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Research article

The complex Hessian quotient flow on compact Hermitian manifolds

Jundong Zhou* **and Yawei Chu**

School of Mathematics and Statistics, Fuyang Normal University, Fuyang 236037, China

* **Correspondence:** Email: zhoujundong109@163.com.

Abstract: In this paper, we consider the parabolic Hessian quotient equation on compact Hermitian manifolds. By setting up a priori estimates of the admissible solutions, we prove the long-time existence of the solution to the parabolic Hessian quotient equation and its convergence. As an application, we show the solvability of a class of complex Hessian quotient equations, which generalizes the relevant results.

Keywords: Hermitian manifolds; complex quotient flow; a priori estimates; long-time existence; convergence
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1. Introduction

Let (M,ω) be a complex n-dimensional compact Hermitian manifold and χ be a smooth real (1,1)-form on (M,ω). Γ_k^ω is the set of all real (1,1)-forms whose eigenvalues belong to the k-positive cone Γ_k. For any $u\in C^2(M)$, we can get a new (1,1)-form

$$\chi_u := \chi + \sqrt{-1}\partial\bar\partial u.$$

In any local coordinate chart, χ_u can be expressed as

$$\chi_u = \sqrt{-1}(\chi_{i\bar j} + u_{i\bar j})dz^i\wedge d\bar z^j.$$

In this article, we study the following form of parabolic Hessian quotient equations

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \log\frac{C_n^k\chi_u^k\wedge\omega^{n-k}}{C_n^l\chi_u^l\wedge\omega^{n-l}} - \log\phi(x,u),\ (x,t)\in M\times[0,T), \\ u(x,0) = u_0(x),\quad x\in M, \end{cases} \tag{1.1}$$

where $0\le l<k\le n$, $[0,T)$ is the maximum time interval in which the solution exists and $\phi(x,z)\in C^\infty(M\times\mathbb{R})$ is a given strictly positive function.

The study of the parabolic flows is motivated by complex equations

$$\chi_u^k \wedge \omega^{n-k} = \frac{C_n^l}{C_n^k}\phi(x,u)\chi_u^l \wedge \omega^{n-l},\ \chi_u \in \Gamma_k^\omega. \tag{1.2}$$

Equation (1.2) include some important geometry equations, for example, complex Monge-Ampère equation and Donaldson equation [6], which have attracted extensive attention in mathematics and physics since Yau's breakthrough in the Calabi conjecture [28]. Since Eq (1.2) are fully nonlinear elliptic, a classical way to solve them is the continuity method. Using this method, the complex Monge-Ampère equation

$$\chi_u^n = \phi(x)\omega^n,\ \chi_u \in \Gamma_n^\omega$$

was solved by Yau [28]. Donaldson equation

$$\chi_u^n = \frac{\int_M \chi^n}{\int_M \chi \wedge \omega^{n-1}}\chi_u \wedge \omega^{n-1},\ \chi_u \in \Gamma_n^\omega$$

was independently solved by Li-Shi-Yao [11], Collins-Szèkelyhidi [3] and Sun [17]. Equation (1.2) also include the complex k-Hessian equation and complex Hessian quotient equation, which, respectively, correspond to

$$C_n^k\chi_u^k \wedge \omega^{n-k} = \phi(x)\omega^n,\ \chi_u \in \Gamma_k^\omega,$$

$$\chi_u^k \wedge \omega^{n-k} = \frac{C_n^l}{C_n^k}\phi(x)\chi_u^l \wedge \omega^{n-l},\ \chi_u \in \Gamma_k^\omega.$$

Dinew and Kolodziej [7] proved a Liouville type theorem for m-subharmonic functions in $\mathbb{C}^n$, and combining with the estimate of Hou-Ma-Wu [10], solved the complex k-Hessian equation by using the continuity method. Under the cone condition, Sun [16] solved the complex Hessian quotient equation by using the continuity method. There have been many extensive studies for complex Monge-Ampère equation, Donaldson equation, the complex k-Hessian equation and the complex Hessian quotient equation on closed complex manifolds, see, e.g., [4, 12, 20, 22, 23, 29, 30]. When the right hand side function ϕ in Eq (1.2) depends on u, that is $\phi = \phi(x,u)$, it is interesting to ask whether we can solve them. We intend to solve (1.2) by the parabolic flow method.

Equation (1.1) covers some of the important geometric flows in complex geometry. If $k = n$ and $l = 0$, (1.1) is known as the complex Monge-Ampère flow

$$\frac{\partial u(x,t)}{\partial t} = \log\frac{\chi_u^n}{\omega^n} - \log\phi(x),\ (x,t) \in M \times [0,T),$$

which is equivalent to the Kähler-Ricci flow. The result of Yau [28] was reproduced by Cao [2] through Kähler-Ricci flow. Using the complex Monge-Ampère flow, similar results on a compact Hermitian manifold and a compact almost Hermitian manifold were proved by Gill [9] and Chu [5], respectively. To study the normalized twisted Chern-Ricci flow

$$\frac{\partial\omega_t}{\partial t} = -Ric(\omega_t) - \omega_t + \eta,$$

which is equivalent to the following Mong-Ampère flow

$$\frac{\partial\varphi}{\partial t} = \log\frac{(\theta_t + dd^c\varphi)^n}{\Omega} - \varphi,$$

Tô [25, 26] considered the following complex Monge-Ampère flow

$$\frac{\partial\varphi}{\partial t} = \log\frac{(\theta_t + dd^c\varphi)^n}{\Omega} - F(t, x, \varphi),$$

where Ω is a smooth volume form on M. From this, we can see that the given function ϕ depends on u in some geometric flows. If $l = 0$, (1.1) is called as the complex k-Hessian flow

$$\frac{\partial u(x,t)}{\partial t} = \log\frac{C_n^k\chi_u^k \wedge \omega^{n-k}}{\omega^n} - \log\phi(x,u),\ (x,t) \in M \times [0,T).$$

The solvability of complex k-Hessian flow was showed by Sheng-Wang [21].

In this paper, our research can be viewed as a generalization of Tô's work in [26] and Sheng-Wang's work in [21]. To solve the complex Hessian quotient flow, the condition of the parabolic C-subsolution is needed. According to Phong and Tô [14], we can give the definition of the parabolic C-subsolution to Eq (1.1).

Definition 1.1. Let $\underline{u}(x,t) \in C^{2,1}(M \times [0,T))$ and $\chi_{\underline{u}} \in \Gamma_k^{\omega}$, if there exist constants δ,$R > 0$, such that for any $(x,t) \in M \times [0,T)$,

$$\log\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} - \partial_t\underline{u} \le \log\phi(x,\underline{u}),\ \ \lambda - \lambda(\underline{u}) + \delta I \in \Gamma_n,$$

implies that

$$|\lambda| < R,$$

then $\underline{u}$ is said to be a parabolic C-subsolution of (1.1), where $\lambda(\underline{u})$ denotes eigenvalue set of $\chi_{\underline{u}}$.

Obviously, we can give the equivalent definition of parabolic C-subsolution of (1.1).

Definition 1.2. Let $\underline{u}(x,t) \in C^{2,1}(M \times [0,T))$ and $\chi_{\underline{u}} \in \Gamma_k^{\omega}$, if there exist constant $\widetilde{\delta} > 0$, for any $(x,t) \in M \times [0,T)$, such that

$$\lim_{\mu\to\infty}\log\frac{\sigma_k(\lambda(\underline{u}) + \mu e_i)}{\sigma_l(\lambda(\underline{u}) + \mu e_i)} > \frac{\partial\underline{u}}{\partial t} + \widetilde{\delta} + \log\phi(x,\underline{u}),\ 1 \le i \le n, \tag{1.3}$$

then $\underline{u}$ is said to be a parabolic C-subsolution of (1.1).

