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Research article

Two new generalized iteration methods for solving absolute value equations using M-matrix

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Abstract: In this paper, we present two new generalized Gauss-Seidel iteration methods for solving absolute value equations $Ax - |x| = b$, where A is an M-matrix. Furthermore, we demonstrate their convergence under specific assumptions. Numerical tests indicate the efficiency of the suggested methods with suitable parameters.

Keywords: absolute value equations; convergence analysis; M-matrix; numerical tests
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1. Introduction

The standard absolute value equation (AVE) is in the form of

$$Ax - |x| = b, \tag{1.1}$$

where $A \in R^{n\times n}$ is an M-matrix, $|x|$ represents all the elements of the vector $x \in R^n$ by absolute value and $b \in R^n$. If "$|x|$" is replaced by "$B|x|$" in (1.1), then the general AVE is obtained, see [24, 30] . The AVE has received considerable attention recently, as it is suitable for a wide variety of optimization problems, e.g., linear programming, linear complementarity problems (LCP) and convex quadratic programming [1–7, 9–16, 23, 25, 26].

In recent years, a wide variety of procedures have been developed for solving AVE (1.1). For example, Wu and Li [34] presented a special shift splitting technique for determining the AVE (1.1) and performed a convergence analysis. Ke and Ma [19] established the SOR-like process to solve the

AVE (1.1). Chen et al. [8] modified the approach of [19] and analyzed the SOR-like approach using optimal parameters. Fakharzadeh and Shams [12] recommended the mixed-type splitting iterative scheme for determining (1.1) and established the convergence properties. Hu with Huang [17] have developed the AVE system as an LCP without any premise and demonstrated the existence and convexity properties. Caccetta et al. [7] studied a smoothing Newton procedure for solving (1.1) and established that the procedure is globally convergent when $\|A^{-1}\| < 1$. Ning and Zhou [40] evaluated improved adaptive differential evolution for AVEs; in this technique, they use local and global search. Salkuyeh [41] addressed the Picard HSS iteration approach and provided sufficient conditions for its convergence, while Edalatpour et al. [11] offered a generalization of the Gauss-Seidel (GGS) approach for AVE (1.1). Cruz et al. [39] utilized the inexact non-smooth Newton approach and designated global linear convergence of the approach. Moosaei et al. [22] proposed two techniques for determining AVE (1.1), namely, the Newton technique with the Armijo step and the Homotopy perturbation technique. For more details, see [18, 20, 27–29, 31–38, 43].

In this article, inspired by the work in [11], based on the GGS iteration method, the new generalized Gauss-Seidel (NGGS) iteration methods are presented to solve the AVE (1.1), and its convergence conditions are discussed in detail. By using some numerical tests, we demonstrate the efficacy of the newly developed methods.

The rest of the article is designed as follows: Section 2 discusses some preliminary information. Section 3 provides details of the proposed methodologies and its convergence conditions. Section 4 reports some tests to indicate the efficiency of the offered methods. Finally, section 5 draws some conclusions.

2. Preliminaries

Here, we will provide some notations, the description of an M-matrix, as well as some helpful lemmas for the later research.

Let $A = (a_{ij}) \in R^{n\times n}$, we represent the absolute value, tridiagonal and infinity norm of A as $|A| = (|a_{ij}|)$, $Trd(A)$ and $\| A \|_\infty$, respectively. The matrix $A \in R^{n\times n}$ is called an Z-matrix if $a_{ij} \le 0$ for $i \ne j$, and an M-matrix if it is a nonsingular Z-matrix and with $A^{-1} \ge 0$.

Lemma 2.1. [33] The matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be strictly diagonally dominant when

$$\sum_{nj=1, j\ne i}^{n} |a_{ij}| < |a_{ii}|, \quad i = 1, 2, \cdots, n.$$

Furthermore, if A is strictly diagonally dominant, then A is invertible.

