Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7519–7527.
DOI:10.3934/math.2022422
Received: 21 October 2021
Revised: 13 December 2021
Accepted: 21 December 2021
Published: 14 February 2022

Research article

Characterizations of local Lie derivations on von Neumann algebras

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Abstract: In this paper, we prove that every local Lie derivation on von Neumann algebras is a Lie derivation; and we show that if $\mathcal{M}$ is a type I von Neumann algebra with an atomic lattice of projections, then every local Lie derivation on $LS(\mathcal{M})$ is a Lie derivation.

Keywords: Lie derivation; local Lie derivation; von Neumann algebra; locally measurable operator
Mathematics Subject Classification: 46L50, 46L57, 47L35

1. Introduction

Let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for each A, B in $\mathcal{A}$. In particular, a derivation δ_M defined by $\delta_M(A) = MA - AM$ for every A in $\mathcal{A}$ is called an *inner derivation*, where M is a fixed element in $\mathcal{M}$.

In [31], S. Sakai proved that every derivation on von Neumann algebras is an inner derivation. In [12], E. Christensen showed that every derivation on nest algebras is an inner derivation. For more information on derivations and inner derivations, we refer to [13, 14, 19].

In [22, 24], R. Kadison, D. Larson and A. Sourour introduced the concept of local derivations. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local derivation* if for every A in $\mathcal{A}$, there exists a derivation δ_A (depending on A) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_A(A)$.

In [22], R. Kadison proved that every continuous local derivation from a von Neumann algebra into its dual Banach module is a derivation. In [24], D. Larson and A. Sourour proved that if X is a Banach space, then every local derivation on $B(X)$ is a derivation. In [21], B. Jonson showed that every local derivation from a C^*-algebra into its Banach bimodule is a derivation. In [15, 16], D. Hadwin and J. Li characterized local derivations on non self-adjoint operator algebras such as nest algebras and CDCSL algebras.

A linear mapping φ from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a *Lie derivation* if $\varphi([A,B]) = [\varphi(A),B] + [A,\varphi(B)]$ for each A, B in $\mathcal{A}$, where $[A,B] = AB - BA$ is the usual Lie product. A Lie derivation φ is said to be standard if it can be decomposed as $\varphi = \delta + \tau$, where δ is a derivation from $\mathcal{A}$ into $\mathcal{M}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A},\mathcal{M})$ such that $\tau([A,B]) = 0$ for each A and B in $\mathcal{A}$, where $\mathcal{Z}(\mathcal{A},\mathcal{M}) = \{M \in \mathcal{M} : AM = MA \text{ for every } A \text{ in } \mathcal{A}\}$.

In [20], B. Johnson proved that every continuous Lie derivation from a C^*-algebra into its Banach bimodule is standard. In [28], M. Mathieu and A. Villena proved that every Lie derivation on a C^*-algebra is standard. In [10], W. Cheung characterized Lie derivations on triangular algebras. In [27], F. Lu proved that every Lie derivation on a completely distributed commutative subspace lattice algebra is standard. In [4], D. Benkovič proved that every Lie derivation on a matrix algebra $M_n(\mathcal{A})$ is standard, where $n \geq 2$ and $\mathcal{A}$ is a unital algebra.

Similar to local derivations, In [9], L. Chen, F. Lu and T. Wang introduced the concept of local Lie derivations. A linear mapping φ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local Lie derivation* if for every A in $\mathcal{A}$, there exists a Lie derivation φ_A (depending on A) from $\mathcal{A}$ into $\mathcal{M}$ such that $\varphi(A) = \varphi_A(A)$.

In [9], L. Chen, F. Lu and T. Wang proved that every local Lie derivation on $B(X)$ is a Lie derivation, where X is a Banach space of dimension exceeding 2. In [8], L. Chen and F. Lu proved that every local Lie derivation on nest algebras is a Lie derivation. In [25, 26], D. Liu and J. Zhang proved that under certain conditions, every local Lie derivation on triangular algebras is a Lie derivation, and every local Lie derivation on factor von Neumann algebras with dimension exceeding 1 is a Lie derivation. In [18], J. He, J. Li, G. An and W. Huang proved that every local Lie derivation on some algebras such as finite von Neumann algebras, nest algebras, Jiang-Su algebra and UHF algebras is a Lie derivation.

