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Qualitative analysis of coupled system of sequential fractional integrodifferential equations

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Abstract: In this article we explore the existence, uniqueness, and stability for a coupled system of sequential fractional integrodifferential equations involving ψ–Hilfer fractional derivative with multi–point boundary conditions. For the uniqueness result we use Banach fixed point theorem, and the Leray–Schauder alternative to obtain the existence result. Further, we investigate various kinds of stability such as Hyers–Ulam stability. Examples are provided to verify our results.

Keywords: ψ–Hilfer fractional derivative; system of integrodifferential equations; existence and uniqueness; stability; controllability; fixed point theorem
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1. Introduction

Fractional calculus is an intransitive branch of applied mathematical techniques that deals with integrals and derivatives of essentially arbitrary orders. Recently, it gained considerable importance and admiration due to its widespread applications in viscoelasticity, biology, fluid dynamics, hydrodynamics, chemistry, control hypotheses, speculation, aerodynamics, information processing system, image processing, etc. [9, 10, 12]. A significant feature of fractional order systems in differentiation with integer order ones is that fractional derivatives (FDs) and integrals have nonlocal nature that helps to trace the hereditary and memory characteristics of the related materials and processes under investigation [2, 11, 16, 19].

Often, it is quite tough to obtain the appropriate solutions of the fractional differential equations (FDEs). Due to this problem, the qualitative presumptions of differential equations (DEs) play a significant role both in ordinary differential equations (ODEs) and FDEs. For boundary value problems (BVPs) of FDEs, the existence of solutions is a main requirement. Moreover, uniqueness of solutions is a significant factor for the more particular action of solutions. From the last few years, these qualitative properties are investigated with different approaches. The qualitative analysis of DEs represents the behavior of solutions of complicated phenomena. In addition, stability analysis of dynamical system of an integer as well as fractional order is very important in various fields of science and engineering. The concept of Hyers–Ulam (HU)–type stability, that began with the seminar work, from 1941 has gained a lot of attention. As a matter of fact, HU–type stability has been taken up by a number of mathematicians and the study of this area has grown to be one of the central subjects in the mathematical analysis. Stability analysis, particularly stability in terms of Ulam and Rassias, is an essential component of the qualitative theory of DEs, as shown by the prior results [8, 22]. This type of stability can be treated with different approaches [15, 17, 18, 23–26].

One of the important approaches is the fixed point (FP) approach. FP theory is an important tool in nonlinear analysis. Particularly, obtaining the existence results for a variety of mathematical problems. Although there are many methods to analyze, under suitable conditions, the existence and uniqueness (EU) of solution of numerous problems with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions and periodic boundary conditions for FDEs [3, 4, 27, 28]. Applications of FP theory in terms of stability analysis of DEs can be found in [14]. There are various definitions of FDs the most popular of Caputo and Riemann–Liouville (RL). A generalization of both Caputo and RL was given by Hilfer [6], known as the Hilfer FD. Some applications and properties of Hilfer derivative can be found in [7]. The FD in the sense of Hilfer, called ψ–Hilfer FD, has been introduced in [21], which unify various fractional operators.

In [20], the authors used the fixed point approach to study the stability of the modified impulsive FED:

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\beta,\rho;\psi}\varpi(\zeta) = f(\zeta, \varpi(\zeta)), \ \zeta \in (s_i, \zeta_{i+1}]; \ i = 0, 1, \dots, m, \\ \\ \varpi(\zeta) = g(\zeta, \varpi(\zeta_i^+)), \ \zeta \in (\zeta_i, s_i]; \ i = 1, 2, \dots, m, \end{cases}$$

where ${}^{H}\mathcal{D}_{0^+}^{\beta,\rho;\psi}(\cdot)$ is the ψ–Hilfer FD with $0 < \beta \leq 1, \ 0 \leq \rho \leq 1$ and $0 = \zeta_0 = s_0 < \zeta_1 \leq s_1 \leq \zeta_2 < \cdots < \zeta_m \leq s_m < \zeta_{m+1} = \mathrm{T}$ are prefixed numbers, $f : [0, \mathrm{T}] \times \Re \to \Re$ is continuous function and $g_i : [\zeta_i, s_i] \times \Re \to \Re$ is continuous for all $i = 1, 2, \dots, m$ which is not an instantaneous impulses.

Abbas *et al.* [1], investigated the existence and attractively of solution for the following problem:

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\beta,\rho;\psi}\frac{\varpi(\zeta)}{g(\zeta,\varpi(\zeta))} = f(\zeta, \varpi(\zeta)), \ \forall\, \zeta \in \Re_+, \\ \\ (\psi(\zeta) - \psi(0))^{1-\varsigma}\,|_{\zeta=0} = \varpi_0; \ \varpi_0 \in \Re, \end{cases}$$

where $\Re_+ := [0, +\infty), \ 0 < \beta < 1, \ 0 \leq \rho \leq 1, \ \varsigma = \beta + \rho(1 - \beta), \ {}^{H}\mathcal{D}_{0^+}^{\beta,\rho;\psi}$ is the ψ–Hilfer FD of order β and type $\rho, \ g : \Re_+ \times \Re \to \Re^*$ and $f : \Re_+ \times \Re \to \Re$ are continuous functions.

Zhou *et al.* [29], explored the existence and stability of solution to the following nonlinear ψ–Hilfer fractional integrodifferential equation

$$\begin{cases} {}^{H}\mathcal{D}_{a^+}^{\beta,\rho;\psi}\varpi(\zeta) = f(\zeta, \varpi(\zeta), {}^{H}\mathcal{D}_{a^+}^{p,q;\psi}\varpi(\zeta)) + \int_a^\zeta K(\zeta, \tau, \varpi(\tau), \varpi(\delta(\tau)))d\tau, \ \forall\, \zeta \in [a, +\infty), \\ \\ I_{a^+}^{1-\gamma}\varpi(a) = 0, \end{cases}$$

where the ψ–Hilfer FD ${}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi}(\cdot)$, ${}^H\mathcal{D}_{a^+}^{p,q;\psi}(\cdot)$ of order $0 < \beta,\ p < 1$ $(p < \beta)$ with type $0 \leq \rho,\ q \leq 1$ and ψ–Riemann–Liouville fractional integral $I_{a^+}^{1-\gamma}(\cdot)$ of order $1-\gamma,\ \gamma = \beta + \rho(1-\beta)$.

Ntouyas and Vivek [13], explored the EU of solution for a new class of multi–point BVP:

$$\begin{cases} \left({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k\,{}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi}\right)\varpi(\zeta) = f(\zeta,\varpi(\zeta)),\ \forall\ \zeta \in [a,b], \\ \varpi(a) = 0,\ \varpi(b) = \sum_{i=1}^{m} \lambda_i \varpi(\theta_i), \end{cases} \tag{1.1}$$

where ${}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi}$ is the ψ–Hilfer FD of order $\beta,\ 1 < \beta \leq 2$ and parameter $\rho,\ 0 \leq \rho \leq 1,\ f \in C([a,b],\Re)$, $0 \leq a < b,\ k,\ \lambda_i \in \Re,\ i = 1,2,\ldots,m$ and $a < \theta_1 < \theta_2 < \cdots < \theta_m < b$. The classical FP theorem was used to show the EU results. The uniqueness result is achieved using Banach's FP theorem, while the existence result were established using nonlinear alternative of Leray–Schauder type and Krasnoselskii's FP theorem for the problem (1.1).

In this study we explore the possibility of such a thing i.e, existence, uniqueness and HU stability criteria for the solution of the nonlocal coupled system of sequential ψ–Hilfer FD of the form:

$$\begin{cases} \left({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k\,{}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi}\right)\varpi(\zeta) = f(\zeta,\varpi(\zeta), I_{a^+}^{\beta;\psi}\varpi(\zeta),\omega(\zeta)),\ \forall\ \zeta \in [a,b], \\ \left({}^H\mathcal{D}_{a^+}^{p,q;\psi} + \nu\,{}^H\mathcal{D}_{a^+}^{p-1,q;\psi}\right)\omega(\zeta) = g(\zeta,\varpi(\zeta),\omega(\zeta), I_{a^+}^{p;\psi}\omega(\zeta)),\ \forall\ \zeta \in [a,b], \\ \varpi(a) = 0,\ \varpi(b) = \sum_{i=1}^{m-2} \lambda_i \omega(\theta_i), \\ \omega(a) = 0,\ \omega(b) = \sum_{j=1}^{n-2} \mu_j \varpi(\xi_j), \end{cases} \tag{1.2}$$

where ${}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi}$, ${}^H\mathcal{D}_{a^+}^{p,q;\psi}$ are the ψ–Hilfer FD of orders β and p, $1 < \beta,\ p \leq 2$ and two parameters $\rho,\ q,\ 0 \leq \rho,\ q \leq 1$, given constants $k,\ \nu,\ \lambda_i,\ \mu_j \in \Re,\ a \geq 0$, the points $a < \theta_1 < \theta_2 < \cdots < \theta_{m-2} < b$, $a < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < b$ and $f,\ g : [a,b] \times \Re \times \Re \times \Re \to \Re$ are continuous functions.

To analyze problem (1.2), we transform it to an analogous FP problem and establish the uniqueness of its solutions using Banach's FP theorem, while obtaining the existence result using the Leray–Schauder alternative [5].

2. Preliminaries

Let $\psi \in C^1([a,b],\Re)$ be an increasing function with $\psi'(\zeta) \neq 0$ for all $\zeta \in [a,b]$.

Definition 2.1. ([9]) Let $\beta > 0$ and $g \in L^1([a,b],\Re)$. The ψ–RL fractional integral of order β to a function g with respect to ψ is defined by

$$I_{a^+}^{\beta;\psi} g(\zeta) = \frac{1}{\Gamma(\beta)} \int_a^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\beta-1} g(s)ds.$$

Definition 2.2. ([21]) Let $n-1 < \beta < n,\ n \in \mathbb{N}$ and $g,\ \psi \in C^n([a,b],\Re)$ such that ψ is an increasing function with $\psi'(\zeta) \neq 0$ for all $\zeta \in [a,b]$. The ψ–Hilfer FD ${}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi}(\cdot)$ of order β to a function g and type $0 \leq \rho \leq 1$, is defined by

$$ {}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} g(\zeta) = I_{a^+}^{\rho(n-\beta);\psi}\left(\frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}\right)^n I_{a^+}^{(1-\rho)(n-\beta);\psi} g(\zeta).$$

Remark 2.1. ([9]) If $\rho = 0$, then we have ψ–RL FD as

$$ {}^{H}\mathcal{D}^{\beta,0;\psi}_{a^+} g(\zeta) := {}^{RL}\mathcal{D}^{\beta;\psi}_{a^+} g(\zeta) = \left(\frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}\right)^n I^{(n-\beta);\psi}_{a^+} g(\zeta) $$

and if $\rho = 1$, we obtain ψ–Caputo FD by

$$ {}^{H}\mathcal{D}^{\beta,1;\psi}_{a^+} g(\zeta) := {}^{C}\mathcal{D}^{\beta;\psi}_{a^+} g(\zeta) = I^{(n-\beta);\psi}_{a^+}\left(\frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}\right)^n g(\zeta). $$

Lemma 2.1. *([14, 21]) Let $\beta,\ \chi > 0$ and $\varrho > 0$ be constants and $\psi \in C^1([a,b],\Re)$ be an increasing function with $\psi'(\zeta) \neq 0$ for all $\zeta \in [a,b]$. Then we have*

1. $I^{\beta;\psi}_{a^+} I^{\chi;\psi}_{a^+} h(\zeta) = I^{\varrho+\chi;\psi}_{a^+} h(\zeta)$,
2. $I^{\beta;\psi}_{a^+}(\psi(\zeta) - \psi(a))^{\varrho-1} = \frac{\Gamma(\varrho)}{\Gamma(\beta+\varrho)}(\psi(\zeta) - \psi(a))^{\beta+\varrho-1}$.