Lemma 2.2. [33] Consider $z, x \in R^n$. Then $\| |z| - |x| \|_\infty \le \|z - x\|_\infty$.

3. Two NGGS methods

Here, we discuss the two NGGS methods: Method I represents the first method, while Method II represents the second method.

3.1. Method I for AVE

By revising the AVE (1.1)

$$Ax - |x| = b.$$

By multiplying λ on both sides, we obtain

$$\lambda Ax - \lambda|x| = \lambda b. \tag{3.1}$$

Let

$$A = D_A - L - U = (\bar{\Omega} + D_A - L) - (\bar{\Omega} + U) \tag{3.2}$$

where, D_A, L and U respectively, are the diagonal, the strictly lower and upper-triangular parts of A. Moreover, $\bar{\Omega} = \Psi(2 - \Psi)(I - D)^{-1}$, where $0 \leq \Psi \leq 2$ and I stands for the identity matrix. Using Eqs (3.1) and (3.2), the Method I is suggested as:

$$(\bar{\Omega} + D_A - \lambda L)x - \lambda|x| = [(1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U)]x + \lambda b. \tag{3.3}$$

Using the scheme, so Eq (3.3) can be written as

$$(\bar{\Omega} + D_A - \lambda L)x^{i+1} - \lambda|x^{i+1}| = [(1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U)]x^i + \lambda b. \tag{3.4}$$

Where $i = 0, 1, 2, ...$, and $0 < \lambda \leq 1$. Note that if $\lambda = 1$ and $\bar{\Omega} = 0$, then the Eq (3.4) is reduces to the GGS method [11].

In order to demonstrate the convergence of Method I, we prove the theorem listed below.

Theorem 3.1. Assume that the diagonal elements of matrix A are all greater than one, and the $D_A - L - I$ matrix is a strict row-wise diagonally dominant matrix. If

$$\|(\bar{\Omega} + D_A - \lambda L)^{-1}[(1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U)]\|_\infty < 1 - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty. \tag{3.5}$$

Then the sequence $\{x^i\}$ of Method I converges to the unique solution $x^\star$ of AVE (1.1).

Proof. We will show first $\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty < 1$. Clearly, if we put $L = 0$, then

$$\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty = \|(\bar{\Omega} + D_A)^{-1}\|_\infty < 1.$$

If we consider that $L \neq 0$, we get

$$0 \leq |\lambda L|t < (\bar{\Omega} + D_A - I)t,$$

if we take

$$|\lambda L|t < (\bar{\Omega} + D_A - I)t.$$

Taking both side by $(\bar{\Omega} + D_A)^{-1}$, we get

$$(\bar{\Omega} + D_A)^{-1}|\lambda L|t < (\bar{\Omega} + D_A)^{-1}((\bar{\Omega} + D_A) - I)t,$$

$$|\lambda(\bar{\Omega} + D_A)^{-1}L|t < (I - (\bar{\Omega} + D_A)^{-1})t,$$

$$|\lambda(\bar{\Omega} + D_A)^{-1}L|t < t - (\bar{\Omega} + D_A)^{-1}t,$$

$$(\bar{\Omega} + D_A)^{-1}t < t - |\lambda(\bar{\Omega} + D_A)^{-1}L|t,$$
$$(\bar{\Omega} + D_A)^{-1}t < (1 - |Q|)t, \tag{3.6}$$

where $Q = \lambda(\bar{\Omega} + D_A)^{-1}L$ and $t = (1, 1, ..., 1)^T$. Also, we have

$$0 \leq |(I - Q)^{-1}| = |I + Q + Q^2 + Q^3 + ... + Q^{n-1}|,$$
$$\leq (I + |Q| + |Q|^2 + |Q|^3 + ... + |Q|^{n-1}) = (I - |Q|)^{-1}. \tag{3.7}$$