Compare with the characterizations of derivations on Banach algebras, investigation of derivations on unbounded operator algebras begin much later.

In [32], I. Segal studied the theory of noncommutative integration, and introduces various classes of non-trivial $*$-algebras of unbounded operators. In this paper, we mainly consider the $*$-algebra $S(\mathcal{M})$ of all measurable operators and the $*$-algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with a von Neumann algebra $\mathcal{M}$. In [32], I. Segal showed that the algebraic and topological properties of the measurable operators algebra $S(\mathcal{M})$ are similar to the von Neumann algebra $\mathcal{M}$. If $\mathcal{M}$ is a commutative von Neumann algebra, then $\mathcal{M}$ is $*$-isomorphic to the algebra $L^\infty(\Omega,\Sigma,\mu)$ of all essentially bounded measurable complex functions on a measure space (Ω,Σ,μ); and $S(\mathcal{M})$ is $*$-isomorphic to the algebra $L^0(\Omega,\Sigma,\mu)$ of all measurable almost everywhere finite complex-valued functions on (Ω,Σ,μ). In [5], A. Ber, V. Chilin and F. Sukochev showed that there exists a derivation on $L^0(0,1)$ is not an inner derivation, and the derivation is discontinuous in the measure topology. This result means that the properties of derivations on $S(\mathcal{M})$ are different from the derivations on $\mathcal{M}$.

In [1, 2], S. Albeverio, S. Ayupov and K. Kudaybergenov studied the properties of derivations on various classes of measurable algebras. If $\mathcal{M}$ is a type I von Neumann algebra, in [1], the authors proved that every derivation on $LS(\mathcal{M})$ is an inner derivation if and only if it is $\mathcal{Z}(\mathcal{M})$ linear; in [2], the authors gave the decomposition form of derivations on $S(\mathcal{M})$ and $LS(\mathcal{M})$; they also prove that if $\mathcal{M}$ is a type I_∞ von Neumann algebra, then every derivation on $S(\mathcal{M})$ or $LS(\mathcal{M})$ is an inner derivation. If $\mathcal{M}$ is a properly infinite von Neumann algebra, in [6], A. Ber, V. Chilin and F. Sukochev proved that every derivation on $LS(\mathcal{M})$ is continuous with respect to the local measure topology $t(\mathcal{M})$; and in [7], A. Ber, V. Chilin and F. Sukochev showed that every derivation on $LS(\mathcal{M})$ is an inner derivation. In [3], S. Albeverio and S. Ayupov gave a characterization of local derivations on $S(\mathcal{M})$, where $\mathcal{M}$ is an abelian von Neumann algebra. In [17], D. Hadwin and J. Li proved that if $\mathcal{M}$ is a von Neumann algebra without abelian direct summands, then every local derivation on $LS(\mathcal{M})$ or $S(\mathcal{M})$ is a derivation. In [11], V. Chilin and I. Juraev showed that every Lie derivation on $LS(\mathcal{M})$ or $S(\mathcal{M})$ is standard.

This paper is organized as follows. In Section 2, we recall the definitions of algebras of measurable operators and local measurable operators.

In Section 3, we generalize the in [18, Corollary 3.2] and prove that every local Lie derivation on von Neumann algebras is a Lie derivation.

In Section 4, we prove that if $\mathcal{M}$ is a type I von Neumann algebra with an atomic lattice of projections, then every local Lie derivation on $LS(\mathcal{M})$ is a Lie derivation.

2. Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ is the center of $\mathcal{M}$, where

$$\mathcal{M}' = \{a \in B(\mathcal{H}) : ab = ba \text{ for every } b \text{ in } \mathcal{M}\}.$$

Denote by $\mathcal{P}(\mathcal{M}) = \{p \in \mathcal{M} : p = p^* = p^2\}$ the lattice of all projections in $\mathcal{M}$ and by $\mathcal{P}_{fin}(\mathcal{M})$ the set of all finite projections in $\mathcal{M}$. For each p and q in $\mathcal{P}(\mathcal{M})$, if we define the inclusion relation $p \subset q$ by $p \leq q$, then $\mathcal{P}(\mathcal{M})$ is a complete lattice. Suppose that $\{p_l\}_{l\in\lambda}$ is a family of projections in $\mathcal{M}$, we denote

$$\sup_{l\in\lambda} p_l = \overline{\bigcup_{l\in\lambda} p_l\mathcal{H}} \text{ and } \inf_{l\in\lambda} p_l = \bigcap_{l\in\lambda} p_l\mathcal{H}.$$

If $\{p_l\}_{l\in\lambda}$ is an orthogonal family of projections in $\mathcal{M}$, then we have that

$$\sup_{l\in\lambda} p_l = \sum_{l\in\lambda} p_l.$$

Let x be a closed densely defined linear operator on $\mathcal{H}$ with the domain $\mathcal{D}(x)$, where $\mathcal{D}(x)$ is a linear subspace of $\mathcal{H}$. x is said to be *affiliated* with $\mathcal{M}$, denote by $x\eta\mathcal{M}$, if $u^*xu = x$ for every unitary element u in $\mathcal{M}'$.

A linear operator affiliated with $\mathcal{M}$ is said to be *measurable* with respect to $\mathcal{M}$, if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow 1$, $p_n(\mathcal{H}) \subset \mathcal{D}(x)$ and $p_n^{\perp} = 1 - p_n \in \mathcal{P}_{fin}(\mathcal{M})$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Denote by $S(\mathcal{M})$ the set of all measurable operators affiliated with the von Neumann algebra $\mathcal{M}$.

A linear operator affiliated with $\mathcal{M}$ is said to be *locally measurable* with respect to $\mathcal{M}$, if there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $z_n \uparrow 1$ and $z_nx \in S(\mathcal{M})$ for every $n \in \mathbb{N}$. Denote by $LS(\mathcal{M})$ the set of all locally measurable operators affiliated with the von Neumann algebra $\mathcal{M}$.

In [29], M. Muratov and V. Chilin proved that $S(\mathcal{M})$ and $LS(\mathcal{M})$ are both unital $*$-algebras and $\mathcal{M} \subset S(\mathcal{M}) \subset LS(\mathcal{M})$; they also showed that if $\mathcal{M}$ is a finite von Neumann algebra or $\dim(\mathcal{Z}(\mathcal{M})) < \infty$, then $S(\mathcal{M}) = LS(\mathcal{M})$; if $\mathcal{M}$ is a type III von Neumann algebra and $\dim(\mathcal{Z}(\mathcal{M})) = \infty$, then $S(\mathcal{M}) = \mathcal{M}$ and $LS(\mathcal{M}) \neq \mathcal{M}$.

3. Local Lie derivations on von Neumann algebras

In this section, we consider local Lie derivations on von Neumann algebras. To prove our main theorem, we need the following lemma.

Lemma 3.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two unital algebras and $\mathcal{A} = \mathcal{A}_1 \bigoplus \mathcal{A}_2$. If the following five conditions hold:
(1) each Lie derivation on $\mathcal{A}$ is standard;
(2) each derivation on $\mathcal{A}$ is inner;
(3) each local derivation on $\mathcal{A}$ is a derivation;
(4) $\mathcal{Z}(\mathcal{A}_1) \cap [\mathcal{A}_1, \mathcal{A}_1] = \{0\}$;
(5) $\mathcal{A}_2 = [\mathcal{A}_2, \mathcal{A}_2]$,
then every local Lie derivation on $\mathcal{A}$ is a Lie derivation.

Proof. Denote the units of $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ by I, P and Q, respectively. For each A in $\mathcal{A}$, we have that $A = PA + QA = A_1 + A_2$, where $A_i \in \mathcal{A}_i,\ i = 1, 2$.

In the following we suppose that φ is a local Lie derivation on $\mathcal{A}$.