Our main result is

Theorem 1.3. *Let (M,g) a compact Hermitian manifold and χ be a smooth real $(1,1)$-form on M. Assume there exists a parabolic C-subsolution $\underline{u}$ for Eq (1.1) and*

$$\partial_t\underline{u} \ge \max\{\sup_M(\log\frac{\sigma_k(\lambda(u_0))}{\sigma_l(\lambda(u_0))} - \log\phi(x,u_0)), 0\}, \tag{1.4}$$

$$\frac{\phi_z(x,z)}{\phi} > c_\phi > 0, \tag{1.5}$$

where c_ϕ is a constant. Then there exits a unique smooth solution $u(x,t)$ to (1.1) all time with

$$\sup_{x\in M}(u_0(x) - \underline{u}(x,0)) = 0. \tag{1.6}$$

Moreover, $u(x,t)$ is C^∞ convergent to a smooth function u_∞, which solves Eq (1.2).

The rest of this paper is organized as follows. In Section 2, we give some important lemmas and estimate on $|u_t(x,t)|$. In Section 3, we prove C^0 estimates of Eq (1.1) by the parabolic C-subsolution condition and the Alexandroff-Bakelman-Pucci maximum principle. In Section 4, using the parabolic C-subsolution condition, we establish the C^2 estimate for Eq (1.1) by the method of Hou-Ma-Wu [10]. In Section 5, we adapt the blowup method of Dinew and Kolodziej [7] to obtain the gradient estimate. In Section 6, we give the proof of the long-time existence of the solution to the parabolic equation and its convergence, that is Theorem 1.3.

2. Preliminaries

In this section, we give some notations and lemmas. In holomorphic coordinates, we can set

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j = \sqrt{-1}\delta_{ij}dz^i \wedge d\bar{z}^j, \ \ \chi = \sqrt{-1}\chi_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

$$\chi_u = \sqrt{-1}(\chi_{i\bar{j}} + u_{i\bar{j}})dz^i \wedge d\bar{z}^j = \sqrt{-1}X_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

$$\chi_{\underline{u}} = \sqrt{-1}(\chi_{i\bar{j}} + \underline{u}_{i\bar{j}})dz^i \wedge d\bar{z}^j = \sqrt{-1}\underline{X}_{i\bar{j}}dz^i \wedge d\bar{z}^j.$$

$\lambda(u)$ and $\lambda(\underline{u})$ denote the eigenvalue set of $\{X_{i\bar{j}}\}$ and $\{\underline{X}_{i\bar{j}}\}$ with respect to $\{g_{i\bar{j}}\}$, respectively. In local coordinates, (1.1) can be written as

$$\partial_t u = \log \frac{\sigma_k(\lambda(u))}{\sigma_l(\lambda(u))} - \log \phi(x,u). \tag{2.1}$$

For simplicity, set

$$F(\lambda(u)) = \log \frac{\sigma_k(\lambda(u))}{\sigma_l(\lambda(u))},$$

then (2.1) is abbreviated as

$$\partial_t u = F(\lambda(u)) - \log \phi(x,u). \tag{2.2}$$

We use the following notation

$$F^{i\bar{j}} = \frac{\partial F}{\partial X_{i\bar{j}}}, \ \ \mathcal{F} = \sum_i F^{i\bar{i}}, \ \ F^{i\bar{j},p\bar{q}} = \frac{\partial^2 F}{\partial X_{i\bar{j}} \partial X_{p\bar{q}}}.$$

For any $x_0 \in M$, we can choose a local holomorphic coordinates such that the matrix $\{X_{i\bar{j}}\}$ is diagonal and $X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}}$, then we have, at $x_0 \in M$,

$$\lambda(u) = (\lambda_1, \cdots, \lambda_n) = (X_{1\bar{1}}, \cdots, X_{n\bar{n}}),$$

$$F^{i\bar{j}} = F^{i\bar{i}}\delta_{ij} = \Big(\frac{\sigma_{k-1}(\lambda|i)}{\sigma_k} - \frac{\sigma_{l-1}(\lambda|i)}{\sigma_l}\Big)\delta_{ij}, \ \ F^{1\bar{1}} \leq \cdots \leq F^{n\bar{n}}.$$

To prove a priori C^0-estimate for solution to Eq (1.1), we need the following variant of the Alexandroff-Bakelman-Pucci maximum principle, which is Proposition 10 in [20].

Lemma 2.1. *[20] Let $v : B(1) \to \mathbb{R}$ be a smooth function, which meets the condition $v(0)+\epsilon \le \inf_{\partial B(1)} v$, where $B(1)$ denotes the unit ball in $\mathbb{R}^n$. Define the set*

$$\Omega = \left\{ \begin{array}{c} x \in B(1) : |Dv(x)| < \frac{\epsilon}{2}, \quad and \\ v(y) \ge v(x) + Dv(x) \cdot (y - x), \forall y \in B(1) \end{array} \right\}.$$

Then there exists a costant $c_0 > 0$ such that

$$c_0 \epsilon^n \le \int_\Omega \det(D^2 v).$$

Next, we give an estimate on $|u_t(x,t)|$.

Lemma 2.2. *Under the assumption of Theorem 1.3, let $u(x,t)$ be a solution to (1.1). Then for any $(x,t) \in M \times [0,T)$, we have*

$$\min\{\inf_M u_t(x,0), 0\} \le u_t(x,t) \le \max\{\sup_M u_t(x,0), 0\}. \tag{2.3}$$

Furthermore, there is a constant $C > 0$ such that

$$\sup_{M\times[0,T)} |\partial_t u(x,t)| \le \sup_M |\partial_t u(x,0)| \le C,$$

where C depends on $H = |u_0|_{C^2(M)}$ and $|\phi|_{C^0(M\times[-H,H])}$.

Proof. Differentiating (2.2) on both sides simultaneously at t, we obtain

$$(u_t)_t = F^{i\bar{j}} X_{i\bar{j}t} - \frac{\phi_z}{\phi} u_t = F^{i\bar{j}} (u_t)_{i\bar{j}} - \frac{\phi_z}{\phi} u_t. \tag{2.4}$$

Set $u_t^\varepsilon = u_t - \varepsilon t$, $\varepsilon > 0$. For any $T' \in (0,T)$, suppose u_t^ε achieves its maximum M_t at $(x_0, t_0) \in M\times[0,T']$. Without loss of generality, we may suppose $M_t \ge 0$. If $t_0 > 0$, From the parabolic maximum principle and (2.4), we get

$$\begin{aligned} 0 \le& (u_t^\varepsilon)_t - F^{i\bar{j}} (u_t^\varepsilon)_{i\bar{j}} + \frac{\phi_z}{\phi} u_t^\varepsilon \\ \le& (u_t)_t - \varepsilon - F^{i\bar{j}} (u_t)_{i\bar{j}} + \frac{\phi_z}{\phi} u_t - \varepsilon \frac{\phi_z}{\phi} t_0 \\ \le& -\varepsilon - \varepsilon \frac{\phi_z}{\phi} t_0. \end{aligned}$$

This is obviously a contradiction, so $t_0 = 0$ and

$$\sup_{M\times[0,T']} u_t^\varepsilon(x,t) = \sup_M u_t(x,0),$$

that is

$$\sup_{M\times[0,T']} u_t(x,t) = \sup_{M\times[0,T']} (u_t^\varepsilon(x,t) + \varepsilon t) \le \sup_M u_t(x,0) + \varepsilon T'.$$

Letting $\varepsilon \to 0$, we obtain

$$\sup_{M\times[0,T']} u_t(x,t) \le \sup_M u_t(x,0).$$

Since $T' \in (0,T)$ is arbitrary, we have

$$\sup_{M\times[0,T)} u_t(x,t) \le \sup_M u_t(x,0). \tag{2.5}$$

Similarly, setting $u_t^\varepsilon = u_t + \varepsilon t,\ \varepsilon > 0$, we obtain

$$\inf_{M\times[0,T)} u_t(x,t) \ge \inf_M u_t(x,0). \tag{2.6}$$

(2.1) yields

$$|u_t(x,0)| = |\log\frac{\sigma_k(\lambda(u_0))}{\sigma_l(\lambda(u_0))} - \log\phi(x,u_0)| \le C. \tag{2.7}$$

Combining (2.5)–(2.7), we complete the proof of Proposition 2.2.