The following lemma contains the compositional property of ψ–RL fractional integral operator with the ψ–Hilfer FD operator.

Lemma 2.2. *([21]) Let $f \in L(a,b)$, $n-1 < \beta \leq n$, $n \in \mathbb{N}$, $0 \leq \rho \leq 1$, $\gamma^* = \beta + n\rho - \beta\rho$, $I^{(n-\beta)(1-\rho);\psi}_{a^+} f \in AC^k[a,b]$. Then*

$$ I^{\beta;\psi}_{a^+} {}^{H}\mathcal{D}^{\beta,\rho;\psi}_{a^+} f(\zeta) = f(\zeta) - \sum_{k=1}^{n} \frac{(\psi(\zeta) - \psi(a))^{\gamma^*-k}}{\Gamma(\gamma^* - k + 1)} f^{[n-k]}_{\psi} I^{(n-\beta)(1-\rho);\psi}_{a^+} f(a), $$

where $f^{[n-k]}_{\psi} = \left(\frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}\right)^{n-k}$.

Lemma 2.3. *Let $\gamma = \beta + 2\rho - \beta\rho$, $\delta = p + 2q - pq$, and $h,\ z \in C([a,b],\Re)$ be given functions. Then the function $\varpi,\ \omega \in C([a,b],\Re)$ is a solution of the boundary value problem*

$$ \begin{cases} \left({}^{H}\mathcal{D}^{\beta,\rho;\psi}_{a^+} + k\ {}^{H}\mathcal{D}^{\beta-1,\rho;\psi}_{a^+}\right)\varpi(\zeta) = h(\zeta),\ \forall\ \zeta \in [a,b],\ 1 < \beta,\ p \leq 2, \\ \left({}^{H}\mathcal{D}^{p,q;\psi}_{a^+} + \nu\ {}^{H}\mathcal{D}^{p-1,q;\psi}_{a^+}\right)\omega(\zeta) = z(\zeta),\ \forall\ \zeta \in [a,b],\ 0 \leq \rho,\ q \leq 1, \\ \varpi(a) = 0,\ \varpi(b) = \sum_{i=1}^{m-2} \lambda_i \omega(\theta_i), \\ \omega(a) = 0,\ \omega(b) = \sum_{j=1}^{n-2} \mu_j \varpi(\xi_j), \end{cases} \tag{2.1} $$

if and only if

$$ \begin{aligned} \varpi(\zeta) &= I^{\beta;\psi}_{a^+} h(\zeta) - kI^{1;\psi}_{a^+}\varpi(\zeta) + \frac{(\psi(\zeta) - \psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg\{\Delta\Bigg[-\nu\sum_{i=1}^{m-2}\lambda_i I^{1;\psi}_{a^+}\omega(\theta_i) \\ &\quad + \sum_{i=1}^{m-2}\lambda_i I^{p;\psi}_{a^+} z(\theta_i) + kI^{1;\psi}_{a^+}\varpi(b) - I^{\beta;\psi}_{a^+} h(b)\Bigg] \end{aligned} \tag{2.2} $$

$$+B\Big[-k\sum_{j=1}^{n-2}\mu_j I_{a^+}^{1;\psi}\varpi(\xi_j)+\sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi}h(\xi_j)+\nu I_{a^+}^{1;\psi}\omega(b)-I_{a^+}^{p;\psi}z(b)\Big]\Big\},$$

and

$$\begin{aligned}\omega(\zeta) &= I_{a^+}^{p;\psi}z(\zeta)-\nu I_{a^+}^{1;\psi}\omega(\zeta)+\frac{(\psi(\zeta)-\psi(a))^{\delta-1}}{\Lambda\Gamma(\delta)}\Big\{A\Big[-k\sum_{j=1}^{n-2}\mu_j I_{a^+}^{1;\psi}\varpi(\xi_j)\\ &\quad+\sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi}h(\xi_i)+\nu I_{a^+}^{1;\psi}\omega(b)-I_{a^+}^{p;\psi}z(b)\Big]\\ &\quad+\Omega\Big[-\nu\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{1;\psi}\omega(\theta_i)+\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{p;\psi}z(\theta_i)+kI_{a^+}^{1;\psi}\varpi(b)-I_{a^+}^{\beta;\psi}h(b)\Big]\Big\},\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}A &= \frac{(\psi(b)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)},\ \ B=\sum_{i=1}^{m-2}\lambda_i\frac{(\psi(\theta_i)-\psi(a))^{\delta-1}}{\Gamma(\delta)},\\ \Omega &= \sum_{j=1}^{n-2}\mu_j\frac{(\psi(\xi_j)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)},\ \ \Delta=\frac{(\psi(b)-\psi(a))^{\delta-1}}{\Gamma(\delta)},\end{aligned}$$

and it is assumed that

$$\Lambda := A\Delta-B\Omega\neq 0. \tag{2.4}$$

Proof. Assume that ϖ is a solution of the nonlocal BVP (2.1) on $[a,b]$. Operating fractional integral on both sides of first equation $I_{a^+}^{\beta;\psi}$ and using Lemma 2.2, we obtain for $\zeta\in[a,b]$,

$$\varpi(\zeta)-\sum_{k=1}^{2}\frac{(\psi(\zeta)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}\varpi_{\psi}^{[2-k]}I_{a^+}^{(1-\rho)(2-\beta);\psi}\varpi(a)+kI_{a^+}^{1;\psi}\varpi(\zeta)=I_{a^+}^{\beta;\psi}h(\zeta).$$

Hence, using the fact that $(1-\rho)(2-\beta)=2-\gamma$, we have

$$\begin{aligned}\varpi(\zeta) &= \frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\Big(\frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}I_{a^+}^{2-\gamma;\psi}\varpi\Big)(a)\\ &\quad+\frac{(\psi(\zeta)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}I_{a^+}^{2-\gamma;\psi}\varpi(a)-kI_{a^+}^{1;\psi}\varpi(\zeta)+I_{a^+}^{\beta;\psi}h(\zeta)\\ &= \frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}{}^{H}\mathcal{D}_{a^+}^{\gamma-1,\rho;\psi}\varpi(a)\\ &\quad+\frac{(\psi(\zeta)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}I_{a^+}^{2-\gamma;\psi}\varpi(a)-kI_{a^+}^{1;\psi}\varpi(\zeta)+I_{a^+}^{\beta;\psi}h(\zeta)\\ &= c_1\frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+c_2\frac{(\psi(\zeta)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}-kI_{a^+}^{1;\psi}\varpi(\zeta)+I_{a^+}^{\beta;\psi}h(\zeta),\end{aligned}$$

where $c_1={}^{H}\mathcal{D}_{a^+}^{\gamma-1,\rho;\psi}\varpi(a)$ and $c_2=I_{a^+}^{2-\gamma;\psi}\varpi(a)$.

From the first boundary condition $\varpi(a) = 0$ we can obtain $c_2 = 0$, since $\lim_{\zeta\to a}(\zeta - a)^{\gamma-2} = \infty$. Then we get

$$\varpi(\zeta) = c_1\frac{(\psi(\zeta) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} - kI^{1;\psi}_{a^+}\varpi(\zeta) + I^{\beta;\psi}_{a^+}h(\zeta),\ \forall\ \zeta \in [a, b]. \tag{2.5}$$

By a similar way we obtain

$$\omega(\zeta) = d_1\frac{(\psi(\zeta) - \psi(a))^{\delta-1}}{\Gamma(\delta)} - \nu I^{1;\psi}_{a^+}\omega(\zeta) + I^{p;\psi}_{a^+}z(\zeta),\ \forall\ \zeta \in [a, b]. \tag{2.6}$$

From the second boundary conditions $\varpi(b) = \sum_{i=1}^{m-2}\lambda_i\omega(\theta_i)$ and $\omega(b) = \sum_{j=1}^{n-2}\mu_j\varpi(\xi_j)$, we get the system

$$\begin{cases} Ac_1 - Bd_1 &= P, \\ -\Omega c_1 + \Delta d_1 &= Q, \end{cases} \tag{2.7}$$

where

$$P = -\nu\sum_{i=1}^{m-2}\lambda_i I^{1;\psi}_{a^+}\omega(\theta_i) + \sum_{i=1}^{m-2}\lambda_i I^{p;\psi}_{a^+}z(\theta_i) + kI^{1;\psi}_{a^+}\varpi(b) - I^{\beta;\psi}_{a^+}h(b)$$

and

$$Q = -k\sum_{j=1}^{n-2}\mu_j I^{1;\psi}_{a^+}\varpi(\xi_j) + \sum_{j=1}^{n-2}\mu_j I^{\beta;\psi}_{a^+}h(\xi_j) + \nu I^{1;\psi}_{a^+}\omega(b) - I^{p;\psi}_{a^+}z(b).$$

Solving the system (2.7), we find that

$$\begin{aligned} c_1 &= \frac{1}{\Lambda}(\Delta P + BQ), \\ d_1 &= \frac{1}{\Lambda}(AQ + \Omega P). \end{aligned}$$

Substituting the value of c_1, d_1 in (2.5) and (2.6) yields the solution (2.2) and (2.3).

