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Research article

Self-adaptive algorithms for the split problem of the quasi-pseudocontractive operators in Hilbert spaces

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Abstract: Self-adaptive algorithms are presented for solving the split common fixed point problem of quasi-pseudocontractive operators in Hilbert spaces. Weak and strong convergence theorems are given under some mild assumptions.

Keywords: split common fixed point; quasi-pseudocontractive operator; self-adaptive algorithm
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1. Introduction

Inverse problems in various disciplines can be expressed as split feasibility problems and their generalizations, such as multiple-sets split feasibility problems and split common fixed point problems, and many iterative algorithms have been presented to solve these problems (see [1–6] for multiple-sets split feasibility problems, [7–10] for split feasibility problems, [11–18] for split common fixed point problems, or [19–24] for a self-adaptive method).

The present article is focusing on the split common fixed point problem by virtue of self-adaptive algorithms such that involved methods are simpler. The split common fixed point problem is a generalization of the split feasibility problem which is a general way to characterize various inverse problems arising in many real-world application problems, such as medical image reconstruction and intensity-modulated radiation therapy.

Recall that the split common fixed point problem is to find a point $u \in H_1$ such that

$$u \in \mathrm{Fix}(T) \quad \text{and} \quad Au \in \mathrm{Fix}(S). \tag{1.1}$$

The split feasibility problem is to find a point satisfying

$$u \in C \quad \text{and} \quad Au \in Q, \tag{1.2}$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator. Problem (1.2) was firstly introduced by Censor and Elfving [25] in finite-dimensional Hilbert spaces. In [26], note that solving (1.1) can be translated to solve the fixed point equation

$$u = S(u - \tau A^*(I - T)Au), \ \tau > 0.$$

Whereafter, Censor and Segal proposed an algorithm for directed operators. Since then, there has been growing interest in the split common fixed point problem. In [27], based on the work [26], Moudafi investigated an algorithm for solving the split common fixed-point problem for the class of demicontractive operators in a Hilbert space. In [28], Kraikaew et al. modified the iterative scheme studied by Moudafi for quasi-nonexpansive operators to obtain strong convergence to a solution of the split common fixed point problem. In [29], based on Halpern's type method, Boikanyo constructed an algorithm for demicontractive operators that produces sequences that always converge strongly to a specific solution of the split common fixed point problem. In [30], Ansari introduced an implicit algorithm and an explicit algorithm for solving the split common fixed point problems. In [31], using the hybrid method and the shrinking projection method in mathematical programming, Takahashi proved strong convergence theorems for finding a solution of the split common fixed point problem in two Banach spaces. In [32], Wang proposed a new algorithm for the split common fixed-point problem that does not need any priori information of the operator norm.

In the case where C and Q in (1.2) are the intersections of finitely many fixed point sets of nonlinear operators, problem (1.2) is called by Censor and Segal ([26]) the split common fixed point problem. More precisely, the split common fixed point problem requires one to seek an element $u \in H_1$ such that

$$u \in \bigcap_{i=1}^{s} \mathrm{Fix}(T_i) \quad \text{and} \quad Au \in \bigcap_{j=1}^{t} \mathrm{Fix}(S_i), \tag{1.3}$$

where $\mathrm{Fix}(S_i)$ and $\mathrm{Fix}(T_i)$ denote the fixed point sets of two classes of nonlinear operators $S_i : H_1 \to H_1$ and $T_i : H_2 \to H_2$, respectively.

In particular, Yao et al. [21] introduced the following new iterative algorithms for the split common fixed point problem of demicontractive operators.

Algorithm 1.1. *Choose an arbitrary initial guess $x_0 \in H_1$. Assume x_n has been constructed. If*

$$\|x_n - Tx_n + A^*(I - S)Ax_n\| = 0, \tag{1.4}$$

then stop; otherwise, continue and construct via the manner

$$x_{n+1} = x_n - \gamma\tau_n(x_n - Sx_n + A^*(I - T)Ax_n),$$

where $\gamma \in (0, \min\{1-\beta, 1-\mu\})$ *is a positive constant and* τ_n *is chosen self-adaptively as*

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2}. \tag{1.5}$$

Algorithm 1.2. *Let* $u \in H_1$ *and choose an arbitrary initial value* $x_0 \in H_1$. *Assume* x_n *has been constructed. If*

$$\|x_n - Tx_n + A^*(I-S)Ax_n\| = 0, \tag{1.6}$$

then stop; otherwise, continue and construct via the manner

$$x_{n+1} = \alpha_n u + (1-\alpha_n)[x_n - \gamma\tau_n(x_n - Tx_n + A^*(I-S)Ax_n)], \tag{1.7}$$

where $\gamma \in (0, \min\{1-\beta, 1-\mu\})$ *is a positive constant and* τ_n *is chosen self-adaptively as*

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2}{\|x_n - Sx_n + A^*(I-T)Ax_n\|^2}. \tag{1.8}$$

Yao et al. obtained the weak and strong convergence of Algorithms 1.1 and 1.2, respectively. It should be pointed out that Algorithm 1.1 and 1.2 do not need any prior information of the operator norm. Inspired by the work in the literature, the main purpose of this paper is to present two new self-adaptive algorithms for approximating a solution of the split common fixed point problem (1.3) for the class of quasi-pseudocontractive operators which is more general than the classes of quasi-nonexpansive operators, directed operators and demicontractive operators. Weak and strong convergence theorems are given under some mild assumptions.