Thus, from Eqs (3.6) and (3.7), we get

$$|(\bar{\Omega} + D_A - \lambda L)^{-1}|t = |(I - Q)^{-1}(\bar{\Omega} + D_A)^{-1}|t \leq |(I - Q)^{-1}||(\bar{\Omega} + D_A)^{-1}|t,$$
$$< (I - |Q|)^{-1}(I - |Q|)t = t.$$

So, we obtain

$$\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty < 1.$$

To show the uniqueness of the solution, let $x^\star$ and $z^\star$ be two not the same solutions of the AVE (1.1). Using Eq (3.4), we get

$$x^\star = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}|x^\star| + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))x^\star + \lambda b], \tag{3.8}$$

$$z^\star = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}|z^\star| + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))z^\star + \lambda b]. \tag{3.9}$$

From Eqs (3.8) and (3.9), we get

$$x^\star - z^\star = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}(|x^\star| - |z^\star|) + (D_A - \lambda L)^{-1}((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))(x^\star - z^\star).$$

Using Lemma 2.2 and Eq (3.5), the above equation can be written as

$$\|x^\star - z^\star\|_\infty \leq \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty\||x^\star| - |z^\star|\|_\infty$$
$$+\|(\bar{\Omega} + D_A - \lambda L)^{-1}((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))\|_\infty\|x^\star - z^\star\|_\infty,$$
$$< \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty\|x^\star - z^\star\|_\infty + (1 - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty)\|x^\star - z^\star\|_\infty,$$
$$\|x^\star - z^\star\|_\infty - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty\|x^\star - z^\star\|_\infty < (1 - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty)\|x^\star - z^\star\|_\infty,$$
$$(1 - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty)\|x^\star - z^\star\|_\infty < (1 - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty)\|x^\star - z^\star\|_\infty,$$
$$\|x^\star - z^\star\|_\infty < \|x^\star - z^\star\|_\infty.$$

The above results are contradictory. Finally, $x^\star = z^\star$.

In order to verify the convergence, let $x^\star$ is a unique solution of (1.1). So, from Eq (3.8) and

$$x^{i+1} = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}|x^{i+1}| + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))x^i + \lambda b],$$

we deduce

$$x^{i+1} - x^\star = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}(|x^{i+1}| - |x^\star|) + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))(x^i - x^\star)].$$

Based on Lemma 2.2 and the infinity norm, we get

$$\|x^{i+1} - x^\star\|_\infty - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty\|x^{i+1} - x^\star\|_\infty$$
$$\leq \|(\bar{\Omega} + D_A - \lambda L)^{-1}((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))\|_\infty\|x^i - x^\star\|_\infty,$$

and since $\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty < 1$ it follows that

$$\|x^{i+1} - x^\star\|_\infty \leq \frac{\|(\bar{\Omega} + D_A - \lambda L)^{-1}((1-\lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))\|_\infty}{1 - \lambda\|(\bar{\Omega} + D_A - \lambda L)^{-1}\|_\infty}\|x^i - x^\star\|_\infty.$$

According to this inequality, the convergence of Method I is possible when condition Eq (3.5) is fulfilled.

3.2. Method II for AVE

Here, we outline Method II of the NGGS method. By using Eqs (3.1) and (3.2), we can formulate Method II to determine AVE (1.1) as follows:

$$(\bar{\Omega} + D_A - \lambda L)x^{i+1} - \lambda|x^{i+1}| = [(1-\lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U)]x^{i+1} + \lambda b, \quad i = 0, 1, 2, ...$$

In order to demonstrate the convergence of Method II, we prove the theorem listed below.

Theorem 3.2. Assume that the diagonal elements of matrix A are all greater than one, and the $D_A - L - I$ matrix is a strict row-wise diagonally dominant matrix. Then the sequence $\{x^i\}$ of Method II converges to the unique solution $x^\star$ of AVE (1.1).