Conversely, suppose that ϖ is the solution of the fractional integral (2.2). Operating fractional derivative ${}^H\mathcal{D}^{\beta,\rho;\psi}_{a^+}$ on both sides of equation (2.2) and using Lemma 2.1, we obtain

$$\begin{aligned} {}^H\mathcal{D}^{\beta,\rho;\psi}_{a^+}\varpi(\zeta) &= h(\zeta) - ({}^H\mathcal{D}^{\beta,\rho;\psi}_{a^+}k)I^{1;\psi}_{a^+}\varpi(\zeta) + \frac{1}{\Lambda}\Big\{\Delta\Big[-\nu\sum_{i=1}^{m-2}\lambda_i I^{1;\psi}_{a^+}\omega(\theta_i) \\ &\quad + \sum_{i=1}^{m-2}\lambda_i I^{p;\psi}_{a^+}z(\theta_i) + kI^{1;\psi}_{a^+}\varpi(b) - I^{\beta;\psi}_{a^+}h(b)\Big] \\ &\quad + B\Big[-k\sum_{j=1}^{n-2}\mu_j I^{1;\psi}_{a^+}\varpi(\xi_j) + \sum_{j=1}^{n-2}\mu_j I^{\beta;\psi}_{a^+}h(\xi_j) + \nu I^{1;\psi}_{a^+}\omega(b) \\ &\quad - I^{p;\psi}_{a^+}z(b)\Big]\Big\}{}^H\mathcal{D}^{\beta,\rho;\psi}_{a^+}\frac{(\psi(\zeta) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\ &= h(\zeta) - k\,{}^H\mathcal{D}^{\beta-1,\rho;\psi}_{a^+}\varpi(\zeta). \end{aligned}$$

Now we prove that ϖ satisfies the boundary condition (2.1). Obviously $\varpi(a) = 0$. For each $i(i = 1, 2, \ldots, m)$, from (2.2), we have

$$\begin{aligned}
\sum_{i=1}^{m-2} \lambda_i \omega(\theta_i) &= \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} z(\theta_i) - \sum_{i=1}^{m-2} \lambda_i \nu I_{a^+}^{1;\psi} \omega(\theta_i) \\
&\quad + \sum_{i=1}^{m-2} \lambda_i \frac{(\psi(\theta_i) - \psi(a))^{\delta-1}}{\Lambda\Gamma(\delta)} \Big\{ A \Big[-k \sum_{j=1}^{n-2} \mu_j I_{a^+}^{1;\psi} \varpi(\xi_j) \\
&\quad + \sum_{j=1}^{n-2} \mu_j I_{a^+}^{\beta;\psi} h(\xi_i) + \nu I_{a^+}^{1;\psi} \omega(b) - I_{a^+}^{p;\psi} z(b) \Big] \\
&\quad + \Omega \Big[-\nu \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{1;\psi} \omega(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} z(\theta_i) + k I_{a^+}^{1;\psi} \varpi(b) - I_{a^+}^{\beta;\psi} h(b) \Big] \Big\} \\
&= \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} z(\theta_i) - \sum_{i=1}^{m-2} \lambda_i \nu I_{a^+}^{1;\psi} \omega(\theta_i) \\
&\quad + \Big[\frac{(\psi(b) - \psi(a))^{\delta-1}}{\Lambda\Gamma(\delta)} - 1 \Big] \Big\{ A \Big[-k \sum_{j=1}^{n-2} \mu_j I_{a^+}^{1;\psi} \varpi(\xi_j) \\
&\quad + \sum_{j=1}^{n-2} \mu_j I_{a^+}^{\beta;\psi} h(\xi_i) + \nu I_{a^+}^{1;\psi} \omega(b) - I_{a^+}^{p;\psi} z(b) \Big] \\
&\quad + \Omega \Big[-\nu \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{1;\psi} \omega(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} z(\theta_i) + k I_{a^+}^{1;\psi} \varpi(b) - I_{a^+}^{\beta;\psi} h(b) \Big] \Big\} \\
&= I_{a^+}^{p;\psi} z(b) - \nu I_{a^+}^{1;\psi} \omega(b) \\
&\quad + \Big[\frac{(\psi(b) - \psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)} - 1 \Big] \Big\{ A \Big[-k \sum_{j=1}^{n-2} \mu_j I_{a^+}^{1;\psi} \varpi(\xi_j) \\
&\quad + \sum_{j=1}^{n-2} \mu_j I_{a^+}^{\beta;\psi} h(\xi_i) + \nu I_{a^+}^{1;\psi} \omega(b) - I_{a^+}^{p;\psi} z(b) \Big] \\
&\quad + \Omega \Big[-\nu \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{1;\psi} \omega(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} z(\theta_i) + k I_{a^+}^{1;\psi} \varpi(b) - I_{a^+}^{\beta;\psi} h(b) \Big] \Big\} \\
&= \varpi(b).
\end{aligned}$$

This completes the proof. □

3. Existence and uniqueness results

We introduce the space $\mathcal{H} = \{\varpi(\zeta) \mid \varpi(\zeta) \in C([a,b], \Re)\}$ endowed with the norm $\|\varpi\| = \sup\{|\varpi(\zeta)|, \ \zeta \in [a,b]\}$. Obviously $(\mathcal{H}, \|\cdot\|)$ is a Banach space. Then the product space $(\mathcal{H} \times \mathcal{H}, \|(\varpi, \omega)\|)$ is also a Banach space equipped with norm $\|(\varpi, \omega)\| = \|\varpi\| + \|\omega\|$.

In view of Lemma 2.3, we define an operator $\mathcal{S} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ by

$$\mathcal{S}(\varpi, \omega)(\zeta) = \Big(\mathcal{S}_1(\varpi, \omega)(\zeta), \mathcal{S}_2(\varpi, \omega)(\zeta)\Big),$$

where

$$\begin{aligned} \mathcal{S}_1(\varpi, \omega)(\zeta) &= I_{a^+}^{\beta;\psi} f_{\varpi\omega}(\zeta) - k I_{a^+}^{1;\psi} \varpi(\zeta) + \frac{(\psi(\zeta) - \psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \Big\{ \Delta\Big[-\nu \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{1;\psi} \omega(\theta_i) \\ &\quad + \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} g_{\varpi\omega}(\theta_i) + k I_{a^+}^{1;\psi} \varpi(b) - I_{a^+}^{\beta;\psi} f_{\varpi\omega}(b) \Big] \\ &\quad + B\Big[-k \sum_{j=1}^{n-2} \mu_j I_{a^+}^{1;\psi} \varpi(\xi_j) + \sum_{j=1}^{n-2} \mu_j I_{a^+}^{\beta;\psi} f_{\varpi\omega}(\xi_j) + \nu I_{a^+}^{1;\psi} \omega(b) - I_{a^+}^{p;\psi} g_{\varpi\omega}(b) \Big] \Big\}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \mathcal{S}_2(\varpi, \omega)(\zeta) &= I_{a^+}^{p;\psi} g_{\varpi\omega}(\zeta) - \nu I_{a^+}^{1;\psi} \omega(\zeta) + \frac{(\psi(\zeta) - \psi(a))^{\delta-1}}{\Lambda\Gamma(\delta)} \Big\{ A\Big[-k \sum_{j=1}^{n-2} \mu_j I_{a^+}^{1;\psi} \varpi(\xi_j) \\ &\quad + \sum_{j=1}^{n-2} \mu_j I_{a^+}^{\beta;\psi} f_{\varpi\omega}(\xi_i) + \nu I_{a^+}^{1;\psi} \omega(b) - I_{a^+}^{p;\psi} g_{\varpi\omega}(b) \Big] \\ &\quad + \Omega\Big[-\nu \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{1;\psi} \omega(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I_{a^+}^{p;\psi} g_{\varpi\omega}(\theta_i) + k I_{a^+}^{1;\psi} \varpi(b) - I_{a^+}^{\beta;\psi} f_{\varpi\omega}(b) \Big] \Big\}, \end{aligned} \tag{3.2}$$

where

$$f_{\varpi\omega} = f(\zeta, \varpi(\zeta), I_{a^+}^{\beta;\psi} \varpi(\zeta), \omega(\zeta)), \text{ and } g_{\varpi\omega} = g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi} \omega(\zeta)), \ \forall \ \zeta \in [a, b].$$

For the computational convenience, we put

$$X_1 = |k|(\psi(b) - \psi(a)) + \frac{|A|}{|\Lambda|}|\Delta||k|(\psi(b) - \psi(a)) + \frac{|A|}{|\Lambda|}|B||k| \sum_{j=1}^{n-2} |\mu_j|(\psi(\xi_j) - \psi(a)), \tag{3.3}$$

$$Y_1 = \frac{|A|}{|\Lambda|}|\Delta||\nu| \sum_{i=1}^{m-2} |\lambda_i|(\psi(\theta_i) - \psi(a)) + \frac{|A|}{|\Lambda|}|B||\nu|(\psi(b) - \psi(a)), \tag{3.4}$$

$$F_1 = \frac{(\psi(b) - \psi(a))^{\beta}}{\Gamma(\beta + 1)}\Big(1 + \frac{|A|}{|\Lambda|}|\Delta|\Big) + \frac{|A|}{|\Lambda|}|B| \sum_{j=1}^{n-2} |\mu_j| \frac{(\psi(\xi_j) - \psi(a))^{\beta}}{\Gamma(\beta + 1)}, \tag{3.5}$$

$$G_1 = \frac{|A|}{|\Lambda|}|\Delta| \sum_{i=1}^{m-2} |\lambda_i| \frac{(\psi(b) - \psi(a))^{p}}{\Gamma(p + 1)} + \frac{|A|}{|\Lambda|}|B| \frac{(\psi(b) - \psi(a))^{p}}{\Gamma(p + 1)}, \tag{3.6}$$

$$X_2 = \frac{|\Delta|}{|\Lambda|}|A||k|\sum_{j=1}^{n-2}|\mu_j|(\psi(\xi_j)-\psi(a)) + \frac{|\Delta|}{|\Lambda|}|\Omega||k|(\psi(b)-\psi(a)), \tag{3.7}$$

$$Y_2 = |\nu|(\psi(b)-\psi(a)) + \frac{|\Delta|}{|\Lambda|}|A||\nu|(\psi(b)-\psi(a)) + \frac{|\Delta|}{|\Lambda|}|\Omega|\sum_{i=1}^{m-2}|\lambda_i|(\psi(\theta_i)-\psi(a)), \tag{3.8}$$

$$F_2 = \frac{|\Delta|}{|\Lambda|}|A|\sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\xi_j)-\psi(a))^{\beta}}{\Gamma(\beta+1)} + \frac{|\Delta|}{|\Lambda|}|\Omega|\frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)}, \tag{3.9}$$

$$G_2 = \frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\Big(1+\frac{|\Delta|}{|\Lambda|}|A|\Big) + \frac{|\Delta|}{|\Lambda|}|\Omega|\sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}. \tag{3.10}$$

The Leray–Schauder alternative provides the basis for our first outcome.

Theorem 3.1. *(**Leray–Schauder alternative** [5]) Let $\mathcal{T} : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let $\mathcal{E}(\mathcal{T}) = \{\varpi \in E : \varpi = \lambda\mathcal{T}(\varpi)$ for some $0 < \lambda < 1\}$. Then either the set $\mathcal{E}(\mathcal{T})$ is unbounded, or $\mathcal{T}$ has at least one fixed point.*

Theorem 3.2. *Assume that*

- *(H_1) $f,\ g : [a,b]\times\Re\times\Re\times\Re \to \Re$ are continuous functions and there exist real constants $p_i,\ q_i \geq 0,\ (i = 1,2,3)$ and $p_0,\ q_0 > 0$ such that, $\forall\ x_i,\ y_i,\ z_i \in \Re,\ (i = 1,2)$,*

$$\begin{aligned} |f(\zeta,x_1,y_1,z_1)| &\leq p_0 + p_1|x_1| + p_2|y_1| + p_3|z_1|, \\ |g(\zeta,x_2,y_2,z_2)| &\leq q_0 + q_1|x_2| + q_2|y_2| + q_3|z_2|. \end{aligned}$$

If

$$\begin{aligned} \mathcal{M}_1 &= [F_1+F_2]p_1 + [F_1+F_2]p_2 I_{a^+}^{\beta;\psi} + [G_1+G_2]q_1 + [X_1+X_2] < 1, \\ \mathcal{M}_2 &= [F_1+F_2]p_3 + [G_1+G_2]q_2 + [G_1+G_2]q_1 I_{a^+}^{p;\psi} + [Y_1+Y_2] < 1, \end{aligned}$$

where $X_i,\ Y_i,\ F_i,\ G_i,\ i = 1,2$ are given by (3.3)-(3.10), then the system (1.2) has at least one solution on $[a,b]$.