2. preliminaries

In this section, we collect some tools including some definitions, some useful inequalities and lemmas which will be used to derive our main results in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \longrightarrow C$ be an operator. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T, that is, $\mathrm{Fix}(T) = \{u | u = Tu, u \in C\}$.

First, we give some definitions related to the involved operators.

Definition 2.1. *An operator* $T : C \longrightarrow C$ *is said to be*

(i) nonexpansive if $\|Tu - Tv\| \le \|u - v\|$ *for all* $u, v \in C$.

(ii) quasi-nonexpansive if $\|Tu - u^*\| \le \|u - u^*\|$ *for all* $u \in C$ *and* $u^* \in \mathrm{Fix}(T)$.

(iii) firmly nonexpansive if $\|Tu - Tv\|^2 \le \|u - v\|^2 - \|(I-T)u - (I-T)v\|^2$ *for all* $u, v \in C$.

(iv) directed(or firmly quasi-nonexpansive) if $\|Tu - u^*\|^2 \le \|u - u^*\|^2 - \|Tu - u\|^2$ *for all* $u \in C$ *and* $u^* \in \mathrm{Fix}(T)$.

Definition 2.2. *An operator* $T : C \longrightarrow C$ *is said to be pseudo-contractive if* $\langle Tu - Tv, u - v \rangle \le \|u - v\|^2$ *for all* $u, v \in C$.

The interest of pseudocontractive operators lies in their connection with monotone operators; namely, T is a pseudocontraction if and only if the complement $I - T$ is a monotone operator. It is well known that T is pseudocontractive if and only if

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|(I - T)u - (I - T)v\|^2,$$

for all $u, v \in C$.

Definition 2.3. *An operator T is said to be strictly pseudocontractive if*

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + k\|(I - T)u - (I - T)v\|^2,$$

for all $u, v \in C$, where $k \in [0, 1)$.

Definition 2.4. *An operator T is said to be demicontractive if there exists a constant $k \in [0, 1)$ such that*

$$\|Tu - u^*\|^2 \leq \|u - u^*\|^2 + k\|Tu - u\|^2,$$

or equivalently,

$$\langle u - Tu, u - u^* \rangle \geq \frac{1-k}{2}\|u - Tu\|^2, \tag{2.1}$$

for all $u \in C$ and $u^* \in \mathrm{Fix}(T)$.

Remark 2.1. From the above definitions, we note that the class of demicontractive operators contains important operators such as the directed operators, the quasi-nonexpansive operators and the strictly pseudocontractive operators with fixed points. Such a class of operators is fundamental because it includes many types of nonlinear operators arising in applied mathematics and optimization.

Definition 2.5. *An operator $T : C \longrightarrow C$ is said to be quasi-pseudocontractive if $\|Tu - u^*\|^2 \leq \|u - u^*\|^2 + \|Tu - u\|^2$ for all $u \in C$ and $u^* \in$ Fix(T).*

Definition 2.6. *An operator $T : C \longrightarrow C$ is said to be L-Lipschitzian if there exists $L > 0$ such that $\|Tu - Tv\| \leq L\|u - v\|$ for all $u, v \in C$.*

Usually, the convergence of fixed point algorithms requires some additional smoothness properties of the mapping T such as demi-closedness.

Definition 2.7. *An operator T is said to be demiclosed if, for any sequence $\{u_n\}$ which weakly converges to u^*, and if $Tu_n \longrightarrow w$, then $Tu^* = w$.*

Definition 2.8. *A sequence $\{u_n\}$ is called Fejér-monotone with respect to a given nonempty set Ω if for every $u \in \Omega$,*

$$\|u_{n+1} - u\| \leq \|u_n - u\|,$$

for all $n \geq 0$.

Recall that the (nearest point or metric) projection from H onto C, denoted by P_C, assigns to each $u \in H$, the unique point $P_C u \in C$ with the property

$$\|u - P_C u\| = \inf\{\|u - v\| : v \in C\}.$$

The metric projection P_C of H onto C is characterized by

$$\langle u - P_C u, v - P_C u\rangle \leq 0, \tag{2.2}$$

for all $u \in H, v \in C$. It is well known that the metric projection $P_C : H \to C$ is firmly nonexpansive, that is,

$$\langle u - v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2,$$
$$\text{or } \|P_C u - P_C v\|^2 \leq \|u - v\|^2 - \|(I - P_C)u - (I - P_C)v\|^2,$$

for all $u, v \in H$.

Next we adopt the following notations:

- $u_n \rightharpoonup u$ means that $\{u_n\}$ converges weakly to u;
- $u_n \to u$ means that $\{u_n\}$ converges strongly to u;
- $\omega_w(u_n)$ stands for the set of cluster points in the weak topology, that is,

$$\omega_w(u_n) = \{u : \exists u_{n_j} \rightharpoonup u\}.$$

Lemma 2.1. *([33]) Let C be a nonempty closed convex subset in H. If the sequence $\{u_n\}$ is Fejér monotone with respect to Ω, then we have the following conclusions:*

(i) $u_n \rightharpoonup u \in \Omega$ iff $\omega_w(u_n) \subset \Omega$;

(ii) the sequence $\{P_\Omega u_n\}$ converges strongly;

(iii) if $u_n \rightharpoonup u \in \Omega$, then $u = \lim_{n\to\infty} P_\Omega u_n$.