Proof. The uniqueness result follows from Theorem 3.1. To demonstrate the convergence, suppose

$$\begin{aligned} x^{i+1} - x^\star = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}|x^{i+1}| + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1-\lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))x^{i+1} + \lambda b] \\ -(\lambda(\bar{\Omega} + D_A - \lambda L)^{-1}|x^\star| + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1-\lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))x^\star + \lambda b]), \end{aligned}$$

$$(\bar{\Omega} + D_A - \lambda L)(x^{i+1} - x^\star) = \lambda(|x^{i+1}| - |x^\star|) + ((1-\lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))(x^{i+1} - x^\star),$$

$$\lambda(D_A - L - U)x^{i+1} - \lambda|x^{i+1}| = \lambda(D_A - L - U)x^\star - \lambda|x^\star|,$$

$$(D_A - L - U)x^{i+1} - |x^{i+1}| = (D_A - L - U)x^\star - |x^\star|. \tag{3.10}$$

By using Eqs (3.2) and (3.10), we get

$$Ax^{i+1} - |x^{i+1}| = Ax^\star - |x^\star|,$$

$$Ax^{i+1} - |x^{i+1}| = b.$$

Therefore, x^{i+1} solves the AVE (1.1).

4. Numerical tests

The purpose of this section is to present a number of numerical tests that demonstrate the effectiveness of new approaches from three perspectives: The iteration steps (Itr), computing time (Time), and norm of absolute residual vectors (RVS). Where, RVS is defined by

$$RVS := \frac{\|Ax^i - |x^i| - b\|_2}{\|b\|_2} \leq 10^{-6}.$$

All calculations are run on Intel (C) Core (TM) i5-3337U, 4 GB RAM, 1.80 GHz, and MATLAB (2016a). Furthermore, the zero vector is the initial vector for Example 4.1.

Problem 4.1. Let

$$A = \begin{cases} 4, & \text{for } j = i, \\ -1, & \text{for } \begin{cases} j = i+1, & i = 1, 2, ..., n-1, \\ j = i-1, & i = 2, ..., n, \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $b = Ax^\star - |x^\star|$, where $x^\star = ((-1)^i, (i = 1, 2, .., n))^T \in R^n$. We describe the suggested methods in comparison with the optimal parameters SOR-like algorithm given in [8] (written as SLM using $\omega = 0.825$), the special shift splitting algorithm presented in [34] (written as SSM), and the GGS technique shown in [11]. In Table 1, we examine the results.

Table 1. The outcomes of Problem 4.1 with $\Psi = 0.5$ and $\lambda = 0.95$.

Methods	n	1000	2000	3000	4000
SLM	Itr	18	18	18	18
	Time	3.0156	13.1249	33.9104	65.1345
	RVS	6.12e–07	6.13e–07	6.13e–07	6.14e–07
SSM	Itr	14	14	14	14
	Time	2.8128	9.0954	17.3028	29.1644
	RVS	8.91e–07	8.92e–07	8.93e–07	8.93e–07
GGS	Itr	9	8	8	8
	Time	2.1924	7.5182	15.3273	24.3822
	RVS	4.02e–07	7.78e–07	6.35e–07	5.49e–07
Method I	Itr	8	8	8	8
	Time	2.0704	5.2615	9.3395	17.6224
	RVS	9.14e–07	6.46e–07	5.28e–07	4.57e–07
Method II	Itr	6	6	6	6
	Time	1.5811	2.9428	3.5738	6.3929
	RVS	9.60e–07	9.61e–07	9.61e–07	9.61e–07

All methods in Table 1 analyze the solution $x^\star$ for various values of n, respectively. Clearly, Method I is more effective than SLM and SSM procedures, and the 'Time' of Method I is less than the GGS. Moreover, Method II demonstrates high computational performance from the perspective of 'Itr' and 'Time'.