Proof. The operator $\mathcal{S}$ is continuous, by the continuity of function f and g. We will show that the operator $\mathcal{S} : \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H}$ is completely continuous. Let $Z_r = \{(\varpi,\omega) \in \mathcal{H}\times\mathcal{H} : \|(\varpi,\omega)\| \leq r\}$ be bounded set. Then, there exist positive constants $\mathcal{L}_i,\ i = 1,2$ such that $|f(\zeta,\varpi(\zeta), I_{a^+}^{\beta;\psi}\varpi(\zeta),\omega(\zeta))| \leq \mathcal{L}_1,\ |g(\zeta,\varpi(\zeta),\omega(\zeta), I_{a^+}^{p;\psi}\omega(\zeta))| \leq \mathcal{L}_2,\ \forall\,(\varpi,\omega) \in Z_r$. Then, for any $(\varpi,\omega) \in Z_r$, we have

$$|\mathcal{S}_1(\varpi,\omega)(\zeta)| \leq I_{a^+}^{\beta;\psi}|f_{\varpi\omega}(\zeta)| + |k| I_{a^+}^{1;\psi}|\varpi(\zeta)| + \frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big\{|\Delta|\Big[|\nu|\sum_{i=1}^{m-2}|\lambda_i| I_{a^+}^{1;\psi}\omega(\theta_i)$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-2}|\lambda_i|I_{a^+}^{p;\psi}|g_{\varpi\omega}(\theta_i)|+|k|I_{a^+}^{1;\psi}|\varpi(b)|+I_{a^+}^{\beta;\psi}|f_{\varpi\omega}(b)|\Big]\\
&+|B|\Big[|k|\sum_{j=1}^{n-2}|\mu_j|I_{a^+}^{1;\psi}\varpi(\xi_j)+\sum_{j=1}^{n-2}|\mu_j|I_{a^+}^{\beta;\psi}|f_{\varpi\omega}(\xi_j)|+|\nu|I_{a^+}^{1;\psi}|\omega(b)|+I_{a^+}^{p;\psi}|g_{\varpi\omega}(b)|\Big]\Big\},\\
\leq\;& \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}\mathcal{L}_1+|k|(\psi(b)-\psi(a))\|\varpi\|_{\mathcal{H}}+\frac{|A|}{|\Lambda|}\Big\{|\Delta|\Big[|\nu|\sum_{i=1}^{m-2}|\lambda_i(\psi(\theta_i)-\psi(a))\|\omega\|_{\mathcal{H}}\\
&+\sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}\mathcal{L}_2+|k|(\psi(b)-\psi(a))\|\varpi\|_{\mathcal{H}}+\frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}\mathcal{L}_1\Big]\\
&+|B|\Big[|k|\sum_{j=1}^{n-2}|\mu_j|(\psi(\xi_j)-\psi(a))\|\varpi\|_{\mathcal{H}}+\sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\xi_j)-\psi(a))^\beta}{\Gamma(\beta+1)}\mathcal{L}_1\\
&+|\nu|(\psi(b)-\psi(a))\|\omega\|_{\mathcal{H}}+\frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}\mathcal{L}_2\Big]\Big\},\\
\leq\;& F_1\mathcal{L}_1+G_1\mathcal{L}_2+X_1\|\varpi\|_{\mathcal{H}}+Y_1\|\omega\|_{\mathcal{H}}.
\end{aligned}
$$

Which implies that

$$\|\mathcal{S}_1(\varpi,\omega)\|_{\mathcal{H}\times\mathcal{H}}\leq F_1\mathcal{L}_1+G_1\mathcal{L}_2+X_1\|\varpi\|_{\mathcal{H}}+Y_1\|\omega\|_{\mathcal{H}}.$$

Similarly,

$$\|\mathcal{S}_2(\varpi,\omega)\|_{\mathcal{H}\times\mathcal{H}}\leq F_2\mathcal{L}_1+G_2\mathcal{L}_2+X_2\|\varpi\|_{\mathcal{H}}+Y_2\|\omega\|_{\mathcal{H}}.$$

From the above mentioned inequalities the operator $\mathcal{S}$ is uniformly bounded, since

$$\|\mathcal{S}(\varpi,\omega)\|_{\mathcal{H}\times\mathcal{H}}\leq (F_1+F_2)\mathcal{L}_1+(G_1+G_2)\mathcal{L}_2+(X_1+X_2)r+(Y_1+Y_2)r.$$

Next, we show that $\mathcal{S}$ is equicontinuous. Let $\zeta_1,\ \zeta_2\in[a,b]$ with $\zeta_1<\zeta_2$. Then we have

$$
\begin{aligned}
&|\mathcal{S}_1(\varpi(\zeta_2),\omega(\zeta_2))-\mathcal{S}_1(\varpi(\zeta_1),\omega(\zeta_1))|\\
\leq\;& \frac{1}{\Gamma(\beta)}\Big|\int_a^{\zeta_1}\psi'(s)[(\psi(\zeta_2)-\psi(s))^{\beta-1}-(\psi(\zeta_1)-\psi(s))^{\beta-1}]f_{\varpi\omega}(s)ds\\
&+\int_{\zeta_1}^{\zeta_2}\psi'(s)(\psi(\zeta_2)-\psi(s))^{\beta-1}f_{\varpi\omega}(s)ds\Big|+|k|\int_{\zeta_1}^{\zeta_2}\psi'(s)|\varpi(s)|ds\\
&+\frac{|(\psi(\zeta_2)-\psi(a))^{\gamma-1}-(\psi(\zeta_1)-\psi(a))^{\gamma-1}|}{|\Lambda|\Gamma(\gamma)}|\Delta P+BQ|\\
\leq\;& \mathcal{L}_1\frac{2(\psi(\zeta_2)-\psi(\zeta_1))^\beta+|(\psi(\zeta_2)-\psi(a))^\beta-(\psi(\zeta_1)-\psi(a))^\beta|}{\Gamma(\beta+1)}+|k|r|\zeta_2-\zeta_1|\\
&+\frac{|(\psi(\zeta_2)-\psi(a))^{\gamma-1}-(\psi(\zeta_1)-\psi(a))^{\gamma-1}|}{|\Lambda|\Gamma(\gamma)}|\Delta P+BQ|.
\end{aligned}
$$

Analogously, we can obtain

$$|\mathcal{S}_2(\varpi(\zeta_2),\omega(\zeta_2))-\mathcal{S}_2(\varpi(\zeta_1),\omega(\zeta_1))|$$

$$
\begin{aligned}
&\leq \mathcal{L}_2 \frac{2(\psi(\zeta_2)-\psi(\zeta_1))^p + |(\psi(\zeta_2)-\psi(b))^p - (\psi(\zeta_1)-\psi(b))^p|}{\Gamma(p+1)} + |\nu| r |\zeta_2 - \zeta_1| \\
&\quad + \frac{|(\psi(\zeta_2)-\psi(a))^{\delta-1} - (\psi(\zeta_1)-\psi(a))^{\delta-1}|}{|\Lambda|\Gamma(\delta)} |AQ + \Omega P|.
\end{aligned}
$$

Therefore, the operator $\mathcal{S}(\varpi,\omega)$ is equicontinuous, and thus the operator $\mathcal{S}(\varpi,\omega)$ is completely continuous.

Last but not least, it will be confirmed that the set $\mathcal{E} = \{(\varpi,\omega) \in \mathcal{H}\times\mathcal{H} | (\varpi,\omega) = \lambda\mathcal{S}(\varpi,\omega),\ 0 \leq \lambda \leq 1\}$ is bounded. Let $(\varpi,\omega) \in \mathcal{E}$ with $(\varpi,\omega) = \lambda\mathcal{S}(\varpi,\omega)$. For $\zeta \in [a,b]$, we have

$$
\varpi(\zeta) = \lambda\mathcal{S}_1(\varpi,\omega), \quad \omega(\zeta) = \lambda\mathcal{S}_2(\varpi,\omega).
$$

Then

$$
\begin{aligned}
|\varpi(\zeta)| &\leq \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}(p_0 + p_1|\varpi(\zeta)| + p_2|I_{a^+}^{\beta;\psi}\varpi(\zeta)| + p_3|\omega(\zeta)|) + |k|(\psi(b)-\psi(a))|\varpi(\zeta)| \\
&\quad + \frac{|A|}{|\Lambda|}\Big\{|\Delta|\Big[|\nu|\sum_{i=1}^{m-2}|\lambda_i|(\psi(\theta_i)-\psi(a))|\omega(\zeta)| \\
&\quad + \sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^p}{\Gamma(p+1)}(q_0 + q_1|\varpi(\zeta)| + q_2|\omega(\zeta)| + q_3|I_{a^+}^{p;\psi}\omega(\zeta)|) + |k|(\psi(b)-\psi(a))|\varpi(\zeta)| \\
&\quad + \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}(p_0 + p_1|\varpi(\zeta)| + p_2|I_{a^+}^{\beta;\psi}\varpi(\zeta)| + p_3|\omega(\zeta)|)\Big] \\
&\quad + |B|\Big[|k|\sum_{j=1}^{n-2}|\mu_j|(\psi(\xi_j)-\psi(a))|\varpi(\zeta)| + \sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\xi_j)-\psi(a))^\beta}{\Gamma(\beta+1)}(p_0 \\
&\quad + p_1|\varpi(\zeta)| + p_2|I_{a^+}^{\beta;\psi}\varpi(\zeta)| + p_3|\omega(\zeta)|) + |\nu|(\psi(b)-\psi(a))|\omega(\zeta)| \\
&\quad + \frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}(q_0 + q_1|\varpi(\zeta)| + q_2|\omega(\zeta)| + q_3|I_{a^+}^{p;\psi}\omega(\zeta)|)\Big]\Big\} \\
&\leq F_1(p_0 + p_1\|\varpi\|_{\mathcal{H}} + p_2\|I_{a^+}^{\beta;\psi}\varpi\|_{\mathcal{H}} + p_3\|\omega\|_{\mathcal{H}}) \\
&\quad + G_1(q_0 + q_1\|\varpi\|_{\mathcal{H}} + q_2\|\omega\|_{\mathcal{H}} + q_3\|I_{a^+}^{p;\psi}\omega\|_{\mathcal{H}}) + X_1\|\varpi\|_{\mathcal{H}} + Y_1\|\omega\|_{\mathcal{H}} \\
&= (F_1p_0 + G_1q_0) + (F_1p_1 + F_1p_2I_{a^+}^{\beta;\psi} + G_1q_1 + X_1)\|\varpi\|_{\mathcal{H}} \\
&\quad + (F_1p_3 + G_1q_2 + G_1q_3I_{a^+}^{p;\psi} + Y_1)\|\omega\|_{\mathcal{H}},
\end{aligned}
$$

and

$$
\begin{aligned}
|\omega(\zeta)| &\leq F_2(p_0 + p_1\|\varpi\|_{\mathcal{H}} + p_2\|I_{a^+}^{\beta;\psi}\omega\|_{\mathcal{H}} + p_3\|\omega\|_{\mathcal{H}}) \\
&\quad + G_2(q_0 + q_1\|\varpi\|_{\mathcal{H}} + q_2\|\omega\|_{\mathcal{H}}) + q_3\|I_{a^+}^{p;\psi}\omega\|_{\mathcal{H}} + X_2\|\varpi\|_{\mathcal{H}} + Y_2\|\omega\|_{\mathcal{H}} \\
&= (F_2p_0 + G_2q_0) + (F_2p_1 + F_2p_2I_{a^+}^{\beta;\psi} + G_2q_1 + X_2)\|\varpi\|_{\mathcal{H}} \\
&\quad + (F_2p_3 + G_2q_2 + G_2q_3I_{a^+}^{p;\psi} + Y_2)\|\omega\|_{\mathcal{H}}.
\end{aligned}
$$