For all $u, v \in H$, the following conclusions hold:

$$\|tu + (1 - t)v\|^2 = t\|u\|^2 + (1 - t)\|v\|^2 - t(1 - t)\|u - v\|^2, \quad t \in [0, 1],$$

$$\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2,$$

and

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle.$$

Lemma 2.2. *([34]) Let H be a Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. If $T : C \to C$ is an L-Lipschitzian operator with $L \geq 1$. Then*

$$Fix(((1 - \delta)I + \delta T)T) = Fix(T((1 - \delta)I + \delta T)) = Fix(T),$$

where $\delta \in (0, \frac{1}{L})$.

Lemma 2.3. *([34]) Let H be a Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. If $T : C \to C$ is an L-Lipschitzian operator with $L \geq 1$ and $I - T$ is demiclosed at 0, then the composition operator*

$$I - T((1 - \delta)I + \delta T),$$

is also demiclosed at 0 provided $\delta \in (0, \frac{1}{L})$.

Lemma 2.4. *([34]) Let H be a Hilbert space and $\emptyset \neq C \subset H$ be a closed convex set. If $T : C \to C$ is an L-Lipschitzian quasi-pseudocontractive operator. Then we have*

$$\|T((1-\zeta)I+\zeta T)u-u^*\|^2 \leq \|u-u^*\|^2+(1-\zeta)\|T((1-\zeta)+\zeta T)u-u\|^2,$$

for all $u \in C$ and $u^* \in Fix(T)$ when $0 < \zeta < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.5. *([35])Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad n \in N,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Results

Throughout the present article, let H_1 and H_2 be two real Hilbert spaces. We use $\langle\cdot,\cdot\rangle$ to denote the inner product, and $\|\cdot\|$ stands for the corresponding norm. Let s and t be positive integers, and let $T_i : H_1 \longrightarrow H_1$ be an L_{1i}-Lipschitzian quasi-pseudocontractive operator with $1 < L_{1i} \leq L_1$ and $S_j : H_2 \longrightarrow H_2$ be an L_{2j}-Lipschitzian quasi-pseudocontractive operator with $1 < L_{2j} \leq L_2$, where $L_1, L_2 > 1$. Denote the fixed point sets of T_i and S_j by $\mathrm{Fix}(T_i)$ and $\mathrm{Fix}(S_j)$, respectively. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint A^*. Throughout, assume

$$\Omega = \{u \in \bigcap_{i=1}^{s} \mathrm{Fix}(T_i) \text{ and } Au \in \bigcap_{j=1}^{t} \mathrm{Fix}(S_j)\} \neq \emptyset.$$

Next we present the following iterative algorithm to solve (1.3).

Algorithm 3.1. *Choose an arbitrary initial value $x_1 \in H_1$. Assume x_n has been constructed. Compute*

$$\begin{aligned} y_n &= T_{i_n}((1-\zeta_n)I+\zeta_n T_{i_n})x_n, \\ z_n &= (I-S_{j_n}((1-\eta_n)I+\eta_n S_{j_n}))Ax_n, \\ u_n &= x_n - y_n + A^*z_n, \quad n \geq 1, \end{aligned} \tag{3.1}$$

where $0 < \zeta_n < \frac{1}{\sqrt{1+L_1^2}+1}$, $0 < \eta_n < \frac{1}{\sqrt{1+L_2^2}+1}$,

$$i_n = \arg\max \{\|x_n - T_i((1-\zeta_n)I+\zeta_n T_i)x_n\| : \ i \in I_1 = \{1,2,\ldots,s\}\},$$

and

$$j_n = \arg\max \{\|(I-S_j((1-\eta_n)I+\eta_n S_j))Ax_n\| : \ j \in I_2 = \{1,2,\ldots,t\}\}.$$

If

$$\|u_n\| = 0, \tag{3.2}$$

then stop (in this case by Remark 3.2 below); otherwise, continue and construct via the manner

$$x_{n+1} = x_n - \tau_n u_n, \tag{3.3}$$

where

$$\tau_n = \lambda_n \frac{\|x_n - y_n\|^2 + \|z_n\|^2}{\|u_n\|^2}, \tag{3.4}$$

in which $\lambda_n > 0$.

Remark 3.1. The equality (3.2) holds if and only if x_n is a solution of (1.3). First, assume that x_n is a solution of (1.3), that is,

$$x_n \in \bigcap_{i=1}^{s} \mathrm{Fix}(T_i) \quad \text{and} \quad Ax_n \in \bigcap_{j=1}^{t} \mathrm{Fix}(S_i).$$

According to Lemma 2.2, we get that

$$T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n = x_n = T_{i_n}x_n,$$

and

$$S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n = Ax_n = S_{j_n}Ax_n.$$

From (3.1), it turns out that $z_n = 0$ and $x_n = y_n$. Therefore, $u_n = x_n - y_n + A^* z_n = 0$.