□

From the concavity of $F(\lambda(u))$ and the condition of the parabolic C-subsolution, we give the following lemma, which plays an important role in the estimation of C^2.

Lemma 2.3. *Under the assumption of Theorem 1.3 and assuming that $X_{1\bar{1}} \ge \cdots \ge X_{n\bar{n}}$, there exists two positive constants N and θ such that we have either*

$$F^{i\bar{i}}(\underline{u}_{i\bar{i}} - u_{i\bar{i}}) - \partial_t(\underline{u} - u) \ge \theta(1+\mathcal{F}) \tag{2.8}$$

or

$$F^{1\bar{1}} \ge \frac{\theta}{N}(1+\mathcal{F}). \tag{2.9}$$

Proof. Since $\underline{u}$ is a parabolic C-subsolution to Eq (1.1), from Definition 1.2, there are uniform constants $\widetilde{\delta} > 0$ and $N > 0$, such that

$$\log\frac{\sigma_k(\lambda(\underline{u}) + Ne_1)}{\sigma_l(\lambda(\underline{u}) + Nue_1)} > \frac{\partial \underline{u}}{\partial t} + \widetilde{\delta} + \log\phi(x,\underline{u}). \tag{2.10}$$

If $\epsilon > 0$ is sufficiently small, it can be obtained from (2.10)

$$\log\frac{\sigma_k(\lambda(\underline{u}) - \epsilon I + Ne_1)}{\sigma_l(\lambda(\underline{u}) - \epsilon I + Nue_1)} \ge \frac{\partial \underline{u}}{\partial t} + \widetilde{\delta} + \log\phi(x,\underline{u}).$$

Set $\lambda' = \lambda(\underline{u}) - \epsilon I + Ne_1$, then

$$F(\lambda') \ge \frac{\partial \underline{u}}{\partial t} + \widetilde{\delta} + \log\phi(x,\underline{u}). \tag{2.11}$$

Using the concavity of $F(\lambda(u))$ gives

$$\begin{aligned} F^{i\bar{i}}(\underline{u}_{i\bar{i}} - u_{i\bar{i}}) &= F^{i\bar{i}}(\{\underline{X}_{i\bar{i}} - X_{i\bar{i}}) \\ &= F^{i\bar{i}}(\{\underline{X}_{i\bar{i}} - \epsilon\delta_{ii} + N\delta_{i1} - X_{i\bar{i}}) + \epsilon\mathcal{F} - NF^{1\bar{1}} \end{aligned} \tag{2.12}$$

$$\geq F(\lambda') - F(\lambda(u)) + \epsilon\mathcal{F} - NF^{1\bar{1}}.$$

From Lemma 2.2 and (1.4), we obtain

$$\underline{u}_t(x,t) \geq u_t(x,t),\ \forall (x,t) \in M \times [0,T). \tag{2.13}$$

In addition, it can be obtained from the condition (1.6)

$$\underline{u}(x,0) \geq u(x,0),\ \forall x \in M \times [0,T). \tag{2.14}$$

(2.13) and (2.14) deduce that

$$\underline{u}(x,t) \geq u(x,t),\ \forall (x,t) \in M \times [0,T). \tag{2.15}$$

It follows from this that

$$\phi(x,\underline{u}) \geq \phi(x,u). \tag{2.16}$$

Combining (2.2), (2.11) and (2.16) gives that

$$\begin{aligned} F(\lambda') - F(\lambda(u)) \geq& \underline{u}_t(x,t) - u_t(x,t) + \widetilde{\delta} + \log\phi(x,\underline{u}) - \log\phi(x,u) \\ \geq& \underline{u}_t(x,t) - u_t(x,t) + \widetilde{\delta}. \end{aligned} \tag{2.17}$$

Put (2.17) into (2.12)

$$F^{i\bar{i}}(\underline{u}_{i\bar{i}} - u_{i\bar{i}}) \geq \underline{u}_t(x,t) - u_t(x,t) + \widetilde{\delta} + \epsilon\mathcal{F} - NF^{1\bar{1}} \geq \widetilde{\delta} + \epsilon\mathcal{F} - NF^{1\bar{1}}.$$

Let

$$\theta = \min\{\frac{\widetilde{\delta}}{2},\ \frac{\epsilon}{2}\}.$$

If $F^{1\bar{1}}N \leq \theta(1+\mathcal{F})$, Inequality (2.8) is obtained, otherwise Inequality (2.9) must be true. □

3. C^0 Estimates

In this section, we prove the C^0 estimates by the existence of the parabolic C-subsolution and the Alexandroff-Bakelman-Pucci maximum principle.

Proposition 3.1. *Under the assumption of Theorem1.3, let $u(x,t)$ be a solution to (1.1). Then there exists a constant $C > 0$ such that*

$$|u(x,t)|_{C^0(M\times[0,T))} \leq C,$$

where C depends on $|u_0|_{C^2(M)}$ and $|\underline{u}|_{C^2(M\times[0,T))}$.

Proof. Combining (2.13), (2.14) and $\frac{\partial\phi(x,z)}{\partial z} \geq 0$ yields

$$\underline{u}_t(x,t) + \log\phi(x,\underline{u}) \geq u_t(x,t) + \log\phi(x,u). \tag{3.1}$$

Let's rewrite Eq (2.2) as

$$F(\lambda(u)) = \partial_t u + \log\phi(x,u). \tag{3.2}$$

when fix $t \in [0,T)$, Eq (3.2) is elliptic. From (3.1), we see that the parabolic C-subsolution $\underline{u}(x,t)$ is a C-subsolution to Eq (3.2) in the elliptic sense. From (2.15), we have

$$\sup_{M\times[0,T)} (u-\underline{u}) = 0.$$

Our goal is to obtain a lower bound for $L = \inf_{M\times t}(u-\underline{u})$. Note that $\lambda(u) \in \Gamma_k$, which implies that $\lambda(u) \in \Gamma_1$, then $\Delta(u-\underline{u}) \geq -\widetilde{C}$, where Δ is the complex Laplacian with respect to ω. According to Tosatti-Weinkove's method [22], we can prove that $\|u-\underline{u}\|_{L^1(M)}$ is bounded uniformly. Let $G : M\times M \to \mathbb{R}$ be the associated Green's function, then, by Yau [28], there is a uniform constant K such that

$$G(x,y)+K \geq 0,\ \forall (x,y) \in M\times M, \text{ and } \int_{y\in M} G(x,y)\omega^n(y) = 0.$$

Since

$$\sup_{M\times[0,T)} (u-\underline{u}) = 0,$$

then for fixed $t \in [0,T)$ there exists a point $x_0 \in M$ such that $(u-\underline{u})(x_0,t) = 0$. Thus

$$\begin{aligned}(u-\underline{u})(x_0,t) &= \int_M (u-\underline{u})d\mu - \int_{y\in M} G(x_0,y)\Delta(u-\underline{u})(y)\omega^n(y)\\ &= \int_M (u-\underline{u})d\mu - \int_{y\in M} (G(x_0,y)+K)\Delta(u-\underline{u})(y)\omega^n(y)\\ &\leq \int_M (u-\underline{u})d\mu + \widetilde{C}K\int_M \omega^n,\end{aligned}$$

that is

$$\int_M (\underline{u}-u)d\mu = \int_M |(u-\underline{u})|d\mu \leq \widetilde{C}K\int_M \omega^n.$$

Let us work in local coordinates, for which the infimum L is achieved at the origin, that is $L = u(0,t) - \underline{u}(0,t)$. We write $B(1) = \{z : |z| < 1\}$. Let $v = u - \underline{u} + \epsilon|z|^2$, for a small $\epsilon > 0$. We have $\inf v = L = v(0)$, and $v(z) \geq L + \epsilon$ for $z \in \partial B(1)$. From Lemma 2.1, we get

$$c_0\epsilon^{2n} \leq \int_\Omega \det(D^2 v). \tag{3.3}$$

At the same time, if $x \in \Omega$, then $D^2 v(x) \geq 0$ implies that

$$u_{i\bar{j}}(x) - \underline{u}_{i\bar{j}}(x) + \epsilon\delta_{ij} \geq 0.$$

If ϵ is sufficiently small, then

$$\lambda(u) \in \lambda(\underline{u}) - \delta I + \Gamma_n.$$

Set $\mu = \lambda(u) - \lambda(\underline{u})$. Since $\lambda(u)$ satisfies Eq (3.2), then

$$F(\lambda(\underline{u})+\mu) = \partial_t u + \log\phi(x,u),\quad \mu+\delta I \in \Gamma_n. \tag{3.4}$$