Hence we have

$$
\|\varpi\|_{\mathcal{H}} \leq (F_1p_0 + G_1q_0) + (F_1p_1 + F_1p_2I_{a^+}^{\beta;\psi} + G_1q_1 + X_1)\|\varpi\|_{\mathcal{H}}
$$

$$+(F_1p_3 + G_1q_2 + G_1q_3I^{p;\psi}_{a^+} + Y_1)\|\omega\|_{\mathcal{H}},$$

and

$$\begin{aligned}\|\omega\|_{\mathcal{H}} &\leq (F_2p_0 + G_2q_0) + (F_2p_1 + F_2p_2I^{\beta;\psi}_{a^+} + G_2q_1 + X_2)\|\varpi\|_{\mathcal{H}}\\ &\quad +(F_2p_3 + G_2q_2 + G_2q_3I^{p;\psi}_{a^+} + Y_2)\|\omega\|_{\mathcal{H}},\end{aligned}$$

which implies that

$$\begin{aligned}\|\varpi\|_{\mathcal{H}} + \|\omega\|_{\mathcal{H}} &\leq [F_1 + F_2]p_0 + [G_1 + G_2]q_0\\ &\quad +\{[F_1 + F_2]p_1 + \{[F_1 + F_2]p_2I^{\beta;\psi}_{a^+} + [G_1 + G_2]q_1 + [X_1 + X_2]\}\|\varpi\|_{\mathcal{H}}\\ &\quad +\{[F_1 + F_2]p_3 + [G_1 + G_2]q_2 + [G_1 + G_2]q_1I^{p;\psi}_{a^+} + [Y_1 + Y_2]\}\|\omega\|_{\mathcal{H}}.\end{aligned}$$

Consequently,

$$\|(\varpi, \omega)\|_{\mathcal{H}\times\mathcal{H}} \leq \frac{[F_1 + F_2]p_0 + [G_1 + G_2]q_0}{\min\{1 - \mathcal{M}_1, 1 - \mathcal{M}_2\}},$$

which proves that $\mathcal{E}$ is bounded. Thus, the operator $\mathcal{S}$, by Lemma 3.1, has at least one fixed point. As a result, there is at least one solution to the BVP (1.2). □

The next theorem uses Banach's contraction mapping principle to show the uniqueness of the system's solution (1.2).

Theorem 3.3. *Assume that*

- *(H_2) $f,\ g : [a, b] \times \Re \times \Re \times \Re \to \Re$ are continuous functions and there exist positive constants $\mathcal{P},\ Q$ such that for all $\zeta \in [a, b]$ and $u_i,\ v_i \in \Re,\ i = 1, 2, 3,$ we have*

$$\begin{aligned}|f(\zeta, u_1, u_2, u_3) - f(\zeta, v_1, v_2, v_3)| &\leq \mathcal{P}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),\\ |g(\zeta, u_1, u_2, u_3) - g(\zeta, v_1, v_2, v_3)| &\leq Q(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|).\end{aligned}$$

Then, the system (1.2) has a unique solution on $[a, b]$, provided that

$$[F_1 + F_2]\mathcal{P} + [G_1 + G_2]Q + [X_1 + X_2] + [Y_1 + Y_2] < 1, \tag{3.11}$$

where $X_i,\ Y_i,\ F_i,\ G_i,\ i = 1, 2$ are given by (3.3)-(3.10).

Proof. Define $\sup_{\zeta\in[a,b]} f(\zeta, 0, 0, 0) = \mathcal{N}_1 < \infty,\ \sup_{\zeta\in[a,b]} g(\zeta, 0, 0, 0) = \mathcal{N}_2 < \infty$ and $r > 0$ such that

$$r > \frac{[F_1 + F_2]\mathcal{N}_1 + [G_1 + G_2]\mathcal{N}_2}{1 - [F_1 + F_2]\mathcal{P} + [G_1 + G_2]Q + [X_1 + X_2] + [Y_1 + Y_2]}.$$

In the first step, we show that $\mathcal{S}Z_r \subset Z_r$, where $Z_r = \{(\varpi, \omega) \in \mathcal{H} \times \mathcal{H} : \|(\varpi, \omega)\| \leq r\}$. By the assumption ($H_2$), for $(\varpi, \omega) \in Z_r,\ \zeta \in [a, b]$, we have

$$\begin{aligned}|f(\zeta, \varpi(\zeta), I^{\beta;\psi}_{a^+}\varpi(\zeta), \omega(\zeta))| &\leq |f(\zeta, \varpi(\zeta), I^{\beta;\psi}_{a^+}\varpi(\zeta), \omega(\zeta)) - f(\zeta, 0, 0, 0)| + |f(\zeta, 0, 0, 0)|\\ &\leq \mathcal{P}(|\varpi(\zeta)| + |I^{\beta;\psi}_{a^+}\varpi(\zeta)| + |\omega(\zeta)|) + \mathcal{N}_1\\ &\leq \mathcal{P}((1 + I^{\beta;\psi}_{a^+})\|\varpi\|_{\mathcal{H}} + \|\omega\|_{\mathcal{H}}) + \mathcal{N}_1\end{aligned}$$

$$\leq \mathcal{P}r + \mathcal{N}_1,$$

and

$$|g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi}\omega(\zeta))| \leq \mathcal{Q}r + \mathcal{N}_2.$$

Using the above estimates, we obtain

$$\begin{aligned}
|\mathcal{S}_1(\varpi,\omega)(\zeta)| \leq\; & I_{a^+}^{\beta;\psi}|f_{\varpi\omega}(\zeta)| + |k|I_{a^+}^{1;\psi}|\varpi(\zeta)| + \frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\Big\{|\Delta|\Big[|\nu|\sum_{i=1}^{m-2}|\lambda_i|I_{a^+}^{1;\psi}\omega(\theta_i) \\
& + \sum_{i=1}^{m-2}|\lambda_i|I_{a^+}^{p;\psi}|g_{\varpi\omega}(\theta_i)| + |k|I_{a^+}^{1;\psi}|\varpi(b)| - I_{a^+}^{\beta;\psi}|f_{\varpi\omega}(b)|\Big] \\
& + |B|\Big[|k|\sum_{j=1}^{n-2}|\mu_j|I_{a^+}^{1;\psi}\varpi(\xi_j) + \sum_{j=1}^{n-2}|\mu_j|I_{a^+}^{\beta;\psi}|f_{\varpi\omega}(\xi_j)| + |\nu|I_{a^+}^{1;\psi}|\omega(b)| - I_{a^+}^{p;\psi}|g_{\varpi\omega}(b)|\Big]\Big\} \\
\leq\; & \frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)}(\mathcal{P}r+\mathcal{N}_1) + |k|(\psi(b)-\psi(a))\|\varpi\|_{\mathcal{H}} \\
& + \frac{|A|}{|\Lambda|}\Big\{|\Delta|\Big[|\nu|\sum_{i=1}^{m-2}|\lambda_i|(\psi(\theta_i)-\psi(a))\|\omega\|_{\mathcal{H}} \\
& + \sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^{p}}{\Gamma(p+1)}(\mathcal{Q}r+\mathcal{N}_2) + |k|(\psi(b)-\psi(a))\|\varpi\|_{\mathcal{H}} \\
& + \frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)}(\mathcal{P}r+\mathcal{N}_1)\Big] + |B|\Big[|k|\sum_{j=1}^{n-2}|\mu_i|(\psi(\xi_i)-\psi(a))\|\varpi\|_{\mathcal{H}} \\
& + \sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\xi_j)-\psi(a))^{\beta}}{\Gamma(\beta+1)}(\mathcal{P}r+\mathcal{N}_1) + |\nu|(\psi(b)-\psi(a))\|\omega\|_{\mathcal{H}} \\
& + \frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}(\mathcal{Q}r+\mathcal{N}_2)\Big]\Big\} \\
\leq\; & [F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1]r + F_1\mathcal{N}_1 + G_1\mathcal{N}_2.
\end{aligned}$$

Hence

$$\|\mathcal{S}_1(\varpi,\omega)\|_{\mathcal{H}\times\mathcal{H}} \leq [F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1]r + F_1\mathcal{N}_1 + G_1\mathcal{N}_2.$$

Similarly,

$$\|\mathcal{S}_2(\varpi,\omega)\|_{\mathcal{H}\times\mathcal{H}} \leq [F_2\mathcal{P} + G_2\mathcal{Q} + X_2 + Y_2]r + F_2\mathcal{N}_1 + G_2\mathcal{N}_2.$$

In consequence, it follows that

$$\begin{aligned}
\|\mathcal{S}(\varpi,\omega)\|_{\mathcal{H}\times\mathcal{H}} \leq\; & [F_1+F_2]\mathcal{P} + [G_1+G_2]\mathcal{Q} + [X_1+X_2] + [Y_1+Y_2]r \\
& + [F_1+F_2]\mathcal{N}_1 + [G_1+G_2]\mathcal{N}_2 \\
\leq\; & r.
\end{aligned}$$

Which shows that $\mathcal{S}Z_r \subset Z_r$.