By the definition of local Lie derivation, we know that for every A_1 in $\mathcal{A}_1$, there exists a Lie derivation φ_{A_1} on $\mathcal{A}$ such that $\varphi(A_1) = \varphi_{A_1}(A_1)$. Since φ_{A_1} is standard and each derivation on $\mathcal{A}$ is inner, we can obtain that

$$\varphi(A_1) = \varphi_{A_1}(A_1) = \delta_{A_1}(A_1) + \tau_{A_1}(A_1) = [A_1, T_{A_1}] + P\tau_{A_1}(A_1) + Q\tau_{A_1}(A_1),$$

where δ_{A_1} is a derivation on $\mathcal{A}$, T_{A_1} is an element in $\mathcal{A}$, and τ_{A_1} is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ such that $\tau_{A_1}([\mathcal{A}, \mathcal{A}]) = 0$.

It means that φ has a decomposition at A_1. Next we show that the decomposition at A_1 is unique. Assume there is another decomposition at A_1, that is

$$\varphi(A_1) = \varphi'_{A_1}(A_1) = \delta'_{A_1}(A_1) + \tau'_{A_1}(A_1) = [A_1, T'_{A_1}] + P\tau'_{A_1}(A_1) + Q\tau'_{A_1}(A_1),$$

where δ'_{A_1} is a derivation on $\mathcal{A}$, T'_{A_1} is an element in $\mathcal{A}$ and τ'_{A_1} is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ such that $\tau'_{A_1}([\mathcal{A}, \mathcal{A}]) = 0$.

Then we have that

$$[A_1, T_{A_1}] + P\tau_{A_1}(A_1) + Q\tau_{A_1}(A_1) = [A_1, T'_{A_1}] + P\tau'_{A_1}(A_1) + Q\tau'_{A_1}(A_1).$$

Thus

$$[A_1, T_{A_1}] - [A_1, T'_{A_1}] = P\tau'_{A_1}(A_1) - P\tau_{A_1}(A_1) + Q\tau'_{A_1}(A_1) - Q\tau_{A_1}(A_1).$$

Since $[A_1, T_{A_1}] - [A_1, T'_{A_1}]$ and $P\tau'_{A_1}(A_1) - P\tau_{A_1}(A_1)$ belong to $\mathcal{A}_1$, and $Q\tau'_{A_1}(A_1) - Q\tau_{A_1}(A_1)$ belongs to $\mathcal{A}_2$, we have that $Q\tau'_{A_1}(A_1) - Q\tau_{A_1}(A_1) = 0$. Moreover, we can obtain that

$$[A_1, T_{A_1}] - [A_1, T'_{A_1}] = [A_1, PT_{A_1}] - [A_1, PT'_{A_1}] \in [\mathcal{A}_1, \mathcal{A}_1],$$

and

$$P\tau'_{A_1}(A_1) - P\tau_{A_1}(A_1) \in \mathcal{Z}(\mathcal{A}_1).$$

From the condition (4), it follows that $[A_1, T_{A_1}] - [A_1, T'_{A_1}] = P\tau'_{A_1}(A_1) - P\tau_{A_1}(A_1) = 0$. It implies that $\delta_{A_1}(A_1) = \delta'_{A_1}(A_1)$ and $\tau_{A_1}(A_1) = \tau'_{A_1}(A_1)$. Hence the decomposition is unique.

Now we have $\varphi|_{\mathcal{A}_1} = \delta_1 + \tau_1$, where δ_1 is a mapping from $\mathcal{A}_1$ into $\mathcal{A}_1$ such that $\delta_1(A_1) = [A_1, S_{A_1}]$ for some element S_{A_1} in $\mathcal{A}_1$, and τ_1 is a mapping from $\mathcal{A}_1$ into $\mathcal{Z}(\mathcal{A})$ such that $\tau_1([\mathcal{A}_1, \mathcal{A}_1]) = 0$.