$\underline{u}$ is a C-subsolution to Eq (3.2) in the elliptic sense, so there is a uniform constant $R > 0$, such that

$$|\mu| \leq R,$$

which means $|v_{i\bar{j}}| \leq C$, for any $x \in \Omega$. As in Blocki [1], for $x \in \Omega$, we have $D^2v(x) \geq 0$ and so

$$D^2v(x) \leq 2^{2n} \det(v_{i\bar{j}})^2 \leq C'.$$

From this and (3.3), we obtain

$$c_0\epsilon^{2n} \leq \int_\Omega det(D^2v) \leq C' \cdot \mathrm{vol}(\Omega). \tag{3.5}$$

On the other hand, by the definition of Ω in Lemma 2.1, for $x \in \Omega$, we get

$$v(0) \geq v(x) - Dv(x) \cdot x > v(x) - \frac{\epsilon}{2},$$

and so

$$|v(x)| > |L + \frac{\epsilon}{2}|.$$

It follows that

$$\int_M |v(x)| \geq \int_\Omega |v(x)| \geq |L + \frac{\epsilon}{2}| \cdot \mathrm{vol}(\Omega). \tag{3.6}$$

Since $\|u - \underline{u}\|_{L^1(M)}$ is bounded uniformly, $\int_M |v(x)|$ is also bounded uniformly. If L is very large, Inequality (3.6) contradicts (3.5), which means that L has a lower bound. For any $t \in [0, T)$, Inequality (3.1) holds, thus

$$|u(x,t)|_{C^0(M\times[0,T))} \leq |L| + \sup_{M\times[0,T)} |\underline{u}| \leq C.$$

□

4. C^2 Estimates

In this section, we prove that the second-order estimates are controlled by the square of the gradient estimate linearly. Our calculation is a parabolic version of that in Hou-Ma-Wu [10].

Proposition 4.1. *Under the assumption of Theorem1.3, let $u(x,t)$ be a solution to (1.1). Then there exists a constant $\widetilde{C}$ such that*

$$\sup_{M\times[0,T)} |\sqrt{-1}\partial\bar{\partial}u| \leq \widetilde{C}(\sup_{M\times[0,T)} |\nabla u|^2 + 1),$$

where $\widetilde{C}$ depends χ, ω, $|\phi|_{C^2(M\times[-C,C])}$, $|\underline{u}|_{C^2(M\times[0,T))}$, $|\partial_t\underline{u}|_{C^0(M\times[0,T))}$ and $|u_0|_{C^2(M)}$.

Proof. Let $\lambda(u) = (\lambda_1, \ldots, \lambda_n)$ and λ_1 is the maximum eigenvalue. For any $T' < T$, we consider the following function

$$W(x,t) = \log\lambda_1 + \varphi(|\nabla u(x,t)|^2) + \psi(u(x,t) - \underline{u}(x,t)),\ (x,t) \in M \times [0, T'], \tag{4.1}$$

where φ and ψ are determined later. We want to apply the maximum principle to the function W. Since the eigenvalues of the matrix $\{X_{i\bar{j}}\}$ with respect to ω need not be distinct at the point where W achieves its maximum, we will perturb $\{X_{i\bar{j}}\}$ following the technique of [20]. Let W achieve its maximum at

$(x_0, t_0) \in M \times [0, T']$. Near (x_0, t_0), we can choose local coordinates such that $\{X_{i\bar{j}}\}$ is diagonal with $X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}}$, and $\lambda(u) = (X_{1\bar{1}}, \cdots, X_{n\bar{n}})$. Let D be a diagonal matrix such that $D^{11} = 0$ and $0 < D^{22} < \cdots < D^{nn}$ are small, satisfying $D^{nn} < 2D^{22}$. Define the matrix $\widetilde{X} = X - D$. At (x_0, t_0), $\widetilde{X}$ has eigenvalues

$$\widetilde{\lambda}_1 = \lambda_1,\ \widetilde{\lambda}_i = \lambda_i - D^{ii},\ n \geq i \geq 2.$$

Since all the eigenvalues of $\widetilde{X}$ are distinct, we can define near (x_0, t_0) the following smooth function

$$\widetilde{W} = \log \widetilde{\lambda}_1 + \varphi(|\nabla u|^2) + \psi(u - \underline{u}), \tag{4.2}$$

where

$$\varphi(s) = -\frac{1}{2}\log(1 - \frac{s}{2K}),\ 0 \leq s \leq K - 1,$$
$$\psi(s) = -E\log(1 + \frac{s}{2L}),\ -L + 1 \leq s \leq L - 1,$$
$$K = \sup_{M\times[0,T']} |\nabla u|^2 + 1,$$
$$L = \sup_{M\times[0,T']} |u| + \sup_{M\times[0,T']} |\underline{u}| + 1,$$
$$E = 2L(C_1 + 1),$$

and $C_1 > 0$ is to be chosen later. Direct calculation yields

$$0 < \frac{1}{4K} \leq \varphi' \leq \frac{1}{2K},\ \ \varphi'' = 2(\varphi')^2 > 0, \tag{4.3}$$

and

$$C_1 + 1 \leq -\psi' \leq 2(C_1 + 1),\ \ \psi'' \geq \frac{4\epsilon}{1-\epsilon}(\psi')^2, \quad \forall\, \epsilon \leq \frac{1}{4E+1}. \tag{4.4}$$

Without loss of generality, we can assume that $\lambda_1 > 1$. From here on, all calculations are done at (x_0, t_0). From the maximum principle, calculating the first and second derivatives of the function $\widetilde{W}$ gives

$$0 = \widetilde{W}_i = \frac{\widetilde{\lambda}_{1,i}}{\lambda_1} + \varphi'(|\nabla u|^2)_i + \psi'(u - \underline{u})_i, \quad 1 \leq i \leq n, \tag{4.5}$$

$$\begin{aligned} 0 \geq \widetilde{W}_{i\bar{i}} =& \frac{\widetilde{\lambda}_{1,i\bar{i}}}{\lambda_1} - \frac{\widetilde{\lambda}_{1,i}\widetilde{\lambda}_{1,\bar{i}}}{\lambda_1^2} + \varphi'(|\nabla u|^2)_{i\bar{i}} + \varphi''|(|\nabla u|^2)_i|^2 \\ &+ \psi'(u - \underline{u})_{i\bar{i}} + \psi''|(u - \underline{u})_i|^2. \end{aligned} \tag{4.6}$$

$$0 \leq \widetilde{W}_t = \frac{\widetilde{\lambda}_{1,t}}{\lambda_1} + \varphi'(|\nabla u|^2)_t + \psi'(u - \underline{u})_t. \tag{4.7}$$

Define

$$\mathcal{L} := F^{i\bar{j}}\nabla_{\frac{\partial}{\partial \bar{z}_j}}\nabla_{\frac{\partial}{\partial z_i}} - \partial_t.$$

Obviously,

$$0 \geq \mathcal{L}\widetilde{W} = \mathcal{L}\log\widetilde{\lambda}_1 + \mathcal{L}\varphi(|\nabla u|^2) + \mathcal{L}\psi(u - \underline{u}). \tag{4.8}$$

Next, we will estimate the terms in (4.8). Direct calculation shows that

$$\mathcal{L}\log\widetilde{\lambda}_1 = F^{i\bar{i}}\frac{\widetilde{\lambda}_{1,i\bar{i}}}{\lambda_1} - F^{i\bar{i}}\frac{|\widetilde{\lambda}_{1,i}|^2}{\lambda_1^2} - \frac{\widetilde{\lambda}_{1,t}}{\lambda_1}. \tag{4.9}$$