We prove that the operator $\mathcal{S}$ is contraction. For $(\varpi_1, \omega_1),\ (\varpi_2, \omega_2) \in \mathcal{H} \times \mathcal{H}$ and for any $\zeta \in [a, b]$, we get

$$
\begin{aligned}
& |\mathcal{S}_1(\varpi_2, \omega_2)(\zeta) - \mathcal{S}_1(\varpi_1, \omega_1)(\zeta)| \\
\leq\ & \frac{(\psi(b) - \psi(a))^\beta}{\Gamma(\beta + 1)} \mathcal{P}(\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}) + |k|(\psi(b) - \psi(a))\|\varpi_2 - \varpi_1\|_{\mathcal{H}} \\
& + \frac{|A|}{|\Lambda|}\Big\{ |\Delta| \Big[|\nu| \sum_{i=1}^{m-2} |\lambda_i|(\psi(\theta_i) - \psi(a))\|\omega_2 - \omega_1\|_{\mathcal{H}} \\
& + \sum_{i=1}^{m-2} |\lambda_i| \frac{(\psi(\theta_i) - \psi(a))^p}{\Gamma(p + 1)} \mathcal{Q}(\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}) + |k|(\psi(b) - \psi(a))\|\varpi_2 - \varpi_1\|_{\mathcal{H}} \\
& \frac{(\psi(b) - \psi(a))^\beta}{\Gamma(\beta + 1)} \mathcal{P}(\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}) \Big] \\
& + |B| \Big[|k| \sum_{j=1}^{n-2} |\mu_i|(\psi(\xi_i) - \psi(a))\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \sum_{j=1}^{n-2} |\mu_j| \frac{(\psi(\xi_j) - \psi(a))^\beta}{\Gamma(\beta + 1)} \mathcal{P}(\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}) \\
& + |\nu|(\psi(b) - \psi(a))\|\omega_2 - \omega_1\| \frac{(\psi(b) - \psi(a))^p}{\Gamma(p + 1)} \mathcal{Q}(\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}) \Big] \Big\} \\
\leq\ & [F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1](\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|y_2 - \omega_1\|_{\mathcal{H}}),
\end{aligned}
$$

and consequently we obtain

$$
\|\mathcal{S}_1(\varpi_2, \omega_2) - \mathcal{S}_1(\varpi_1, \omega_1)\|_{\mathcal{H}\times\mathcal{H}} \leq [F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1](\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}). \tag{3.12}
$$

Similarly. we obtain

$$
\|\mathcal{S}_2(\varpi_2, \omega_2) - \mathcal{S}_2(\varpi_1, \omega_1)\|_{\mathcal{H}\times\mathcal{H}} \leq [F_2\mathcal{P} + G_2\mathcal{Q} + X_2 + Y_2](\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}). \tag{3.13}
$$

It follows from above two inequalities (3.12) and (3.13) that

$$
\begin{aligned}
\|\mathcal{S}(\varpi_2, \omega_2) - \mathcal{S}(\varpi_1, \omega_1)\|_{\mathcal{H}\times\mathcal{H}} \leq\ & \{[F_1 + F_2]\mathcal{P} + [G_1 + G_2]\mathcal{Q} + [X_1 + X_2] \\
& + [Y_1 + Y_2]\}(\|\varpi_2 - \varpi_1\|_{\mathcal{H}} + \|\omega_2 - \omega_1\|_{\mathcal{H}}),
\end{aligned}
$$

As a result, the operator $\mathcal{S}$ is a contraction. As a result of Banach's FP theorem, the operator $\mathcal{S}$ has a unique FP, which is the unique solution of problem (1.2). □

4. Hyers–Ulam stability

We introduce the HU stability idea for problem (1.2) in this part. The definitions that follow are taken from [23].

Let $\epsilon_\beta,\ \epsilon_p > 0,\ f,\ g : [a, b] \times \Re^3 \to \Re$ be continuous functions and $\vartheta_\beta,\ \vartheta_p : [a, b] \to \Re^+$ are nondecreasing functions. Consider the following inequalities:

$$
\begin{cases}
\left| \left({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k\, {}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi} \right) \varpi(\zeta) - f(\zeta, \varpi(\zeta), I_{a^+}^{\beta;\psi}\varpi(\zeta), \omega(\zeta)) \right| \leq \epsilon_\beta,\ \forall\ \zeta \in [a, b], \\
\left| \left({}^H\mathcal{D}_{a^+}^{p,q;\psi} + \nu\, {}^H\mathcal{D}_{a^+}^{p-1,q;\psi} \right) \omega(\zeta) - g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi}\omega(\zeta)) \right| \leq \epsilon_p,\ \forall\ \zeta \in [a, b],
\end{cases} \tag{4.1}
$$

$$\begin{cases} \left| \left({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k\, {}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi} \right) \varpi(\zeta) - f(\zeta, \varpi(\zeta), I_{a^+}^{\beta;\psi} \varpi(\zeta), \omega(\zeta)) \right| \leq \vartheta_\beta(\zeta)\epsilon_\beta, \ \forall\, \zeta \in [a,b], \\ \left| \left({}^H\mathcal{D}_{a^+}^{p,q;\psi} + \nu\, {}^H\mathcal{D}_{a^+}^{p-1,q;\psi} \right) \omega(\zeta) - g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi} \omega(\zeta)) \right| \leq \vartheta_p(\zeta)\epsilon_p, \ \forall\, \zeta \in [a,b], \end{cases} \tag{4.2}$$

and

$$\begin{cases} \left| \left({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k\, {}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi} \right) \varpi(\zeta) - f(\zeta, \varpi(\zeta), I_{a^+}^{\beta;\psi} \varpi(\zeta), \omega(\zeta)) \right| \leq \vartheta_\beta(\zeta), \ \forall\, \zeta \in [a,b], \\ \left| \left({}^H\mathcal{D}_{a^+}^{p,q;\psi} + \nu\, {}^H\mathcal{D}_{a^+}^{p-1,q;\psi} \right) \omega(\zeta) - g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi} \omega(\zeta)) \right| \leq \vartheta_p(\zeta), \ \forall\, \zeta \in [a,b]. \end{cases} \tag{4.3}$$

Definition 4.1. The coupled system (1.2) is HU stable, if there is $C_{\beta,p} = (C_\beta, C_p) > 0$ such that for some $\epsilon = (\epsilon_\beta, \epsilon_p) > 0$ and for each solution $(\varpi, \omega) \in \mathcal{E} \times \mathcal{E}$ of (4.1) there is a solution $(\varpi^*, \omega^*) \in \mathcal{E} \times \mathcal{E}$ of the coupled system (1.2) which satisfies the following inequalities

$$|(\varpi^*, \omega^*)(\zeta) - (\varpi, \omega)(\zeta)| \leq C_{\beta,p}\epsilon, \ \forall\, \zeta \in [a,b].$$

Definition 4.2. The coupled system (1.2) is generalized HU stable, if there is $\Theta \in C(\Re^+, \Re^+)$ with $\Theta(0) = 0$, such that for each solution $(\varpi, \omega) \in \mathcal{E} \times \mathcal{E}$ of (4.1) there is a solution $(\varpi^*, \omega^*) \in \mathcal{E} \times \mathcal{E}$ of the coupled system (1.2) which satisfies the following inequalities

$$|(\varpi^*, \omega^*)(\zeta) - (\varpi, \omega)(\zeta)| \leq \Theta(\epsilon), \ \forall\, \zeta \in [a,b].$$

Definition 4.3. The coupled system (1.2) is HU–Rassias stable with respect to $\vartheta_{\beta,p} = (\vartheta_\beta, \vartheta_p) \in C^1([a,b], \Re)$, if there is a constant $C_{\vartheta_\beta,\vartheta_p} = (C_{\vartheta_\beta}, C_{\vartheta_p}) > 0$ such that for some $\epsilon = (\epsilon_\beta, \epsilon_p) > 0$ and for each solution $(\varpi, \omega) \in \mathcal{E} \times \mathcal{E}$ of (4.2), there is a solution $(\varpi^*, \omega^*) \in \mathcal{E} \times \mathcal{E}$ of the system (1.2) with

$$|(\varpi^*, \omega^*)(\zeta) - (\varpi, \omega)(\zeta)| \leq C_{\vartheta_\beta,\vartheta_p}\vartheta_{\beta,p}(\zeta)\epsilon, \ \forall\, \zeta \in [a,b].$$

Definition 4.4. The coupled system (1.2) is generalized HU–Rassias stable with respect to $\vartheta_{\beta,p} = (\vartheta_\beta, \vartheta_p) \in C^1([a,b], \Re)$, if there is a constant $C_{\vartheta_\beta,\vartheta_p} = (C_{\vartheta_\beta}, C_{\vartheta_p}) > 0$ such that for each solution $(\varpi, \omega) \in \mathcal{E} \times \mathcal{E}$ of (4.3), there is a solution $(\varpi^*, \omega^*) \in \mathcal{E} \times \mathcal{E}$ of the system (1.2) with

$$|(\varpi^*, \omega^*)(\zeta) - (\varpi, \omega)(\zeta)| \leq C_{\vartheta_\beta,\vartheta_p}\vartheta_{\beta,p}(\zeta), \ \forall\, \zeta \in [a,b].$$

Remark 4.1. We say that $(\varpi, \omega) \in \mathcal{E}\times\mathcal{E}$ is a solution of (4.1) there is a functions $\Psi_\beta, \ \Psi_p \in C([a,b], \Re)$ which depend upon $\varpi, \ \omega$, respectively such that

- (a) $|\Psi_\beta(\zeta)| \leq \epsilon_\beta, \ |\Psi_p(\zeta)| \leq \epsilon_p, \ \forall\, \zeta \in [a,b]$,
- (b)

$$\begin{cases} \left({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k\, {}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi} \right) \varpi(\zeta) = f(\zeta, \varpi(\zeta), I_{a^+}^{\beta;\psi} \varpi(\zeta), \omega(\zeta)) + \Psi_\beta(\zeta), \ \forall\, \zeta \in [a,b], \\ \left({}^H\mathcal{D}_{a^+}^{p,q;\psi} + \nu\, {}^H\mathcal{D}_{a^+}^{p-1,q;\psi} \right) \omega(\zeta) = g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi} \omega(\zeta)) + \Psi_p(\zeta), \ \forall\, \zeta \in [a,b]. \end{cases}$$

Theorem 4.1. *Consider $(\varpi, \omega) \in \mathcal{E} \times \mathcal{E}$ be the solution of the inequality (4.1), then*

$$\begin{cases} |\varpi(\zeta) - \overline{\varpi}(\zeta)| \leq S_\beta \epsilon_\beta, \ \forall \ \zeta \in [a,b], \\ |\omega(\zeta) - \overline{\omega}(\zeta)| \leq S_p \epsilon_p, \ \forall \ \zeta \in [a,b], \end{cases}$$