Next we prove that δ_1 and τ_1 are linear mappings. For each A_1 and B_1 in $\mathcal{A}_1$, we have that

$$\varphi(A_1) = \delta_1(A_1) + \tau_1(A_1) = [A_1, S_{A_1}] + \tau_1(A_1),$$

$$\varphi(B_1) = \delta_1(B_1) + \tau_1(B_1) = [B_1, S_{B_1}] + \tau_1(B_1),$$

and

$$\varphi(A_1 + B_1) = \delta_1(A_1 + B_1) + \tau_1(A_1 + B_1) = [A_1 + B_1, S_{A_1+B_1}] + \tau_1(A_1 + B_1).$$

Since φ is additive, through a discussion similar to that before, it implies that

$$[A_1 + B_1, S_{A_1+B_1}] = [A_1, S_{A_1}] + [B_1, S_{B_1}]$$

and

$$\tau_1(A_1 + B_1) = \tau_1(A_1) + \tau_1(B_1).$$

It means that δ_1 and τ_1 are additive mappings. Using the same technique, we can prove that δ_1 and τ_1 are homogeneous. Hence δ_1 and τ_1 are linear mappings.

For every A_2 in $\mathcal{A}_2$, we have that

$$\varphi(A_2) = \varphi_{A_2}(A_2) = \delta_{A_2}(A_2) + \tau_{A_2}(A_2) = [A_2, T_{A_2}] + \tau_{A_2}(A_2),$$

where δ_{A_2} is a derivation on $\mathcal{A}$, T_{A_2} is an element in $\mathcal{A}$ and τ_{A_2} is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ such that $\tau_{A_2}([\mathcal{A}, \mathcal{A}]) = 0$. By condition (5), we have that $\tau_{A_2}(A_2) = 0$. Thus $\varphi(A_2) = [A_2, T_{A_2}] = [A_2, QT_{A_2}]$.

Let $\varphi|_{\mathcal{A}_2} = \delta_2$. Then we have $\delta_2(A_2) = [A_2, S_{A_2}]$ for some element S_{A_2} in $\mathcal{A}_2$. And obviously, δ_2 is linear.

Define two linear mappings as follows:

$$\delta(A) = \delta_1(A_1) + \delta_2(A_2), \quad \tau(A) = \tau_1(A_1),$$

for all $A = A_1 + A_2 \in \mathcal{A}$. By the previous discussion, τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ such that $\tau([\mathcal{A}, \mathcal{A}]) = 0$. In addition,

$$\delta(A) = \delta_1(A_1) + \delta_2(A_2) = [A_1, S_{A_1}] + [A_2, S_{A_2}] = [A_1 + A_2, S_{A_1} + S_{A_2}] = [A, S_{A_1} + S_{A_2}].$$

It means that δ is a local derivation. By condition (3), δ is a derivation. Notice that

$$\varphi(A) = \varphi(A_1) + \varphi(A_2) = \delta_1(A_1) + \tau_1(A_1) + \delta_2(A_2) = \delta(A) + \tau(A).$$

Hence φ is a standard Lie derivation. □

By Lemma 3.1, we have the following result.

Theorem 3.2. Every local Lie derivation on a von Neumann algebra is a Lie derivation.

Proof. Let $\mathcal{A}$ be a von Neumann algebra. It is well known that $\mathcal{A} = \mathcal{A}_1 \bigoplus \mathcal{A}_2$, where $\mathcal{A}_1$ is a finite von Neumann algebra, and $\mathcal{A}_2$ is a proper infinite von Neumann algebra.

By [28, Theorem 1.1], we know that every Lie derivation on $\mathcal{A}$ is standard, by [31, Theorem 1], we have that every derivation on $\mathcal{A}$ is inner, and by [21, Theorem 5.3], it follows that every local derivation on $\mathcal{A}$ is a derivation. Since $\mathcal{A}_2$ is a proper infinite von Neumann algebra, we known that $\mathcal{A}_2 = [\mathcal{A}_2, \mathcal{A}_2]$ (see in [33]).

