac{\max\limits_{i \in N} \frac{p_i(A)}{|a_{ii}|}}{\min\limits_{i \in N} \sum\limits_{j \in N \setminus \{i\}} \frac{r_j(A) - p_j(A)}{|a_{jj}|} |a_{ij}|} \leq \frac{1}{\min\limits_{1 \leq i \leq n} (|a_{ii}| - r_i(A))}.$$

□

The following Example 5 also illustrates the Theorem 10.

Example 5. Considering the following the following SDD matrix

$$A_4 = \begin{bmatrix} 4 & 1 & 1 & 0 \\ 2 & 8 & 2 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 2 & 0 & 4 \end{bmatrix}.$$

By the Varah bound (2.4) of Theorem 5, we obtain that $\|A_4^{-1}\|_\infty \leq 1$.

By calculation, we obtain that

$$p_1(A_4) = 1.25, \ p_2(A_4) = 2.5, \ p_3(A_4) = 1.5 \ and \ \ p_4(A_4) = 1.5,$$

then,

$$\frac{p_1(A_4)}{|a_{11}|} = 0.3125, \ \frac{p_2(A_4)}{|a_{22}|} = 0.3125, \ \frac{p_3(A_4)}{|a_{33}|} = 0.375 \ and \ \frac{p_4(A_4)}{|a_{44}|} = 0.375,$$

thus,

$$\sum_{j \in N \setminus \{1\}} \frac{r_j(A_4) - p_j(A_4)}{|a_{jj}|} |a_{1j}| = 0.5625, \ \sum_{j \in N \setminus \{2\}} \frac{r_j(A_4) - p_j(A_4)}{|a_{jj}|} |a_{2j}| = 1.125,$$

$$\sum_{j\in N\setminus\{3\}} \frac{r_j(A_4)-p_j(A_4)}{|a_{jj}|}|a_{3j}| = 0.5625 \; and \; \sum_{j\in N\setminus\{4\}} \frac{r_j(A_4)-p_j(A_4)}{|a_{jj}|}|a_{4j}| = 0.5625.$$

It is easy to verify that

$$\max_{i\in N} \frac{p_i(A_4)}{|a_{ii}|} \min_{i\in N}(|a_{ii}| - r_i(A_4)) = 0.375 < \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{r_j(A_4)-p_j(A_4)}{|a_{jj}|}|a_{ij}| = 0.5625 < 1 = \min_{i\in N}(|a_{ii}| - r_i(A_4)),$$

that is, the matrix A_4 satisfies the conditions of Theorem 10, thus by the bound of Theorem 10, we obtain

$$||A_4^{-1}||_\infty \leq 0.6667.$$

In fact, $||A_4^{-1}||_\infty \approx 0.4019$. Obviously, $||A_4^{-1}||_\infty \approx 0.4019 < 0.6667 < 1$, which means that the bound in Corollary 2 is better than Varah bound of Theorem 5 in some conditions.

3. Conclusions

In this paper, a new proof that SDD_1 matrices is a subclass of H-matrices is given and based on the new proof, some upper bounds of the infinity norm of inverse of SDD_1 matrices are established, and some new upper bounds of the infinity norm of inverse of SDD matrices are also obtained. Moreover, we show that these new upper bounds of the infinity norm of inverse of SDD matrices are better than well-known Varah bound under some cases. In addition, some numerical examples are given to illustrate the corresponding results.

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Conflict of interest

The authors declare that they have no competing interests.

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