Proof. Using (b) of Remark 4.1, we have

$$\begin{cases} \left({}^{H}\mathcal{D}_{a^+}^{\beta,\rho;\psi} + k \, {}^{H}\mathcal{D}_{a^+}^{\beta-1,\rho;\psi}\right)\varpi(\zeta) = f(\zeta, \varpi(\zeta), I_{a^+}^{\beta;\psi}\varpi(\zeta), \omega(\zeta)) + \Psi_\beta(\zeta), \ \forall \ \zeta \in [a,b], \\ \left({}^{H}\mathcal{D}_{a^+}^{p,q;\psi} + \nu \, {}^{H}\mathcal{D}_{a^+}^{p-1,q;\psi}\right)\omega(\zeta) = g(\zeta, \varpi(\zeta), \omega(\zeta), I_{a^+}^{p;\psi}\omega(\zeta)) + \Psi_p(\zeta), \ \forall \ \zeta \in [a,b], \\ \varpi(a) = 0, \ \varpi(b) = \sum_{i=1}^{m-2} \lambda_i \omega(\theta_i), \\ \omega(a) = 0, \ \omega(b) = \sum_{j=1}^{n-2} \mu_j \varpi(\xi_j), \end{cases} \tag{4.4}$$

So by Lemma 2.2, the solution of (4.4) will be in the given form

$$\begin{aligned} \varpi(\zeta) &= I_{a^+}^{\beta;\psi} f_{\varpi\omega}(\zeta) + I_{a^+}^{\beta;\psi}\Psi_\beta(\zeta) - kI_{a^+}^{1;\psi}\varpi(\zeta) + \frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big\{\Delta\Big[-\nu\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{1;\psi}\omega(\theta_i) \\ &\quad + \sum_{i=1}^{m-2}\lambda_i I_{a^+}^{p;\psi} g_{\varpi\omega}(\theta_i) + kI_{a^+}^{1;\psi}\varpi(b) - I_{a^+}^{\beta;\psi} f_{\varpi\omega}(b) - I_{a^+}^{\beta;\psi}\Psi_\beta(b)\Big] \\ &\quad + B\Big[-k\sum_{j=1}^{n-2}\mu_j I_{a^+}^{1;\psi}\varpi(\xi_j) + \sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi} f_{\varpi\omega}(\xi_j) + \sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi}\Psi_\beta(\xi_j) + \nu I_{a^+}^{1;\psi}\omega(b) - I_{a^+}^{p;\psi} g_{\varpi\omega}(b)\Big]\Big\}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \omega(\zeta) &= I_{a^+}^{p;\psi} g_{\varpi\omega}(\zeta) + I_{a^+}^{p;\psi}\Psi_p(\zeta) - \nu I_{a^+}^{1;\psi}\omega(\zeta) + \frac{(\psi(\zeta)-\psi(a))^{\delta-1}}{\Lambda\Gamma(\delta)}\Big\{A\Big[-k\sum_{j=1}^{n-2}\mu_j I_{a^+}^{1;\psi}\varpi(\xi_j) \\ &\quad + \sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi} f_{\varpi\omega}(\xi_i) + \nu I_{a^+}^{1;\psi}\omega(b) - I_{a^+}^{p;\psi} g_{\varpi\omega}(b) - I_{a^+}^{p;\psi}\Psi_p(b)\Big] \\ &\quad + \Omega\Big[-\nu\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{1;\psi}\omega(\theta_i) + \sum_{i=1}^{m-2}\lambda_i I_{a^+}^{p;\psi} g_{\varpi\omega}(\theta_i) + \sum_{i=1}^{m-2}\lambda_i I_{a^+}^{p;\psi}\Psi_p(\theta_i) + kI_{a^+}^{1;\psi}\varpi(b) - I_{a^+}^{\beta;\psi} f_{\varpi\omega}(b)\Big]\Big\}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} A &= \frac{(\psi(b)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}, \ \ B = \sum_{i=1}^{m-2}\lambda_i \frac{(\psi(\theta_i)-\psi(a))^{\delta-1}}{\Gamma(\delta)}, \\ \Omega &= \sum_{j=1}^{n-2}\mu_j \frac{(\psi(\xi_j)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}, \ \ \Delta = \frac{(\psi(b)-\psi(a))^{\delta-1}}{\Gamma(\delta)}. \end{aligned}$$

From (4.5), we have

$$|\varpi(\zeta) - \overline{\varpi}(\zeta)| \leq I_{a^+}^{\beta;\psi}|\Psi_\beta(\zeta)| + \frac{|A|}{|\Lambda|}\Big\{|\Delta|\Big[-I_{a^+}^{\beta;\psi}|\Psi_\beta(b)|\Big]$$

$$+|B|\Big[\sum_{j=1}^{n-2}|\mu_j|I_{a^+}^{\beta;\psi}|\Psi_\beta(\xi_j)|\Big]\Big\},$$

where

$$\begin{aligned}\overline{\varpi}(\zeta) &= I_{a^+}^{\beta;\psi}f_{\varpi\omega}(\zeta)-kI_{a^+}^{1;\psi}\varpi(\zeta)+\frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big\{\Delta\Big[-\nu\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{1;\psi}\omega(\theta_i)\\ &\quad+\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{p;\psi}g_{\varpi\omega}(\theta_i)+kI_{a^+}^{1;\psi}\varpi(b)-I_{a^+}^{\beta;\psi}f_{\varpi\omega}(b)\Big]\\ &\quad+B\Big[-k\sum_{j=1}^{n-2}\mu_j I_{a^+}^{1;\psi}\varpi(\xi_j)+\sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi}f_{\varpi\omega}(\xi_j)+\nu I_{a^+}^{1;\psi}\omega(b)-I_{a^+}^{p;\psi}g_{\varpi\omega}(b)\Big]\Big\}.\end{aligned}$$

Using (a) of Remark 4.1, we obtain

$$|\varpi(\zeta)-\overline{\varpi}(\zeta)|\le S_\beta\epsilon_\beta,$$

where

$$S_\beta = \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}+\frac{|A|}{|\Lambda|}\Big\{|\Delta|\Big[\frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}\Big]+|B|\Big[\sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\xi_j)-\psi(a))^\beta}{\Gamma(\beta+1)}\Big]\Big\}.$$

Repeat the similar procedure for (4.6) with (a) of Remark 4.1, we have

$$|\omega(\zeta)-\overline{\omega}(\zeta)|\le S_p\epsilon_p,$$

where

$$S_p = \frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}+\frac{|A|}{|\Lambda|}\Big\{|\Delta|\Big[\frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}\Big]+|B|\Big[\sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^p}{\Gamma(p+1)}\Big]\Big\}.$$

□

Theorem 4.2. *If the hypothesis* (H_2) *hold, with*

$$\Delta = 1-\mathcal{K}_\beta\mathcal{K}_p>0.$$

then system (1.2) is stable, in the sense of HU.

Proof. Suppose $(\varpi,\omega)\in\mathcal{E}\times\mathcal{E}$ be the solution of the inequality (4.1) and $(\varpi^*,\omega^*)\in\mathcal{E}\times\mathcal{E}$ is the solution of the given system

$$\begin{cases}\Big({}^H\mathcal{D}_{a^+}^{\beta,\rho;\psi}+k\,{}^H\mathcal{D}_{a^+}^{\beta-1,\rho;\psi}\Big)\varpi^*(\zeta)=f(\zeta,\varpi^*(\zeta),I_{a^+}^{\beta;\psi}\varpi^*(\zeta),\omega^*(\zeta)),\ \forall\,\zeta\in[a,b],\\ \Big({}^H\mathcal{D}_{a^+}^{p,q;\psi}+\nu\,{}^H\mathcal{D}_{a^+}^{p-1,q;\psi}\Big)\omega^*(\zeta)=g(\zeta,\varpi^*(\zeta),\omega^*(\zeta),I_{a^+}^{p;\psi}\omega^*(\zeta)),\ \forall\,\zeta\in[a,b],\\ \varpi^*(a)=0,\ \varpi^*(b)=\sum_{i=1}^{m-2}\lambda_i\omega^*(\theta_i),\\ \omega^*(a)=0,\ \omega^*(b)=\sum_{j=1}^{n-2}\mu_j\varpi^*(\xi_j).\end{cases}\tag{4.7}$$

Then in view of Lemma 2.2, the solution of (4.7) is

$$
\begin{aligned}
\varpi^*(\zeta) &= I_{a^+}^{\beta;\psi} f_{\varpi^*\omega^*}(\zeta) - kI_{a^+}^{1;\psi}\varpi^*(\zeta) + \frac{(\psi(\zeta)-\psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Big\{\Delta\Big[-\nu\sum_{i=1}^{m-2}\lambda_i I_{a^+}^{1;\psi}\omega^*(\theta_i) \\
&\quad + \sum_{i=1}^{m-2}\lambda_i I_{a^+}^{p;\psi} g_{\varpi^*\omega^*}(\theta_i) + kI_{a^+}^{1;\psi}\varpi^*(b) - I_{a^+}^{\beta;\psi} f_{\varpi^*\omega^*}(b)\Big] \\
&\quad + B\Big[-k\sum_{j=1}^{n-2}\mu_j I_{a^+}^{1;\psi}\varpi^*(\xi_j) + \sum_{j=1}^{n-2}\mu_j I_{a^+}^{\beta;\psi} f_{\varpi^*\omega^*}(\xi_j) + \nu I_{a^+}^{1;\psi}\omega(b) - I_{a^+}^{p;\psi} g_{\varpi^*\omega^*}(b)\Big]\Big\}.
\end{aligned}
$$

Consider

$$
\begin{aligned}
|\varpi(\zeta)-\varpi^*(\zeta)| &\leq |\varpi(\zeta)-\overline{\varpi}(\zeta)| + |\overline{\varpi}(\zeta)-\varpi^*(\zeta)| \\
&\leq \frac{S_\beta}{\epsilon_\beta} + [F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1](\|\varpi-\varpi^*\|_{\mathcal{H}} + \|\omega-\omega^*\|_{\mathcal{H}}) \\
&\leq \aleph_\beta\epsilon_\beta + \mathcal{K}_\beta\|\omega-\omega^*\|_{\mathcal{H}}.
\end{aligned}
$$

Hence

$$
\|\varpi-\varpi^*\|_{\mathcal{H}} \leq \aleph_\beta\epsilon_\beta + \mathcal{K}_\beta\|\omega-\omega^*\|_{\mathcal{H}}, \tag{4.8}
$$

where

$$
\begin{aligned}
\aleph_\beta &= \frac{S_\beta}{1-[F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1]}, \\
\mathcal{K}_\beta &= \frac{[F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1]}{1-[F_1\mathcal{P} + G_1\mathcal{Q} + X_1 + Y_1]}.
\end{aligned}
$$

Similarly,

$$
\|\omega-\omega^*\|_{\mathcal{H}} \leq \aleph_p\epsilon_p + \mathcal{K}_p\|\varpi-\varpi^*\|_{\mathcal{H}}, \tag{4.9}
$$

where

$$
\begin{aligned}
\aleph_p &= \frac{S_p}{1-[F_2\mathcal{P} + G_2\mathcal{Q} + X_2 + Y_2]}, \\
\mathcal{K}_p &= \frac{[F_2\mathcal{P} + G_2\mathcal{Q} + X_2 + Y_2]}{1-[F_2\mathcal{P} + G_2\mathcal{Q} + X_2 + Y_2]}.
\end{aligned}
$$