According to Inequality (78) in [20], we have

$$\widetilde{\lambda}_{1,i\bar{i}} \geq X_{i\bar{i}1\bar{1}} - 2Re(X_{i\bar{1}1}\overline{T^1_{i1}}) - C_0\lambda_1, \tag{4.10}$$

where C_0 depending χ, ω, $|\phi|_{C^2(M\times[-C,C])}$, $|\underline{u}|_{C^2(M\times[0,T))}$, $|\partial_t\underline{u}|_{C^0(M\times[0,T))}$ and $|u_0|_{C^2(M)}$). From here on, C_0 can always absorb the constant it represents before, and can change from one line to the next, but it does not depend on the parameter we choose later. By calculating the covariant derivatives of (4.7) in the direction $\frac{\partial}{\partial z^1}$ and $\frac{\partial}{\partial \bar{z}^1}$, we obtain

$$u_{t1} = F^{i\bar{i}}X_{i\bar{i}1} - (\log\phi)_1 - (\log\phi)_u u_1, \tag{4.11}$$

and

$$\begin{aligned} u_{t1\bar{1}} =& F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}} + F^{i\bar{i}}X_{i\bar{i}1\bar{1}} - (\log\phi)_{1\bar{1}} - (\log\phi)_{1u}u_{\bar{1}} \\ &- (\log\phi)_{u\bar{1}}u_1 - (\log\phi)_{uu}|u_1|^2 - (\log\phi)_u u_{1\bar{1}}. \end{aligned} \tag{4.12}$$

Notice that

$$\begin{aligned} X_{1\bar{1}i} =& \chi_{1\bar{1}i} + u_{1\bar{1}i} \\ =& (\chi_{11i} - \chi_{i11} + T^p_{i1}\chi_{p\bar{1}}) + X_{i\bar{1}1} - T^1_{i1}\lambda_1, \end{aligned} \tag{4.13}$$

therefore

$$|X_{1\bar{1}i}|^2 \leq |X_{i\bar{1}1}|^2 - 2\lambda_1 Re(X_{i\bar{1}1}\overline{T^1_{i1}}) + C_0(\lambda_1^2 + |X_{1\bar{1}i}|). \tag{4.14}$$

Combining (4.14) with

$$\widetilde{\lambda}_{1,i} = X_{1\bar{1}i} - (D^{11})_i,$$

gives

$$\begin{aligned} -F^{i\bar{i}}\frac{|\widetilde{\lambda}_{1,i}|^2}{\lambda_1^2} =& -F^{i\bar{i}}\frac{|X_{1\bar{1}i}|^2}{\lambda_1^2} + \frac{2}{\lambda_1^2}F^{i\bar{i}}Re(X_{1\bar{1}i}(D^{11})_{\bar{i}}) - \frac{F^{i\bar{i}}|(D^{11})_i|^2}{\lambda_i^2} \\ \geq& -F^{i\bar{i}}\frac{|X_{1\bar{1}i}|^2}{\lambda_1^2} - \frac{C_0}{\lambda_1^2}F^{i\bar{i}}|X_{1\bar{1}i}| - C_0\mathcal{F} \\ \geq& -F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} + \frac{2}{\lambda_1}F^{i\bar{i}}Re(X_{i\bar{1}1}\overline{T^1_{i1}}) - \frac{C_0}{\lambda_1^2}F^{i\bar{i}}|X_{1\bar{1}i}| - C_0\mathcal{F}. \end{aligned} \tag{4.15}$$

Let's set $\lambda_1 \geq K$, otherwise the proof is completed. Putting 4.10–4.12 and (4.15) into (4.9) yields

$$\mathcal{L}\log\widetilde{\lambda}_1 \geq \frac{-F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}}}{\lambda_1} - F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} - \frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} - C_0\mathcal{F}$$

$$
\begin{aligned}
&+\frac{(\log\phi)_{1\bar{1}}+(\log\phi)_{1u}u_{\bar{1}}+(\log\phi)_{u\bar{1}}u_1+(\log\phi)_{uu}|u_1|^2-(\log\phi)_u\chi_{1\bar{1}}}{\lambda_1}\\
\geq&\frac{-F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}}}{\lambda_1}-F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2}-\frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1}-C_0\mathcal{F}-C_0
\end{aligned}\tag{4.16}
$$

A simple computation gives

$$\mathcal{L}\varphi(|\nabla u|^2)=\varphi' F^{i\bar{i}}(|\nabla u|^2)_{i\bar{i}}+\varphi'' F^{i\bar{i}}|(|\nabla u|^2)_i|^2-\varphi'(|\nabla u|^2)_t.\tag{4.17}$$

Next, we estimate the formula (4.17). Differentiating Eq (2.2), we have

$$(u_t)_p=F^{i\bar{i}}X_{i\bar{i}p}-(\log\phi)_p-(\log\phi)_u u_p,\tag{4.18}$$

and

$$(u_t)_{\overline{p}}=F^{i\bar{i}}X_{i\bar{i}\overline{p}}-(\log\phi)_{\overline{p}}-(\log\phi)_u u_{\overline{p}}.\tag{4.19}$$

It follows from (4.18) and (4.19) that

$$
\begin{aligned}
\partial_t|\nabla u|^2=&\sum_p u_{tp}u_{\overline{p}}+\sum_p u_p u_{\overline{t p}}\\
=&F^{i\bar{i}}\sum_p(X_{i\bar{i}p}u_{\overline{p}}+X_{i\bar{i}\overline{p}}u_p)-\sum_p(\log\phi)_p u_{\overline{p}}\\
&-\sum_p(\log\phi)_{\overline{p}}u_p-2\sum_p(\log\phi)_u|\nabla u|^2.
\end{aligned}\tag{4.20}
$$

By commuting derivatives, we have the identity

$$
\begin{aligned}
F^{i\bar{i}}X_{i\bar{i}p}=&F^{i\bar{i}}u_{i\bar{i}p}+F^{i\bar{i}}\chi_{i\bar{i}p}\\
=&F^{i\bar{i}}u_{pi\bar{i}}-F^{i\bar{i}}T^q_{pi}u_{q\bar{i}}-F^{i\bar{i}}u_q R_{i\bar{i}p}{}^{q}+F^{i\bar{i}}\chi_{i\bar{i}p}.
\end{aligned}\tag{4.21}
$$

Direct calculation gives

$$F^{i\bar{i}}(|\nabla u|^2)_{i\bar{i}}=\sum_p F^{i\bar{i}}(u_{pi\bar{i}}u_{\overline{p}}+u_{\overline{p}i\bar{i}}u_p)+\sum_p F^{i\bar{i}}(u_{pi}u_{\overline{p}\bar{i}}+u_{p\bar{i}}u_{\overline{p}i}).\tag{4.22}$$

It follows from (4.21) that

$$
\begin{aligned}
&F^{i\bar{i}}u_{pi\bar{i}}u_{\overline{p}}-F^{i\bar{i}}X_{i\bar{i}p}u_{\overline{p}}\\
=&F^{i\bar{i}}T^q_{pi}u_{q\bar{i}}u_{\overline{p}}+F^{i\bar{i}}u_{\overline{p}}u_q R_{i\bar{i}p}{}^{q}-F^{i\bar{i}}\chi_{i\bar{i}p}u_{\overline{p}}\\
\geq&-C_0K^{\frac{1}{2}}F^{i\bar{i}}X_{i\bar{i}}-C_0K^{\frac{1}{2}}\mathcal{F}-C_0K\mathcal{F}.
\end{aligned}\tag{4.23}
$$

Noticing that

$$F^{i\bar{i}}X_{i\bar{i}}=\frac{\sigma_{k-1}(\lambda|i)}{\sigma_k}\lambda_i-\frac{\sigma_{l-1}(\lambda|i)}{\sigma_l}\lambda_i=k-l,\tag{4.24}$$

from this and (4.23), we obtain

$$F^{i\bar{i}}u_{pi\bar{i}}u_{\overline{p}}-F^{i\bar{i}}X_{i\bar{i}p}u_{\overline{p}}\geq -C_0K^{\frac{1}{2}}-C_0K^{\frac{1}{2}}\mathcal{F}-C_0K\mathcal{F}.\tag{4.25}$$