Hence it is sufficient to prove that $\mathcal{Z}(\mathcal{A}_1) \cap [\mathcal{A}_1, \mathcal{A}_1] = \{0\}$. Since $\mathcal{A}_1$ is finite and by [23, Theorem 8.2.8], it follows that there is a center-valued trace τ on $\mathcal{A}_1$ such that $\tau(Z) = Z$ for every Z in $\mathcal{Z}(\mathcal{A}_1)$ and $\tau([A, B]) = 0$ for each A and B in $\mathcal{A}_1$. Suppose that $A \in \mathcal{Z}(\mathcal{A}_1) \cap [\mathcal{A}_1, \mathcal{A}_1]$, then we have that $\tau(A) = A$ and $\tau(A) = 0$, it implies that $A = 0$.

By Lemma 3.1, we know that every local Lie derivation on a von Neumann algebra is a Lie derivation. □

4. Local Lie derivations on algebras of locally measurable operators

In this section, we mainly consider local Lie derivations on algebras of all locally measurable operators affiliated with a type I von Neumann algebra. To prove the main result, we need the following lemmas.

Lemma 4.1. Suppose that $\mathcal{A}$ is a commutative unital algebra and $\mathcal{J} = M_n(\mathcal{A})$. Then $\mathcal{Z}(\mathcal{J}) \cap [\mathcal{J}, \mathcal{J}] = \{0\}$.

Proof. Let $\{e_{i,j}\}_{i,j=1}^n$ be the system of matrix units in $M_n(\mathcal{A})$. Then for every element A in $\mathcal{J}$, we have that $A = \sum_{i,j=1}^n a_{ij} e_{ij}$, where $a_{ij} \in \mathcal{A}$.

Define a linear mapping τ from $\mathcal{J}$ into $\mathcal{A}$ by $\tau(A) = \sum_{i=1}^n a_{ii}$ for every $A = \sum_{i,j=1}^n a_{ij} e_{ij} \in \mathcal{J}$. Since $\mathcal{A}$ is commutative, it is not difficult to verify that $\tau([A, B]) = 0$ for each A and B in $\mathcal{J}$.

It should be noticed that $\mathcal{Z}(\mathcal{J}) = \{A : A = \sum_{i=1}^n a e_{ii},\ a \in \mathcal{A}\}$. Suppose that $A = \sum_{i=1}^n a e_{ii}$ is an element in $\mathcal{Z}(\mathcal{J}) \cap [\mathcal{J}, \mathcal{J}]$, then by the definition of τ, we have that $\tau(A) = na$ and $\tau(A) = 0$. It implies that $A = 0$. □

Lemma 4.2. Suppose that $\mathcal{A} = \prod_{i \in \Lambda} \mathcal{A}_i$. If $\mathcal{Z}(\mathcal{A}_i) \cap [\mathcal{A}_i, \mathcal{A}_i] = \{0\}$ for every $i \in \Lambda$, then we have that $\mathcal{Z}(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}] = \{0\}$.

Proof. Let $A = \{a_i\}_{i \in \Lambda}$ be an element in $\mathcal{Z}(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}]$. Then for every $i \in \Lambda$, we have that $a_i \in \mathcal{Z}(\mathcal{A}_i) \cap [\mathcal{A}_i, \mathcal{A}_i]$. From the assumption, it follows that $a_i = 0$. Hence $A = 0$. □

Lemma 4.3. Suppose that $\mathcal{M}$ is a type I_∞ von Neumann algebra. Then $LS(\mathcal{M}) = [LS(\mathcal{M}), LS(\mathcal{M})]$.

Proof. By [30], we know that for every x in $LS(\mathcal{M})$, there exists a sequence $\{z_n\}$ of mutually orthogonal central projections in $\mathcal{M}$ with $\sum_{n=1}^\infty z_n = I$, such that $x = \sum_{n=1}^\infty z_n x$, and $z_n x \in \mathcal{M}$ for every $n \in \mathbb{N}$. Since $\mathcal{M}$ is a proper infinite von Neumann algebra, it is well known that $\mathcal{M} = [\mathcal{M}, \mathcal{M}]$. Thus we have that $z_n x = \sum_{i=1}^k [a_i^n, b_i^n]$, where $a_i^n, b_i^n \in \mathcal{M}$ for each n and i.