In the sequel, we assume that the equality (3.2) holds. For any $u \in \Omega$, we have

$$\begin{aligned}
0 &= \langle u_n, x_n - u\rangle \\
&= \langle x_n - y_n + A^* z_n, x_n - u\rangle \\
&= \langle x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n, x_n - u\rangle \\
&\quad + \langle A^*(I - S_{j_n}((1-\eta_n)u + \eta_n S_{j_n}))Ax_n, x_n - u\rangle \\
&= \langle x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n, x_n - u\rangle \\
&\quad + \langle (I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n, Ax_n - Au\rangle.
\end{aligned} \tag{3.5}$$

By Lemma 2.4, $T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})$ and $S_{j_n}((1-\eta_n)I + \eta_n S_{j_n})$ are demicontractive, from (2.1), we deduce

$$\begin{aligned}
\langle x_n &- T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n, x_n - u\rangle \\
&\geq \frac{1-(1-\zeta_n)}{2}\|x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n\|^2 \\
&= \frac{\zeta_n}{2}\|x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n\|^2,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\langle (I- &S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n, Ax_n - Au\rangle \\
&\geq \frac{1-(1-\eta_n)}{2}\|(I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n\|^2 \\
&= \frac{\eta_n}{2}\|(I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n\|^2.
\end{aligned} \tag{3.7}$$

By(3.5)–(3.7), we get

$$\begin{aligned}0 &= \langle u_n, x_n - u\rangle \\ &\geq \frac{\zeta_n}{2}\|x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n\|^2 \\ &\quad + \frac{\eta_n}{2}\|(I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n\|^2.\end{aligned} \tag{3.8}$$

Since $\zeta_n, \eta_n \in (0,1)$, we deduce

$$\|x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n\| = 0, \tag{3.9}$$

and

$$\|(I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n\| = 0. \tag{3.10}$$

According to the definitions of i_n and j_n, it follows from (3.9) and (3.10) that

$$\|x_n - T_i((1-\zeta_n)I + \zeta_n T_i)x_n\| = 0, \tag{3.11}$$

for all $i \in I_1$ and

$$\|(I - S_j((1-\eta_n)I + \eta_n S_j))Ax_n\| = 0, \tag{3.12}$$

for all $j \in I_2$. Hence, by Lemma 2.2, we have

$$x_n \in \bigcap_{i=1}^{s} \mathrm{Fix}(T_i((1-\zeta_n)I + \zeta_n T_i)) = \bigcap_{i=1}^{s} \mathrm{Fix}(T_i),$$

and

$$Ax_n \in \bigcap_{j=1}^{t} \mathrm{Fix}(S_j((1-\eta_n)I + \eta_n S_j)) = \bigcap_{j=1}^{t} \mathrm{Fix}(S_j).$$

Therefore, $x_n \in \Omega$.

Assume that the sequence $\{x_n\}$ generated by Algorithm 3.1 is infinite. In other words, Algorithm 3.1 does not terminate in a finite number of iterations. Next, we demonstrate the convergence analysis of the sequence $\{x_n\}$ generated by Algorithm 3.1.

Theorem 3.1. *Suppose that $I - T_i$ (for all $i \in I_1$) and $I - S_j$ (for all $j \in I_2$) are demiclosed at zero. If $\Omega \neq \emptyset$ and the following conditions are satisfied:*

(C_1) $0 < \zeta \leq \zeta_n < \frac{1}{\sqrt{1+L_1^2}+1}$;

(C_2) $0 < \eta \leq \eta_n < \frac{1}{\sqrt{1+L_2^2}+1}$;

(C_3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\zeta, \eta\}$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to a solution $z^*(= \lim_{n\to\infty} P_\Omega(x_n))$ of problem (1.3).

Proof. Firstly, we prove that the sequence $\{x_n\}$ is Fejér-monotone with respect to Ω. Picking up $z \in \Omega$, from (3.8), we have

$$\begin{aligned}
&\langle u_n, x_n - z\rangle \\
&= \langle x_n - y_n + A^* z_n, x_n - z\rangle \\
&\geq \frac{\zeta}{2}\|x_n - T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n\|^2 \\
&\quad + \frac{\eta}{2}\|(I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n\|^2 \\
&\geq \frac{1}{2}\min\{\zeta, \eta\}(\|x_n - y_n\|^2 + \|z_n\|^2).
\end{aligned} \tag{3.13}$$

According to (3.1), (3.3), (3.4) and (3.13), we derive

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|x_n - z - \tau_n u_n\|^2 \\
&= \|x_n - z\|^2 - 2\tau_n\langle u_n, x_n - z\rangle + \tau_n^2\|u_n\|^2 \\
&\leq \|x_n - z\|^2 - \min\{\zeta, \eta\}\frac{\lambda_n(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2} \\
&\quad + \frac{\lambda_n^2(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2} \\
&= \|x_n - z\|^2 - \lambda_n(\theta - \lambda_n)\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2},
\end{aligned} \tag{3.14}$$

where

$$\theta = \min\{\zeta, \eta\}.$$

By virtue of (3.14), we deduce that the sequence $\{x_n\}$ is Fejér-monotone with respect to Ω. Next, we show that every weak cluster point of the sequence $\{x_n\}$ belongs to the solution set of problem (1.3), i.e. $\omega_w(x_n) \subset \Omega$.