In the same way, we can get

$$F^{i\bar{i}}u_{\bar{p}i\bar{i}}u_p - F^{i\bar{i}}X_{i\bar{i}\bar{p}}u_p \geq -C_0K^{\frac{1}{2}} - C_0K^{\frac{1}{2}}\mathcal{F} - C_0K\mathcal{F}. \tag{4.26}$$

Using (4.20)–(4.26) in (4.17), we have

$$\begin{aligned}\mathcal{L}\varphi(|\nabla u|^2) \geq&\varphi^{''}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 + \varphi^{'}(-C_0K^{\frac{1}{2}} - C_0K^{\frac{1}{2}}\mathcal{F} - C_0K\mathcal{F})\\ &+\varphi^{'}\sum_p((\log\phi)_pu_{\bar{p}} + (\log\phi)_{\bar{p}}u_p + 2(\log\phi)_u|\nabla u|^2)\\ &+\sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2)\\ \geq&\varphi^{''}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 + \sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2) - C_0 - C_0\mathcal{F}.\end{aligned} \tag{4.27}$$

A simple calculation gives

$$\mathcal{L}\psi(u-\underline{u}) = \psi^{''}F^{i\bar{i}}|(u-\underline{u})_i|^2 + \psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})]. \tag{4.28}$$

Substituting (4.27), (4.28) and (4.16) into (4.8),

$$\begin{aligned}0\geq&\frac{-F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}}}{\lambda_1} - F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} - \frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1}\\ &+\varphi^{''}F^{i\bar{i}}|(|\nabla u|^2)_i|^2 + \varphi'\sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2) - C_0 - C_0\mathcal{F}\\ &+\psi^{''}F^{i\bar{i}}|(u-\underline{u})_i|^2 + \psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})].\end{aligned} \tag{4.29}$$

Let $\delta > 0$ be a sufficiently small constant to be chosen later and satisfy

$$\delta \leq \min\{\frac{1}{1+4E}, \frac{1}{2}\}. \tag{4.30}$$

We separate the rest of the calculations into two cases.

Case 1: $\lambda_n < -\delta\lambda_1$.

Using(4.5), we find that

$$\begin{aligned}-\frac{F^{i\bar{i}}|X_{1\bar{1}i}|^2}{\lambda_1^2} =&-F^{i\bar{i}}\Big|\varphi^{'}(|\nabla u|^2)_i + \psi^{'}(u-\underline{u})_i - \frac{(D^{11})_i}{\lambda_1}\Big|^2\\ \geq&-2(\varphi^{'})^2F^{i\bar{i}}\Big|(|\nabla u|^2)_i\Big|^2 - 2F^{i\bar{i}}\Big|\psi^{'}(u-\underline{u})_i - \frac{(D^{11})_i}{\lambda_1}\Big|^2\\ \geq&-2(\varphi^{'})^2F^{i\bar{i}}\Big|(|\nabla u|^2)_i\Big|^2 - C_0|\psi^{'}|^2K\mathcal{F} - C_0\mathcal{F}.\end{aligned} \tag{4.31}$$

From (4.13), we have

$$\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2} \leq \frac{|X_{1\bar{1}i}|^2}{\lambda_1^2} + C_0(1 + \frac{|X_{1\bar{1}i}|}{\lambda_1}). \tag{4.32}$$

Combining (4.31) with (4.32), we conclude that

$$-\frac{F^{i\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1^2} \geq -2(\varphi^{'})^2 F^{i\bar{i}}\Big|(|\nabla u|^2)_i\Big|^2 - C_0|\psi^{'}|^2 K\mathcal{F} - C_0\mathcal{F} - C_0 F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1}. \tag{4.33}$$

Note that the operator F is concave, which implies that

$$\frac{-F^{i\bar{j},p\bar{q}}X_{i\bar{j}1}X_{p\bar{q}\bar{1}}}{\lambda_1} \geq 0. \tag{4.34}$$

Applying (4.33) and (4.34) to (4.29) and using $\varphi'' = 2(\varphi')^2$ yield that

$$\begin{aligned} 0 \geq & - C_0 F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} - \frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} + \varphi' \sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\overline{p}i}|^2) \\ & - C_0|\psi^{'}|^2 K\mathcal{F} - C_0\mathcal{F} - C_0 + \psi^{''} F^{i\bar{i}}|(u-\underline{u})_i|^2 \\ & + \psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})]. \end{aligned} \tag{4.35}$$

Note that the fact

$$\frac{|X_{1\bar{1}i}|}{\lambda_1} = \Big| -\varphi'(u_{pi}u_{\overline{p}} + u_p u_{\overline{p}i}) - \psi'(u-\underline{u})_i + \frac{(D^{11})_i}{\lambda_1}\Big|,$$

It follows that

$$-C_0 F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} \geq -C_0\varphi' K^{-\frac{1}{2}} F^{i\bar{i}}(|u_{pi}| + |u_{\overline{p}i}|) + C_0\psi' K^{\frac{1}{2}}\mathcal{F} - C_0\mathcal{F}. \tag{4.36}$$

Using the following inequality

$$K^{-\frac{1}{2}}(|u_{pi}| + |u_{\overline{p}i}|) \leq \frac{1}{4C_0}(|u_{pi}|^2 + |u_{\overline{p}i}|^2) + C_0 K.$$

deduces

$$-C_0 F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} \geq -\frac{1}{4}\varphi' F^{i\bar{i}}(|u_{pi}|^2 + |u_{\overline{p}i}|^2) + C_0\psi' K^{\frac{1}{2}}\mathcal{F} - C_0\mathcal{F}. \tag{4.37}$$

Note that $\lambda_1 > 1$, we have

$$-\frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1} \geq -\frac{1}{4}\varphi' F^{i\bar{i}}(|u_{pi}|^2 + |u_{\overline{p}i}|^2) + C_0\psi' K^{\frac{1}{2}}\mathcal{F} - C_0\mathcal{F}. \tag{4.38}$$

Since $\psi^{''} > 0$, which implies that

$$\psi^{''} F^{i\bar{i}}|(u-\underline{u})_i|^2 \geq 0. \tag{4.39}$$

According to Lemma 2.3, there are at most two possibilities:

(1) If (2.8) holds true, then

$$\psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})] \geq \theta(1+\mathcal{F})|\psi'|. \tag{4.40}$$

Substituting (4.37)—(4.40) into (4.35) and using $\varphi' \geq \frac{1}{4K}$ yield that

$$\begin{aligned} 0 \geq & \frac{1}{8K}\sum_p F^{i\bar{i}}(|u_{pi}|^2 + |u_{\bar{p}i}|^2) - C_0|\psi^{'}|^2 K\mathcal{F} - C_0(\mathcal{F}+1) + \theta(1+\mathcal{F})|\psi'| \\ \geq & \frac{1}{8K}F^{i\bar{i}}\lambda_i^2 - C_0(C_1+1)^2 K\mathcal{F} + \theta(C_1+1)(\mathcal{F}+1) - C_0(\mathcal{F}+1) \\ \geq & \frac{\delta^2\lambda_1^2}{8nK}\mathcal{F} - C_0(C_1+1)^2 K\mathcal{F} + \theta(C_1+1)(\mathcal{F}+1) - C_0(\mathcal{F}+1). \end{aligned} \tag{4.41}$$

We may set $\theta C_1 \geq C_0$. It follows from (4.41) that $\lambda_1 \leq \widetilde{C}K$.