Set $s_i = \sum_{n=1}^\infty z_n a_i^n$ and $t_i = \sum_{n=1}^\infty z_n b_i^n$. By the definition of locally measurable operators, it is easy to show that s_i and t_i are two elements in $LS(\mathcal{M})$.

Since that $\{z_n\}$ are mutually orthogonal central projections, we can obtain that

$$[s_i, t_i] = [\sum_{n=1}^{\infty} z_n a_i^n, \sum_{n=1}^{\infty} z_n b_i^n] = \sum_{n=1}^{\infty} z_n [a_i^n, b_i^n],$$

moreover, we have that

$$\sum_{i=1}^{k} [s_i, t_i] = \sum_{i=1}^{k} \sum_{n=1}^{\infty} z_n [a_i^n, b_i^n] = \sum_{n=1}^{\infty} z_n (\sum_{i=1}^{k} [a_i^n, b_i^n]) = \sum_{n=1}^{\infty} z_n x = x.$$

It follows that $x \in [LS(\mathcal{M}), LS(\mathcal{M})]$. □

In the following we show the main result of this section.

Theorem 4.4. Suppose that $\mathcal{M}$ is a type I von Neumann algebra with an atomic lattice of projections. Then every local Lie derivation from $LS(\mathcal{M})$ into itself is a Lie derivation.

Proof. By [23, Theorem 6.5.2], we know that $\mathcal{M} = \mathcal{M}_1 \bigoplus \mathcal{M}_2$, where $\mathcal{M}_1$ is a type I_{finite} von Neumann algebra and $\mathcal{M}_2$ is a type I_∞ von Neumann algebra. Hence by [2, Proposition 1.1], we have that $LS(\mathcal{M}) \cong LS(\mathcal{M}_1) \bigoplus LS(\mathcal{M}_2)$.

In the following we will verify the conditions (1) to (5) in Lemma 3.1 one by one.

By [11, Theorem 1], we know that every Lie derivation on $LS(\mathcal{M})$ is standard; by [2, Corollary 5, 12], we know that every derivation on $LS(\mathcal{M})$ is inner for a von Neumann algebra with atomic lattice of projections.

It is proved in [16] that every local derivation on $LS(\mathcal{M})$ is a derivation for a von Neumann algebra without abelian direct summands. While for an abelian von Neumann algebra with atomic lattice of projections, by [3, Theorem 3.8] we know that every local derivation on $LS(\mathcal{M})$ is a derivation. Associated the two results, we can obtain each local derivation on $LS(\mathcal{M})$ is a derivation for a von Neumann algebra with atomic lattice of projections.

Since $\mathcal{M}_1$ is a type I_{finite} von Neumann algebra, we know that $\mathcal{M}_1 = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$, where each $\mathcal{A}_n$ is a homogenous type I_n von Neumann algebra. Hence $LS(\mathcal{M}_1) \cong \prod_{n=1}^{\infty} LS(\mathcal{A}_n)$. Since $\mathcal{A}_n$ is a homogenous type I_n von Neumann algebra, by [2] we know that $LS(\mathcal{A}_n) \cong M_n(\mathcal{Z}(LS(\mathcal{A}_n)))$. By Lemmas 4.1 and 4.2, we know that the condition (4) in Lemma 3.1 holds. And by Lemma 4.3, the condition (5) in Lemma 3.1 holds. □

5. Conclusions

In this paper, we prove that every local Lie derivation on von Neumann algebras is a Lie derivation; and we show that if $\mathcal{M}$ is a type I von Neumann algebra with an atomic lattice of projections, then every local Lie derivation on $LS(\mathcal{M})$ is a Lie derivation.

Acknowledgments

The authors thank the referee for his or her suggestions. This research was partially supported by the National Natural Science Foundation of China (Grant No. 11801342, 11801005, 11801050) and the Natural Science Foundation of Chongqing (cstc2020jcyj-msxmX0723). We are grateful to the anonymous reviewers and editors for their valuable comments which have enabled to improve the original version of this paper.

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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