From (4.8) and (4.9), we write as

$$
\begin{aligned}
\|\varpi-\varpi^*\|_{\mathcal{H}} &\leq \aleph_\beta\epsilon_\beta + \mathcal{K}_\beta\|\omega-\omega^*\|_{\mathcal{H}}, \\
\|\omega-\omega^*\|_{\mathcal{H}} &\leq \aleph_p\epsilon_p + \mathcal{K}_p\|\varpi-\varpi^*\|_{\mathcal{H}},
\end{aligned}
$$

$$
\begin{bmatrix} 1 & -\mathcal{K}_\beta \\ -\mathcal{K}_p & 1 \end{bmatrix}
\begin{bmatrix} \|\varpi-\varpi^*\|_{\mathcal{H}} \\ \|\omega-\omega^*\|_{\mathcal{H}} \end{bmatrix}
\leq
\begin{bmatrix} S_\beta\epsilon_\beta \\ S_p\epsilon_p \end{bmatrix}.
$$

Solving the above inequality, we have

$$\begin{bmatrix} \|\varpi - \varpi^*\|_{\mathcal{H}} \\ \\ \|\omega - \omega^*\|_{\mathcal{H}} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Delta} & \frac{\mathcal{K}_\beta}{\Delta} \\ \\ \frac{\mathcal{K}_p}{\Delta} & \frac{1}{\Delta} \end{bmatrix} \begin{bmatrix} S_\beta \epsilon_\beta \\ \\ S_p \epsilon_p \end{bmatrix},$$

where

$$\Delta = 1 - \mathcal{K}_\beta \mathcal{K}_p > 0.$$

Further simplification gives

$$\begin{aligned} \|\varpi - \varpi^*\|_{\mathcal{H}} &\leq \frac{S_\beta \epsilon_\beta}{\Delta} + \frac{\mathcal{K}_\beta \aleph_p \epsilon_\beta}{\Delta}, \\ \|\omega - \omega^*\|_{\mathcal{H}} &\leq \frac{S_p \epsilon_p}{\Delta} + \frac{\mathcal{K}_p \aleph_\beta \epsilon_p}{\Delta}, \end{aligned}$$

from which we have

$$\|\varpi - \varpi^*\|_{\mathcal{H}} + \|\omega - \omega^*\|_{\mathcal{H}} \leq \frac{S_\beta \epsilon_\beta}{\Delta} + \frac{\mathcal{K}_\beta \aleph_p \epsilon_\beta}{\Delta} + \frac{S_p \epsilon_p}{\Delta} + \frac{\mathcal{K}_p \aleph_\beta \epsilon_p}{\Delta}. \tag{4.10}$$

Let $\max\{\epsilon_\beta, \epsilon_p\} = \epsilon$, then from (4.10), we get

$$\|(\varpi, \omega) - (\varpi^*, \omega^*)\|_{\mathcal{H} \times \mathcal{H}} \leq C_{\beta,p} \epsilon,$$

where

$$C_{\beta,p} = \frac{S_\beta}{\Delta} + \frac{\mathcal{K}_\beta \aleph_p}{\Delta} + \frac{S_p}{\Delta} + \frac{\mathcal{K}_p \aleph_\beta}{\Delta}.$$

□

Remark 4.2. By setting $\Theta(\epsilon)C_{\beta,p}\epsilon$, $\Theta(0) = 0$ in (4.10), then by Definition 4.2, the proposed system (1.2) is generalized HU stable.

- (H_3) Let Φ_β, $\Phi_p \in C([a, b], \Re^+)$ be an increasing functions and there exist Λ_{Φ_β}, $\Lambda_{\Phi_p} > 0$, such that for each $\zeta \in [a, b]$, the given integral inequalities

  $$I_{a^+}^{\beta;\psi} \Phi_\beta \leq \Lambda_{\Phi_\beta} \Phi_\beta(\zeta) \text{ and } I_{a^+}^{\beta-1;\psi} \Phi_\beta \leq \Lambda_{\Phi_\beta} \Phi_\beta(\zeta),$$

  and

  $$I_{a^+}^{p;\psi} \Phi_p \leq \Lambda_{\Phi_p} \Phi_p(\zeta) \text{ and } I_{a^+}^{p-1;\psi} \Phi_p \leq \Lambda_{\Phi_p} \Phi_p(\zeta),$$

  holds.

Remark 4.3. Under the hypothesis (H_3), (4.7) with Theorem 4.1, the proposed system (1.2) will be HU–Rassias and generalized HU–Rassias stable.

5. Example

Consider the following system

$$
\begin{cases}
\left({}^{H}\mathcal{D}^{\frac{3}{2},\frac{1}{3};(e^{\zeta})} + \frac{1}{55}\ {}^{H}\mathcal{D}^{\frac{1}{2},\frac{1}{3};(e^{\zeta})}\right)\varpi(\zeta) = f(\zeta, \varpi(\zeta), \omega(\zeta), z(\zeta)),\ \zeta \in [\frac{1}{4}, \frac{5}{2}], \\
\left({}^{H}\mathcal{D}^{\frac{4}{3},\frac{1}{2};(e^{\zeta})} + \frac{1}{58}\ {}^{H}\mathcal{D}^{\frac{1}{3},\frac{1}{2};(e^{\zeta})}\right)\omega(\zeta) = g(\zeta, \varpi(\zeta), \omega(\zeta), z(\zeta)),\ \zeta \in [\frac{1}{4}, \frac{5}{2}], \\
\varpi(\frac{1}{4}) = 0,\ \varpi(\frac{5}{2}) = \frac{1}{3}\omega(\frac{3}{4}) + \frac{1}{6}\omega(\frac{3}{2}) + \frac{1}{9}\omega(2), \\
\omega(\frac{1}{4}) = 0,\ \omega(\frac{5}{2}) = \frac{1}{5}\varpi(\frac{1}{2}) + \frac{2}{7}\varpi(\frac{5}{4}) + \frac{3}{8}\varpi(\frac{7}{4}) + \frac{4}{11}\varpi(\frac{9}{4}).
\end{cases}
\tag{5.1}
$$

Here $\psi(\zeta) = e^{\zeta}$, $\beta = \frac{3}{2}$, $\rho = \frac{1}{3}$, $p = \frac{4}{3}$, $q = \frac{1}{2}$, $k = \frac{1}{55}$, $\nu = \frac{1}{58}$, $\gamma = \frac{5}{3}$, $\delta = \frac{5}{3}$, $\lambda_1 = \frac{1}{10}$, $\lambda_2 = \frac{1}{9}$, $\lambda_3 = \frac{1}{7}$, $\mu_1 = \frac{1}{15}$, $\mu_2 = \frac{2}{17}$, $\mu_3 = \frac{1}{8}$, $\mu_4 = \frac{1}{11}$, $\theta_1 = \frac{1}{8}$, $\theta_2 = \frac{1}{6}$, $\theta_3 = \frac{1}{12}$, $\xi_1 = \frac{1}{6}$, $\xi_2 = \frac{2}{11}$, $\xi_3 = \frac{1}{14}$, $\xi_4 = \frac{1}{25}$, $a = \frac{1}{4}$, $b = \frac{5}{2}$, $m = 5$, $n = 6$.

From the given data, we can calculate $A \approx 5.44520$, $\Delta \approx 5.44520$, $\Omega \approx 3.41528$, $B \approx 1.13875$, $\Lambda = 25.76105$, $X_1 \approx 0.47925$, $Y_1 \approx 0.01130$, $F_1 \approx 60.58302$, $G_1 \approx 19.15967$, $X_2 \approx 0.06355$, $Y_2 \approx 0.49452$, $F_2 \approx 30.85423$, $G_2 \approx 52.60651$.

(i) Let the nonlinear functions f and g be defined on $[\frac{1}{4}, \frac{5}{2}]$ by

$$
\begin{aligned}
|f(\zeta, \omega_1, \omega_2, \omega_3)| &\leq \frac{1}{101(\zeta^2+1)} + \frac{1}{250(1+\zeta^2)^2}\left(3\omega_2 + \frac{|\omega_1|}{|\omega_1|+1}\right) + \frac{1}{105} sin\omega_3, \\
|g(\zeta, \omega_1, \omega_2, \omega_3)| &\leq \frac{\zeta}{56(\zeta^4+1)} + \frac{1}{206(1+\zeta^2)}\left(\frac{\omega_2}{3} + 3\omega_1\right) + \frac{1}{155} cos\omega_3.
\end{aligned}
$$

It is obvious to check that the above functions satisfy

$$
\begin{aligned}
|f(\zeta, \omega_1, \omega_2, \omega_3)| &\leq \frac{1}{101} + \frac{1}{250}(\omega_2 + \omega_1) + \frac{1}{105}\omega_3, \\
|g(\zeta, \omega_1, \omega_2, \omega_3)| &\leq \frac{1}{56} + \frac{1}{201}(\omega_2 + \omega_1) + \frac{1}{155}\omega_3,
\end{aligned}
$$

which can set $p_0 = \frac{1}{101}$, $p_1 = p_2 = \frac{1}{250}$, $p_3 = \frac{1}{105}$, $q_0 = \frac{1}{56}$, $q_1 = q_2 = \frac{1}{206}$, $q_3 = \frac{1}{155}$ as in the hypothesis (H_1) of Theorem 3.2. Then we can find that

$$\mathcal{M}_1 \approx 0.91469 < 1,\ \mathcal{M}_2 \approx 0.86329.$$

Thus all assumptions of Theorem 3.2 are satisfy. The conclusion of Theorem 3.2 implies that problem (5.1) has at least one solution on $[\frac{1}{4}, \frac{5}{2}]$. Now checking the Lipschitz condition for f and g, we obtain

$$
\begin{aligned}
|f(\zeta, \omega_1, \omega_2, \omega_3) - f(\zeta, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)| &\leq \frac{1}{285}(|\omega_1 - \bar{\omega}_1| + |\omega_2 - \bar{\omega}_2| + |\omega_3 - \bar{\omega}_3|), \\
|g(\zeta, \omega_1, \omega_2, \omega_3) - g(\zeta, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)| &\leq \frac{1}{250}(|\omega_1 - \bar{\omega}_1| + |\omega_2 - \bar{\omega}_2| + |\omega_3 - \bar{\omega}_3|).
\end{aligned}
$$

Then, by setting $\mathcal{P} = \frac{1}{285}$ and $Q = \frac{1}{250}$ the condition (H_2) of Theorem 3.3 is fulfilled. In addition we find that

$$[F_1 + F_2]\mathcal{P} + [G_1 + G_2]Q + [X_1 + X_2] + [Y_1 + Y_2] \approx 0.70614 < 1.$$

Therefore, the system (5.1) has a unique solution on $[\frac{1}{4}, \frac{5}{2}]$, by the benefit of the Theorem 3.3. Moreover,

$$\Delta = 1 - \mathcal{K}_\beta \mathcal{K}_p \approx 0.70244 > 0$$

is also satisfied. Thus system (5.1) is HU stable, generalized HU stable, HU–Rassias stable and generalized HU–Rassias stable.

Conflict of interest

The authors declare that they have no competing interest regarding this research work.

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