(2) If (2.9) holds true,

$$F^{1\bar{1}} > \frac{\theta}{N}(1+\mathcal{F}). \tag{4.42}$$

According to $\psi^{'} < 0$ and the concavity of the operator F, we have

$$\begin{aligned} \psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})] = & \psi'[F^{i\bar{i}}(X_{i\bar{i}} - \underline{X}_{i\bar{i}}) - \partial_t(u-\underline{u})] \\ \geq & \psi'[F(\chi_u) - F(\chi_{\underline{u}}) - \partial_t u + \partial_t\underline{u}] \\ = & \psi'[\phi(x,u) + \partial_t\underline{u} - F(\chi_{\underline{u}})] \\ \geq & C_0\psi'. \end{aligned} \tag{4.43}$$

Using (4.37)—(4.39) and (4.43) in (4.35), together with (4.42), we find that

$$\begin{aligned} 0 \geq & \frac{1}{8K}F^{i\bar{i}}\lambda_i^2 - C_0|\psi^{'}|^2 K\mathcal{F} + C_0\psi' - C_0(\mathcal{F}+1) \\ \geq & \frac{\theta\lambda_1^2}{8NK}(1+\mathcal{F}) + \frac{\delta^2\lambda_1^2}{8nK}\mathcal{F} - C_0(C_1+1)^2 K\mathcal{F} \\ & - C_0(C_1+1) - C_0(\mathcal{F}+1). \end{aligned} \tag{4.44}$$

Let λ_1 be sufficiently large, so that

$$\frac{\theta\lambda_1^2}{8NK}(1+\mathcal{F}) - C_0(C_1+1) - C_0(\mathcal{F}+1) \geq 0,$$

It follows from (4.44)that $\lambda_1 \leq \widetilde{C}K$.

Case 2: $\lambda_n \geq -\delta\lambda_1$.

Let

$$I = \{i \in \{1,\cdots,n\} | F^{i\bar{i}} > \delta^{-1}F^{1\bar{1}}\}.$$

Let us first treat those indices which are not in I. Similar to (4.31), we obtain

$$-\sum_{i\notin I}\frac{F^{i\bar{i}}|X_{1\bar{1}i}|^2}{\lambda_1^2} \geq -2(\varphi^{'})^2\sum_{i\notin I}F^{i\bar{i}}\left|(|\nabla u|^2)_i\right|^2 - \frac{C_0K}{\delta}|\psi^{'}|^2 F^{1\bar{1}} - C_0\mathcal{F}. \tag{4.45}$$

Using (4.32) yields that

$$-\sum_{i\notin I}\frac{F^{i\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1^2}\geq -2(\varphi')^2\sum_{i\notin I}F^{i\bar{i}}\Big|(|\nabla u|^2)_i\Big|^2-\frac{C_0K}{\delta}|\psi'|^2F^{1\bar{1}}$$
$$-C_0\sum_{i\notin I}\frac{F^{i\bar{i}}|X_{1\bar{1}i}|}{\lambda_1}-C_0\mathcal{F}. \tag{4.46}$$

Since

$$-F^{i\bar{1},1\bar{i}}=\frac{F^{i\bar{i}}-F^{1\bar{1}}}{X_{1\bar{1}}-X_{i\bar{i}}}\text{ and }\quad \lambda_i\geq\lambda_n\geq-\delta\lambda_1,$$

which implies that

$$-\sum_{i\in I}F^{i\bar{1},1\bar{i}}\geq\frac{1-\delta}{1+\delta}\frac{1}{\lambda_1}\sum_{i\in I}F^{i\bar{i}},$$

It follows that

$$-\frac{F^{i\bar{1},1\bar{i}}|X_{i\bar{1}1}|^2}{\lambda_1}\geq\frac{1-\delta}{1+\delta}\sum_{i\in I}F^{i\bar{i}}\frac{|X_{i\bar{1}1}|^2}{\lambda_1^2}. \tag{4.47}$$

Recalling $\varphi''=2(\varphi')^2$ and $0<\delta\leq\frac{1}{2}$, we obtain from (4.5) that

$$\begin{aligned}&\sum_{i\in I}\varphi''F^{i\bar{i}}\Big|(|\nabla u|^2)_i\Big|^2\\ =&2\sum_{i\in I}F^{i\bar{i}}\Big|\frac{X_{i\bar{1}1}}{\lambda_1}+\psi'(u-\underline{u})_i+\frac{\chi_{11i}-\chi_{i11}+T^p_{i1}\chi_{p\bar{1}}-(D^{11})_i}{\lambda_1}\Big|^2\\ \geq&2\sum_{i\in I}F^{i\bar{i}}\Big(\delta\Big|\frac{X_{i\bar{1}1}}{\lambda_1}\Big|^2-\frac{2\delta}{1-\delta}(\psi')^2|(u-\underline{u})_i|^2-C_0\Big)\\ \geq&2\delta\sum_{i\in I}F^{i\bar{i}}\Big|\frac{X_{i\bar{1}1}}{\lambda_1}\Big|^2-\frac{4\delta}{1-\delta}(\psi')^2F^{i\bar{i}}|(u-\underline{u})_i|^2-C_0\mathcal{F}.\end{aligned} \tag{4.48}$$

Notice that $\psi''\geq\frac{4\epsilon}{1-\epsilon}(\psi')^2$ if $\epsilon=\frac{1}{4E+1}$. Since $\frac{1}{4E+1}\geq\delta$, we get that

$$\psi''F^{i\bar{i}}|(u-\underline{u})_i|^2-\frac{4\delta}{1-\delta}(\psi')^2F^{i\bar{i}}|(u-\underline{u})_i|^2\geq 0. \tag{4.49}$$

Take (4.46)–(4.49) into (4.29),

$$0\geq-C_0\sum_{i\notin I}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1}-\frac{C_0}{\lambda_1}F^{i\bar{i}}\frac{|X_{1\bar{1}i}|}{\lambda_1}+\varphi'\sum_pF^{i\bar{i}}(|u_{pi}|^2+|u_{\bar{p}i}|^2)$$
$$-C_0\mathcal{F}-C_0-\frac{C_0K}{\delta}|\psi'|^2F^{1\bar{1}}+\psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}}-\partial_t(u-\underline{u})]. \tag{4.50}$$

Similar to (4.37) and (4.38), by using the third term of (4.50) to absorb the first two terms of it, We get that

$$0\geq\frac{1}{8K}\sum_pF^{i\bar{i}}(|u_{pi}|^2+|u_{\bar{p}i}|^2)-\frac{C_0K}{\delta}|\psi'|^2F^{1\bar{1}}$$

$$
\begin{aligned}
&- C_0\mathcal{F} - C_0 + \psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})] \\
\geq &\frac{1}{8K}\sum_p F^{i\bar{i}}\lambda_i^2 - \frac{C_0K}{\delta}|\psi^{'}|^2F^{1\bar{1}} - C_0(\mathcal{F}+1) \\
&+ \psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})].
\end{aligned}
\tag{4.51}
$$

According to Lemma2.3, there are at most two possibilities:

(1) If (2.8) holds true, then

$$
\psi'[F^{i\bar{i}}(u-\underline{u})_{i\bar{i}} - \partial_t(u-\underline{u})] \geq \theta(1+\mathcal{F})|\psi'|. \tag{4.52}
$$

Put (4.52) into (4.51)

$$
\begin{aligned}
0 \geq &\frac{1}{8K}\sum_p F^{i\bar{i}}\lambda_i^2 - \frac{C_0K}{\delta}|\psi^{'}|^2F^{1\bar{1}} - C_0(\mathcal{F}+1) + \theta(1+\mathcal{F})|\psi'| \\
\geq &\frac{1}{8K}\sum_p F^{1\bar{1}}\lambda_1^2 - \frac{C_0K}{\delta}(1+C_1)^2F^{1\bar{1}} - C_0(\mathcal{F}+1) + \theta(1+\mathcal{F})(1+C_1).
\end{aligned}
\tag{4.53}
$$

Here, C_1 is determined finally, such that

$$
\theta C_1 \geq C_0.
$$

It follows from (4.53) that

$$
\lambda_1 \leq \widetilde{C}K.
$$

(2) If (2.9) holds true,

$$
F^{1\bar{1}} > \frac{\theta}{N}(1+\mathcal{F}). \tag{4.54}
$$

Substituting (4.43) into (4.51) and using (4.54) give that

$$
0 \geq \frac{1}{8K}\lambda_1^2 - \frac{C_0K}{\delta}|(1+C_1)|^2 - \frac{N}{\theta}C_0(1+C_1) - \frac{N}{\theta}C_0 \tag{4.55}
$$

It follows that

$$
\lambda_1 \leq \widetilde{C}K.
$$

□

5. C^1 Estimates

To obtain the gradient estimates, we adapt the blow-up method of Dinew and Kolodziej [7] and reduce the problem to a Liouville type theorem which is proved in [7].