From the Fejér-monotonicity of $\{x_n\}$ it follows that the sequence $\{x_n\}$ is bounded, and so are the sequences $\{Ax_n\}$, $\{T_i x_n\}(i \in I_1)$ and $\{S_j Ax_n\}(j \in I_2)$. Further, from (3.14), we obtain

$$\lambda_n(\theta - \lambda_n)\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2} \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2, \tag{3.15}$$

which implies that

$$\lim_{n\to\infty}\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2} = 0. \tag{3.16}$$

This together with the boundedness of the sequence $\{u_n\}$ implies that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0,$$

and

$$\lim_{n\to\infty}\|z_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|x_n - T_i((1-\zeta_n)I + \zeta_n T_i)x_n\| = 0,$$

for all $i \in I_1$ and

$$\lim_{n\to\infty} \|Ax_n - S_j((1-\eta_n)I + \eta_n S_j)Ax_n\| = 0,$$

for all $j \in I_2$. It follows that

$$\begin{aligned}
&\|x_n - T_i x_n\| \\
&\quad \le \|x_n - T_i((1-\zeta_n)I + \zeta_n T_i)x_n\| \\
&\qquad + \|T_i((1-\zeta_n)I + \zeta_n T_i))x_n - T_i x_n\| \\
&\quad \le \|x_n - T_i((1-\zeta_n)I + \zeta_n T_i)x_n\| \\
&\qquad + \zeta_n L_{1i}\|x_n - T_i x_n\| \\
&\quad \le \|x_n - T_i((1-\zeta_n)I + \zeta_n T_i)x_n\| \\
&\qquad + \zeta_n L_1\|x_n - T_i x_n\|,
\end{aligned} \tag{3.17}$$

for all $i \in I_1$ and

$$\begin{aligned}
&\|Ax_n - S_j Ax_n\| \\
&\quad \le \|Ax_n - S_j((1-\eta_n)I + \eta_n S_j)Ax_n\| \\
&\qquad + \|S_j((1-\eta_n)I + \eta_n S_j)Ax_n - S_j Ax_n\| \\
&\quad \le \|Ax_n - S_j((1-\eta_n)I + \eta_n S_j)Ax_n\| \\
&\qquad + \eta_n L_{2j}\|Ax_n - S_j Ax_n\| \\
&\quad \le \|Ax_n - S_j((1-\eta_n)I + \eta_n S_j)Ax_n\| \\
&\qquad + \eta_n L_2\|Ax_n - S_j Ax_n\|,
\end{aligned} \tag{3.18}$$

for all $j \in I_2$. By (C_1), we have $\zeta_n(\sqrt{1+L_1^2}+1) < 1$ and so $\zeta_n L_1 < 1 - \zeta_n \le 1 - \zeta$. This together with (3.17) implies that

$$\begin{aligned}
\|x_n - T_i x_n\| &\le \frac{1}{1-\zeta_n L_1}\|x_n - T_i((1-\zeta_n)I + \zeta_n T_i))x_n\| \\
&\le \frac{1}{\zeta}\|x_n - T_i((1-\zeta_n)I + \zeta_n T_i))x_n\|,
\end{aligned} \tag{3.19}$$

for all $i \in I_1$. Employing a similar way, we obtain

$$\begin{aligned}
\|Ax_n - S_j Ax_n\| &\le \frac{1}{1-\eta_n L_2}\|Ax_n - S_j((1-\eta_n)I + \eta_n S_j))Ax_n\| \\
&\le \frac{1}{\eta}\|Ax_n - S_j((1-\eta_n)I + \eta_n S_j))Ax_n\|,
\end{aligned} \tag{3.20}$$

for all $j \in I_2$. Therefore,

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0,$$

for all $i \in I_1$, and

$$\lim_{n\to\infty} \|Ax_n - S_j Ax_n\| = 0,$$

for all $j \in I_2$. By the demiclosedness (at zero) of $I - T_i$ (for all $i \in I_1$) and $I - S_j$ (for all $j \in I_2$), we deduce immediately $\omega_w(x_n) \subset \Omega$. To this end, the conditions of Lemma 2.1 are all satisfied. Consequently, $x_n \rightharpoonup z^*(= \lim_{n\to\infty} P_\Omega x_n)$. The proof is completed. □

Algorithm 3.1 has only weak convergence. Now, we present a new algorithm with strong convergence.

Algorithm 3.2. *Let $u \in H_1$ and choose an arbitrary initial value $x_1 \in H_1$. Assume x_n has been constructed. Compute*

$$\begin{aligned} y_n &= T_{i_n}((1-\zeta_n)I + \zeta_n T_{i_n})x_n, \\ z_n &= (I - S_{j_n}((1-\eta_n)I + \eta_n S_{j_n}))Ax_n, \\ u_n &= x_n - y_n + A^* z_n, \quad n \geq 1, \end{aligned} \tag{3.21}$$

where $0 < \zeta_n < \frac{1}{\sqrt{1+L_1^2}+1}$, $0 < \eta_n < \frac{1}{\sqrt{1+L_2^2}+1}$,

$$i_n = \arg\max \{\|x_n - T_i((1-\zeta_n)I + \zeta_n T_i)x_n\| : \ i \in I_1 = \{1, 2, \ldots, s\}\},$$

and

$$j_n = \arg\max \{\|(I - S_j((1-\eta_n)I + \eta_n S_j))Ax_n\| : \ j \in I_2 = \{1, 2, \ldots, t\}\}.$$

If

$$\|u_n\| = 0, \tag{3.22}$$

then stop; otherwise, continue and construct via the manner

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \tau_n u_n), \tag{3.23}$$

where $\alpha_n \in (0, 1)$ *and*

$$\tau_n = \lambda_n \frac{\|x_n - y_n\|^2 + \|z_n\|^2}{\|u_n\|^2}, \tag{3.24}$$

in which $\lambda_n > 0$.