Problem 4.2. Let $A = M + I \in R^{n\times n}$ and the vector $b = Ax^\star - |x^\star| \in R^n$, such that

$$M = Trd(-1.5I_n, H_n, -0.5I_n) \in R^{n\times n}, \quad x^\star = (1, 2, 1, 2, ..., 1, 2)^T \in R^n,$$

where $H_n = Trd(-1.5, 4, -0.5) \in R^{v\times v}$ and $n = v^2$. Here, use the same initial vector and stopping criteria described in [12]. We compare the presented techniques with the AOR method [21], the mixed-type splitting (MT) iterative scheme [12] and the GGS method [11]. Table 2 provides the numerical data.

In Table 2, we present the numeric outcomes of the AOR method, MT method, GGS method, Method I and Method II, respectively. We can conclude from these outcomes that our proposed methods are more efficient than AOR and MT and GGS techniques.

Problem 4.3. Let $A = M + 4I \in R^{n\times n}$ and the vector $b = Ax^\star - |x^\star| \in R^n$, such that

$$M = Trd(-I_n, H_n, -I_n) \in R^{n\times n}, \quad x_i^\star = ((-1)^i, (i = 1, 2, .., n))^T \in R^n,$$

where $H_n = Trd(-1, 4, -1) \in R^{v\times v}$, $I \in R^{v\times v}$ is the unit matrix and $n = v^2$. In this problem, we use the same initial vector and stopping criteria described in [12]. We compare the offered procedures with the AOR method [21], the mixed-type splitting (MT) iterative scheme [12], and the technique presented in [14] (expressed by SISA). The computational outcomes are listed in Table 3.

All methods in Table 3 analyze the solution $x^\star$ for various values of n, respectively. Clearly, Method I is more effective than AOR and MT procedures, and the 'Time' of Method I is less than the SISA method. Moreover, Method II demonstrates high computational performance from the perspective of 'Itr' and 'Time'.

Problem 4.4. Let

$$A = Trd(-1, 8, -1) \in R^{n\times n}, \quad x_i^\star = ((-1)^i, \quad (i = 1, 2, ..., n))^T \in R^n$$

and $b = Ax^\star - |x^\star| \in R^n$. Using the same initial vector and the stopping criteria described in [14]. We compare the novel approaches with the technique offerd in [14] (expressed by SISA using $\omega = 1.0455$), the SOR-like method proposed in [19] (written by SOR) and the modulus-based SOR method presented in [42] (written as MSOR). The outcomes are listed in Table 4.

It is clear from Table 4 that all the tested methods provide a quick calculation of AVE (1.1). We observe that the 'Itr' and 'Time' of the recommended methods are less than the existing techniques. The results of our study indicate that our suggested methods for AVEs are feasible and highly effective.

Table 2. The outcomes of Problem 4.2 with $\Psi = 0.7$ and $\lambda = 0.98$.

Methods	n	100	400	900	1600	4900
	Itr	97	190	336	706	384
AOR	Time	0.4721	2.8203	3.2174	6.3887	9.2344
	RVS	9.80e-07	9.61e-07	9.73e-07	9.84e-07	9.36e-07
	Itr	88	157	250	386	342
MT	Time	0.4041	1.7953	3.0219	5.7626	8.8965
	RVS	8.91e-07	9.65e-07	9.18e-07	9.56e-07	9.89e-07
	Itr	34	52	67	81	95
GGS	Time	0.2207	0.5346	1.0472	1.7328	2.7612
	RVS	9.53e-07	8.40e-07	8.42e-07	8.35e-07	9.89e-07
	Itr	32	49	63	76	84
Method I	Time	0.1971	0.3177	0.9243	1.3922	1.9920
	RVS	9.59e-07	8.20e-07	8.40e-07	8.60e-07	7.42e-07
	Itr	20	31	41	49	62
Method II	Time	0.1329	0.1936	0.8341	1.0271	1.3872
	RVS	9.55e-07	8.39e-07	6.22e-07	9.32e-07	8.90e-07

Table 3. The outcomes of Problem 4.3 with $\Psi = 0.7$ and $\lambda = 0.98$.