Proposition 5.1. *Under the assumption of Theorem 1.3, let $u(x,t)$ be a solution to (1.1). Then there exists a uniform constant $\widetilde{C}$ such that*

$$
\sup_{M\times[0,T)} |\nabla u| \leq \widetilde{C}. \tag{5.1}
$$

Proof. Suppose that the gradient estimate (5.1) does not hold. Then there exists a sequence $(x_m, t_m) \in M \times [0, T)$ with $t_m \to T$ such that

$$\sup_{M\times[0,t_m]} |\nabla u(x,t)| = |\nabla u(x_m, t_m)| \text{ and } \lim_{m\to\infty} |\nabla u(x_m, t_m)| = \infty.$$

After passing to a subsequence, we may assume that $\lim_{m\to\infty} x_m = x_0 \in M$. We choose a coordinate chart $\{U, (z_1, \cdots, z_n)\}$ at x_0, which we identify with an open set in $\mathbb{C}^n$, and such that $\omega(0) = \beta = \sqrt{-1}\sum_i dz^i \wedge d\overline{z}^i$. We may assume that the open set contains $\overline{B_1(0)}$ and m is sufficiently large so that $z_m = z(x_m) \in B_1(0)$. Define

$$|\nabla u(x_m, t_m)| = C_m, \ \ \widetilde{u}_m(z) = u(\frac{z}{C_m}, t_m).$$

From this and Proposition 4.1, we have

$$\sup_M |\nabla \widetilde{u}_m| = 1, \ \sup_M |\sqrt{-1}\,\overline{\partial}\partial\widetilde{u}_m| \le \widetilde{C}.$$

This yields that $\widetilde{u}_m$ is contained in the Hölder space $C^{1,\gamma(\mathbb{C}^n)}$ with a uniform. Along with a standard application of Azela-Ascoli theorem, we may suppose $\widetilde{u}_m$ has a limit $\widetilde{u} \in C^{1,\gamma}(\mathbb{C}^n)$ with

$$|\widetilde{u}| + |\nabla\widetilde{u}| < C \text{ and } |\nabla\widetilde{u}(0)| \neq 0, \tag{5.2}$$

in particular $\widetilde{u}$ is not constant. On the other hand, similar to the method of Dinew and Kolodziej [7], we have

$$\begin{aligned}&\Big[\chi_u(\frac{z}{C_m})\Big]^k \wedge \Big[\omega(\frac{z}{C_m})\Big]^{n-k}\\ =&e^{\partial_t u}\phi_m(\frac{z}{C_m}, u)\Big[\chi_u(\frac{z}{C_m})\Big]^l \wedge \Big[\omega(\frac{z}{C_m})\Big]^{n-l}.\end{aligned}$$

Fixing z, we obtain

$$\begin{aligned}&C_m^{2(k-l)}\Big[O(\frac{1}{C_m^2})\beta + \sqrt{-1}\,\overline{\partial}\partial\widetilde{u}_m(z)\Big]^k \wedge \Big[(1 + O(\frac{|z|^2}{C_m^2}))\beta\Big]^{n-k}\\ =&e^{\partial_t u_m}\phi_m(\frac{z}{C_m}, u_m)\Big[O(\frac{1}{C_m^2})\beta + \sqrt{-1}\,\overline{\partial}\partial\widetilde{u}_m(z)\Big]^l \wedge \Big[(1 + O(\frac{|z|^2}{C_m^2}))\beta\Big]^{n-l}.\end{aligned} \tag{5.3}$$

Lemma 2.2 gives $\partial_t u$ is bounded uniformly. Since

$$\phi_m(\frac{z}{C_m}, u_m) \le \sup_{M\times[-C,C]} \phi,$$

which implies that $\phi_m(\frac{z}{C_m}, u_m)$ is bounded uniformly. Taking the limits on both sides of 5.3 by $m \to \infty$ yields that

$$(\sqrt{-1}\partial\overline{\partial}\widetilde{u})^k \wedge \beta^{n-k} = 0. \tag{5.4}$$

which is in the pluripotential sense. Moreover, a similar reasoning tells us that for any $1 \le p \le k$,

$$(\sqrt{-1}\partial\overline{\partial}\widetilde{u})^p \wedge \beta^{n-p} \ge 0. \tag{5.5}$$

Then, (5.4) and (5.5) imply $\widetilde{u}$ is a k-subharmonic. By a result of Blocki [1], $\widetilde{u}$ is a maximal k-subharmonic function in $\mathbb{C}^n$. Applying the Liouville theorem in [7], we find that $\widetilde{u}$ is a constant, which contradicts with (5.2).

□

6. Long-time existence and convergence

In this section, we shall give a proof of the long-time existence to the flow and its convergence, that is Theorem 1.3.

From Lemma 2.2, Proposition 3.1, Propositions 4.1 and 5.1, we conclude that Eq 1.1 is uniformly parabolic. Therefore by Evans-Krylov's regularity theory [8, 13, 19, 27] for uniformly parabolic equation, we obtain higher order derivative estimates. By the a priori estimates which don't depend on time, one can prove that the short time existence on $[0, T)$ extends to $[0, \infty)$, that is the smooth solution exists at all time $t > 0$. After proving C^∞ estimates on $[0, \infty)$, we are able to show the convergence of the solution flow.

Let $v = e^{\gamma t} u_t$, where $0 < \gamma < c_\phi$. Commuting derivative of v with respect to t and using (2.4), we obtain

$$\begin{aligned} v_t =& e^{\gamma t} u_{tt} + \gamma v \\ =& \gamma v + e^{\gamma t}(F^{i\bar{j}} u_{ti\bar{j}} - \frac{\phi_z}{\phi} u_t) \\ =& F^{i\bar{j}} v_{i\bar{j}} + (\gamma - \frac{\phi_z}{\phi}) v. \end{aligned}$$

Using the condition (1.5) yields $\gamma - \frac{\phi_z}{\phi} < 0$, According to the parabolic maximum principle, it follows that

$$\sup_{M \times [0,\infty)} |v(x, t)| \leq \sup_M |u_t(x, 0)| \leq \sup_M |F(\lambda(u_0)) - \phi(x, u_0)| \leq C,$$

which means that $|u_t|$ decreases exponentially, in particular

$$\partial_t (u + \frac{C}{\gamma} e^{-\gamma t}) \leq 0.$$

According to Proposition 3.1, it follows that $u + \frac{C}{\gamma} e^{-\gamma t}$ is bounded uniformly and decreasing in t. Thus it converges to a smooth function u_∞. From the higher order prior estimates, we can see that the function $u(x, t)$ converges smoothly to u_∞. Letting $t \to \infty$ in Eq (2.1),

$$\frac{\sigma_k(\lambda(u_\infty))}{\sigma_l(\lambda(u_\infty))} = \phi(x, u_\infty).$$

7. Conclusions

In this paper, we have considered the parabolic Hessian quotient equation (1.1), in which the right hand side function ϕ depends on u. Firstly, we prove C^0 estimates of Eq (1.1) by the parabolic C-subsolution condition and the Alexandroff-Bakelman-Pucci maximum principle. Secondly, we establish the C^2 estimate for Eq (1.1) by using the parabolic C-subsolution condition. Thirdly, we obtain the gradient estimate by adapting the blowup method. Finally we give the proof of the long-time existence of the solution to the parabolic equation and its convergence. As an application, we show the solvability of a class of complex Hessian quotient equations, which generalizes the relevant results.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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