Assume that the sequence $\{x_n\}$ generated by Algorithm 3.2 is infinite. In other words, Algorithm 3.2 does not terminate in a finite number of iterations.

Theorem 3.2. *Suppose that $I - T_i$ (for all $i \in I_1$) and $I - S_j$ (for all $j \in I_2$) are demiclosed at zero. If $\Omega \neq \emptyset$ and the following conditions are satisfied:*

(C_1) $0 < \zeta \leq \zeta_n < \frac{1}{\sqrt{1+L_1^2}+1}$;

(C_2) $0 < \eta \leq \eta_n < \frac{1}{\sqrt{1+L_2^2}+1}$;

(C_3) $\lim_{n\to\infty} \alpha_n = 0 \ \sum_{n=1}^{\infty} \alpha_n = +\infty$;

(C_4) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \min\{\zeta, \eta\}$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to a solution $z(= P_\Omega(u))$ of problem (1.3).

Proof. Set $v_n = x_n - \tau_n u_n$ for all $n \geq 0$. By (3.14),we have

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - \lambda_n(\theta - \lambda_n)\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2}. \tag{3.25}$$

In particular, we have $\|v_n - z\| \leq \|x_n - z\|$. Thus, from (3.23), we obtain

$$\begin{aligned}\|x_{n+1} &- z\| \\ &= \|\alpha_n u + (1-\alpha_n)v_n - z\| \\ &\leq \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_n - z\|\}.\end{aligned} \tag{3.26}$$

By induction, we derive

$$\|x_{n+1} - z\| \leq \max\{\|u - z\|, \|x_0 - z\|\}.$$

Hence, $\{x_n\}$ is bounded and so are the sequences $\{Ax_n\}$, $\{T_i x_n\}(i \in I_1)$ and $\{S_j Ax_n\}(j \in I_2)$.

From (3.23), we have

$$\begin{aligned}\|x_{n+1} &- z\|^2 \\ &= \|\alpha_n(u - z) + (1-\alpha_n)(v_n - z)\|^2 \\ &\leq (1-\alpha_n)\|v_n - z)\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle.\end{aligned} \tag{3.27}$$

By virtue of (3.25) and (3.27), we deduce

$$\begin{aligned}\|x_{n+1} &- z\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\quad - (1-\alpha_n)\lambda_n(\theta - \lambda_n)\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2} \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n[2\langle u - z, x_{n+1} - z\rangle \\ &\quad - \frac{1-\alpha_n}{\alpha_n}\lambda_n(\theta - \lambda_n)\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2}].\end{aligned} \tag{3.28}$$

Set $\varpi_n = \|x_n - z\|^2$ and

$$\delta_n = 2\langle u - z, x_{n+1} - z\rangle - \frac{1-\alpha_n}{\alpha_n}\lambda_n(\theta - \lambda_n)\frac{(\|x_n - y_n\|^2 + \|z_n\|^2)^2}{\|u_n\|^2}$$

for all $n \geq 1$. Then, from (3.28), we have

$$0 \leq \varpi_{n+1} \leq (1-\alpha_n)\varpi_n + \alpha_n\delta_n, \quad n \geq 1. \tag{3.29}$$

It is obvious that

$$\delta_n \leq 2\langle u - z, x_{n+1} - z\rangle \leq 2\|u - z\| \cdot \|x_{n+1} - z\|.$$

So, $\limsup_{n\to\infty}\delta_n < \infty$. Next, we show that $\limsup_{n\to\infty}\delta_n \geq -1$ by contradiction. Assume that $\limsup_{n\to\infty}\delta_n < -1$. Then there exists m such that $\delta_n \leq -1$ for all $n \geq m$. It follows from (3.29) that

$$\begin{aligned}\varpi_{n+1} &\leq (1-\alpha_n)\varpi_n + \alpha_n\delta_n \\ &= \varpi_n + \alpha_n(\delta_n - \varpi_n) \\ &\leq \varpi_n - \alpha_n,\end{aligned} \tag{3.30}$$

for all $n \geq m$. Thus,

$$\varpi_{n+1} \leq \varpi_m - \sum_{i=m}^{n} \alpha_i.$$

Hence, by taking lim sup as $n \to \infty$ in the last inequality, we obtain

$$0 \leq \limsup_{n\to\infty} \varpi_{n+1} \leq \varpi_m - \sum_{i=m}^{\infty} \alpha_i = -\infty,$$

which is a contradiction. Therefore, $\limsup_{n\to\infty} \delta_n > -1$ and it is finite. Consequently, we can take a subsequence $\{n_i\}$ such that

$$\begin{aligned} \limsup_{n\to\infty} \delta_n &= \lim_{i\to\infty} \delta_{n_i} \\ &= \lim_{i\to\infty} [-\frac{1-\alpha_{n_i}}{\alpha_{n_i}} \lambda_{n_i}(\theta - \lambda_{n_i}) \frac{(\|x_{n_i} - y_{n_i}\|^2 + \|z_{n_i}\|^2)^2}{\|u_{n_i}\|^2} \\ &\quad + 2\langle u - z, x_{n_i+1} - z\rangle]. \end{aligned} \tag{3.31}$$