Methods	n	64	256	1024	4096
AOR	Itr	14	14	15	35
	Time	0.3483	1.9788	2.3871	5.8097
	RVS	5.21e-07	6.29e-07	6.54e-07	8.74e-07
MT	Itr	14	14	15	25
	Time	0.3168	1.0952	1.9647	2.2194
	RVS	4.31e-07	5.46e-07	5.06e-07	9.38e-07
SISA	Itr	12	12	12	12
	Time	0.3299	1.8322	2.027	3.446
	RVS	5.03e–07	7.58e–07	8.77e–07	9.28e–07
Method I	Itr	11	12	12	12
	Time	0.1928	0.8374	1.5738	2.0733
	RVS	7.30e-07	6.01e-07	7.93e-07	8.88e-07
Method II	Itr	5	5	5	5
	Time	0.1372	0.3871	0.9622	1.7482
	RVS	6.51e-08	6.52e-08	6.23e-08	5.99e-08

Table 4. The outcomes of Problem 4.4 with $\Psi = 0.3$ and $\lambda = 0.95$.

Methods	*n*	1000	2000	3000	4000	5000
SISA	Itr	13	13	14	14	14
	Time	3.9928	8.8680	24.4031	51.3946	73.3394
	RVS	6.04e–07	8.54e–07	2.33e–07	2.69e–07	3.01e-07
SOR	Itr	12	13	13	13	13
	Time	1.5136	3.3817	6.1262	7.1715	9.5261
	RVS	9.45e–08	2.69e–08	3.29e–08	3.80e–08	4.25e–07
MSOR	Itr	10	10	10	10	10
	Time	3.9996	9.2833	29.3747	59.3392	82.3477
	RVS	4.14e–07	5.86e–07	7.18e–07	8.29e–07	9.27e–07
Method I	Itr	9	9	9	9	9
	Time	1.2751	2.6184	5.7322	6.8911	7.3618
	RVS	5.11e–07	5.12e–07	5.12e–07	5.12e–07	5.78e–07
Method II	Itr	6	6	6	6	6
	Time	0.2283	0.4829	0.9572	1.4829	2.0038
	RVS	3.62e–08	5.12e–08	6.27e–08	7.24e–08	8.01e–08

5. Conclusions

In this work, two NGGS methods (Method I and Method II) are presented to solve the AVEs. The convergence properties of the strategies are examined. A number of experiments have been conducted in order to establish the effectiveness of the new approaches.

The GGS technique has been successfully extended by two additional parameters when A is an M-matrix. The cases for more general coefficient matrices are the next issue to be considered.

Appendix

The following is an explanation of how our proposed techniques can be implemented. From $Ax - |x| = b$, we have

$$x = A^{-1}(|x| + b).$$

Thus, we can approximate x^{i+1} as follows,

$$x^{i+1} \approx A^{-1}(|x^i| + b).$$

This process is known as the Picard technique [31]. Now, we examine the procedure for Method I.

Algorithm for Method I. (1) Choose the parameters, an starting vector $x^0 \in R^n$ and set $i = 0$.
(2) Compute $y^i = x^{i+1} \approx A^{-1}(|x^i| + b)$,
(3) Calculate $x^{i+1} = \lambda(\bar{\Omega} + D_A - \lambda L)^{-1}|y^i| + (\bar{\Omega} + D_A - \lambda L)^{-1}[((1 - \lambda)(\bar{\Omega} + D_A) + \lambda(\bar{\Omega} + U))x^i + \lambda b]$.
(4) If $x^{i+1} = x^i$, then stop. Else, apply $i = i + 1$ and repeat step 2.
For Method II, follow the same steps.

Conflict of interest

The authors declare there is no conflicts of interest.

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