Since $\langle u - z, x_{n_i+1} - z\rangle$ is a bounded real sequence, without loss of generality, we may assume the limit $\lim_{i\to\infty}\langle u - z, x_{n_i+1} - z\rangle$ exists. Consequently, from (3.31), the following limit also exists

$$\lim_{i\to\infty} \frac{1-\alpha_{n_i}}{\alpha_{n_i}} \lambda_{n_i}(\theta - \lambda_{n_i}) \frac{(\|x_{n_i} - y_{n_i}\|^2 + \|z_{n_i}\|^2)^2}{\|u_{n_i}\|^2}.$$

This together with conditions (C_3) and (C_4) implies that

$$\lim_{i\to\infty} \frac{(\|x_{n_i} - y_{n_i}\|^2 + \|z_{n_i}\|^2)^2}{\|u_{n_i}\|^2} = 0,$$

which yields $\lim_{i\to\infty} \|x_{n_i} - y_{n_i}\| = 0$ and $\lim_{i\to\infty} \|z_{n_i}\| = 0$. By a similar proof as in Theorem 3.3, we conclude that any weak cluster point of $\{x_{n_i}\}$ belongs to Ω. Note that

$$\begin{aligned} &\|x_{n_i+1} - x_{n_i}\| \\ &\quad = \|\alpha_{n_i} u + (1 - \alpha_{n_i}) v_{n_i} - x_{n_i}\| \\ &\quad \leq \alpha_{n_i} \|u - x_{n_i}\| + (1 - \alpha_{n_i}) \|v_{n_i} - x_{n_i}\| \\ &\quad \leq \alpha_{n_i} \|u - x_{n_i}\| + \tau_{n_i} \|u_{n_i}\| \\ &\quad \leq \alpha_{n_i} \|u - x_{n_i}\| + \lambda_{n_i} \frac{\|x_{n_i} - y_{n_i}\|^2 + \|z_{n_i}\|^2}{2\|u_{n_i}\|} \\ &\quad \to 0. \end{aligned} \tag{3.32}$$

This indicates that $\omega_w(x_{n_i+1}) \subset \Omega$. Without loss of generality, we assume that x_{n_i+1} converges weakly to $x^\dagger \in \Omega$. Now by (3.31), we infer that

$$\begin{aligned}\limsup_{n\to\infty}\delta_n &= \lim_{i\to\infty}\delta_{n_i}\\ &= \lim_{i\to\infty}[-\frac{1-\alpha_{n_i}}{\alpha_{n_i}}\lambda_{n_i}(\theta-\lambda_{n_i})\frac{(\|x_{n_i}-y_{n_i}\|^2+\|z_{n_i}\|^2)^2}{\|u_{n_i}\|^2}\\ &\quad + 2\langle u-z, x_{n_i+1}-z\rangle]\\ &\le 2\lim_{i\to\infty}\langle u-z, x_{n_i+1}-z\rangle\\ &= 2\langle u-z, x^\dagger - z\rangle\\ &\le 0,\end{aligned} \tag{3.33}$$

due to the fact that $z = P_\Omega u$ and (2.2). Finally, applying Lemma 2.5 to (3.29), we conclude that $x_n \to z$. This completes the proof. □

4. Numerical illustrations

In this section, some numerical results are presented. The MATLAB codes run in MATLAB version 9.5 (R2018b) on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30 GHz 2.40 GHz, RAM 8.00 GB. In all examples y-axes shows the value of x_n while the x-axis indicates to the number of iterations.

Example 4.1. Let $T_1 : R \to R$ be odd function and defined by

$$T_1(x) = \begin{cases} x, & x \in [0,1],\\ -4x+5, & \in (1,2),\\ 4x-11, & \in [2,3),\\ x-2, & \in [3,+\infty).\end{cases} \tag{4.1}$$

Let $S_1 : R \to R$ be odd function and defined by

$$S_1(x) = \begin{cases} x, & x \in [0,1],\\ -5x+6, & \in (1,2),\\ 5x-14, & \in [2,3),\\ x-2, & \in [3,+\infty).\end{cases} \tag{4.2}$$

Let $T_2 : R \to R$ and $S_2 : R \to R$ be defined by $T_2(x) = T_1(x+1) - 1$ and $S_2(x) = S_1(x+1) - 1$, respectively. It is obvious that $\mathrm{Fix}(T_1) = [-1,1]$, $\mathrm{Fix}(S_1) = [-1,1]$, $\mathrm{Fix}(T_2) = [-2,0]$ and $\mathrm{Fix}(S_2) = [-2,0]$. We can easily see that T_1, T_2, S_1 and S_2 are quasi-pseudocontractive operators but neither pseudocontractive nor quasi-nonexpansive operators. We also observe that T_1, T_2, S_1 and S_2 are Lipschitzian operators. The values of control parameters for Algorithm 3.1 and Algorithm 3.2 are $\eta_n = \zeta_n = 0.125$, $\lambda_n = 0.1$, $\alpha_n = \frac{1}{n}$, $x_1 = 3$ and $u = 3$.

The numerical results regarding Example 4.1 is reported in Figures 1 and 2.

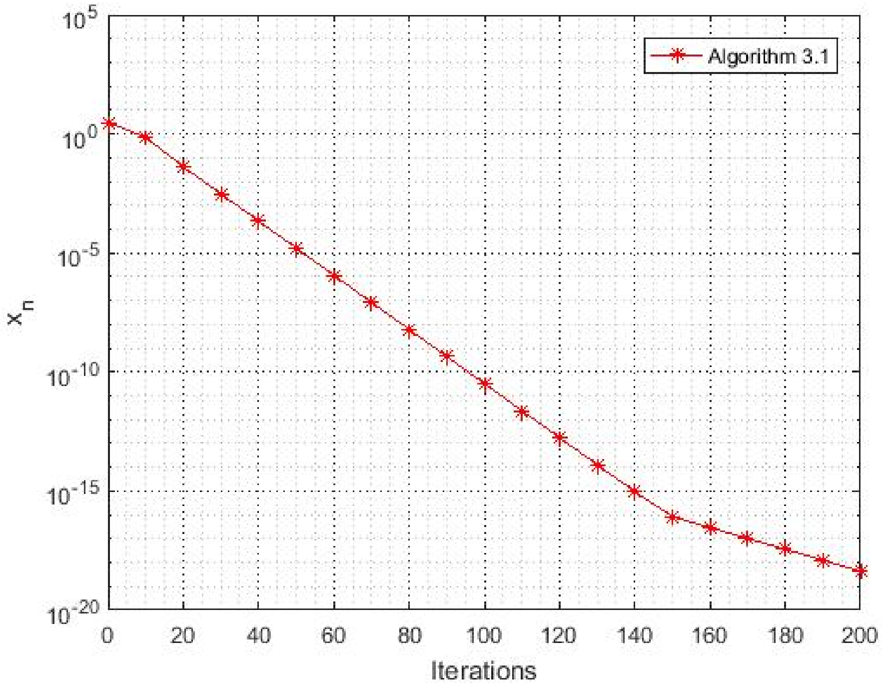

Figure 1. Example 4.1: Numerical behaviour of Algorithm 3.1.

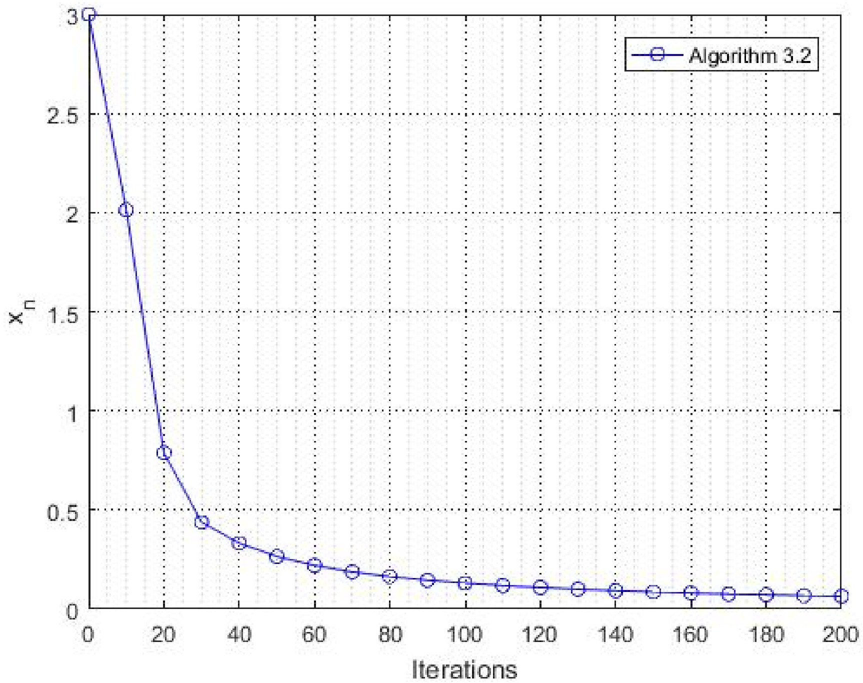

Figure 2. Example 4.1: Numerical behaviour of Algorithm 3.2.

Remark 4.1. In finite dimensional Hilbert spaces, weak convergence and strong convergence are equivalent. As can be seen from Figures 1 and 2, Algorithm 3.1 is more efficient than Algorithm 3.2 for finite dimensional Hilbert spaces. However, for infinite dimensional Hilbert spaces, in order to ensure strong convergence, generally speaking, we have to use Algorithm 3.2.

5. Conclusions

In this paper, we considered a class of the split common fixed point problem. we present two new self-adaptive algorithms for approximating a solution of the split common fixed point problem (1.3) for the class of quasi-pseudocontractive operators. Besides, weak and strong convergence theorems are established under some mild assumptions. Numerical findings have been documented to compare the numerical efficiency of Algorithms 3.1 and 3.2.

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Conflict of interest

The authors declare that they have no competing interests.

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