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*Research article*

## **Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform**

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**Abstract:** In this paper, we find the solution of the time-fractional Newell-Whitehead-Segel equation with the help of two different methods. The newell-Whitehead-Segel equation plays an efficient role in nonlinear systems, describing the stripe patterns' appearance in two-dimensional systems. Four case study problems of Newell-Whitehead-Segel are solved by the proposed methods with the aid of the Antagana-Baleanu fractional derivative operator and the Laplace transform. The numerical results obtained by suggested techniques are compared with an exact solution. To show the effectiveness of the proposed methods, we show exact and analytical results compared with the help of graphs and tables, which are in strong agreement with each other. Also, the results obtained by implementing the suggested methods at various fractional orders are compared, which confirms that the solution gets closer to the exact solution as the value tends from fractional-order towards integer order. Moreover, proposed methods are interesting, easy and highly accurate in solving various nonlinear fractional-order partial differential equations.

**Keywords:** Antagana-Baleanu fractional derivative; Laplace transform decomposition method; variational iteration transform method; time fractional Newell-Whitehead-Segel equation

**Mathematics Subject Classification:** 34A34, 35A20, 35A22, 44A10, 33B15

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### **1. Introduction**

Fractional calculus (FC) is a subject that dates back to 1695 and is regarded to be as old as ordinary calculus. Ordinary calculus made it impossible to model nonlinear real-world phenomenon in nature, hence fractional calculus became popular among researchers. The fractional derivative is the derivative of arbitrary order in applied mathematics and mathematical analysis. In the nineteenth century,

Riemann Liouville [1] implemented the fractional derivative when simulating real-world problems. Some of the Nobel contributions of mathematicians are listed here, Caputo [2], Kemple and Beyer [3], Abbasbandy [4], Jafari and Seifi [5, 6], Miller and Ross [7], Podlubny [8], Kilbas and Trujillo [9], Diethelm et al. [10], Hayat et al. [11], Debanth [12], Momani and Shawagfeh [13], etc. Fractional calculus has become one of the fascinating science and engineering research areas in recent years. Viscoelasticity and damping, diffusion and wave propagation, electromagnetism and heat transfer, biology, signal processing, robotics system classification, physics, mechanics, chemistry, and control theory are the most important scientific fields that use fractional calculus at the moment.

Researchers are interested in FC because of its wide applications in physics, engineering, and real-life sciences. Fractional differential equations accurately represent these physical facts. Fractional differential equations are significantly greater generalizations of integer-order differential equations. Fractional differential equations (FDEs) have generated much interest in recent years. As a result of their frequent appearance in diverse applications, such as quantum mechanics [14], chaotic dynamics [15], plasma physics [16, 17], theory of long-range interaction [18], mechanics of non-Hamiltonian systems [19], physical kinetics [20], anomalous diffusion and transport theory [21], mechanics of fractional media [22], astrophysics [23], and so on. FDEs have been the subject of numerous investigations. Many works have been dedicated to developing efficient methods for solving FDEs, but it is important to remember that finding an analytical or approximate solution is difficult, therefore, accurate methods for obtaining FDE solutions are still being researched. In the literature, there are several analytical and numerical approaches for solving FDEs. For example, the generalized differential transform method (GDTM) [24], adomian decomposition method (ADM) [25], homotopy analysis method (HAM) [26], variational iteration method (VIM) [27], homotopy perturbation method (HPM) [28], Elzaki transform decomposition method (ETDM) [29], iterative Laplace transform method (ILTM) [30], fractional wavelet method (FWM) [31, 32], residual power series method (RPSM) [33, 34].

In this paper, we used two powerful techniques with the aid of the Antagana-Baleanu fractional derivative operator and Laplace transform for solving time fractional NWSEs. The two well-known methods that we implement are LTDM and VITM. The suggested techniques give series form solutions having quick convergence towards the exact solutions. Four non-linear NWSEs case study issues are resolved using the given methodology. Newell and Whitehead [35] developed the non-linear NWSE. The diffusion term's influence interacts with the reaction term's nonlinear effect in the Newell-Whitehead-Segel equation model. The fractional NWSE is written in the following way:

$$D_{\tau}^{\beta} \mu(\xi, \tau) = k D_{\xi}^2 \mu(\xi, \tau) + g\mu - h\mu^r, \quad (1.1)$$

where  $r$  is a positive integer and  $g, h$  are real numbers with  $k > 0$ . The first term  $D_{\tau}^{\beta} \mu(\xi, \tau)$  on the left hand side in (1.1) shows the deviations of  $\mu(\xi, \tau)$  with time at a fixed location, while the right hand side first term  $D_{\xi}^2 \mu(\xi, \tau)$  shows the deviations with spatial variable  $\xi$  of  $\mu(\xi, \tau)$  at a specific time and the right hand side remaining terms  $g\mu - h\mu^r$ , is the source terms.  $\mu(\xi, \tau)$  is a function of the spatial variable  $\xi$  and the temporal variable  $\tau$  in (1.1), with  $\xi \in R$  and  $\tau \geq 0$ . The function  $\mu(\xi, \tau)$  might be considered the (nonlinear) temperature distribution in an infinitely thin and long rod or as the fluid flow velocity in an infinitely long pipe with a small diameter. Many researchers find the analytical solution of NWSEs [36,37] due to their wide range of applications in mechanical and chemical engineering, ecology, biology, and bioengineering.

## 2. Preliminaries

Some basic definitions related to fractional calculus are expressed here in this section.

**Definition 2.1.** The Caputo fractional-order derivative is given as

$${}^{LC}D_{\tau}^{\mathfrak{J}}\{g(\tau)\} = \frac{1}{(n - \mathfrak{J})} \int_0^{\tau} (\tau - k)^{n-\mathfrak{J}-1} g^n(k) dk, \quad (2.1)$$

where  $n < \mathfrak{J} \leq n + 1$ .

**Definition 2.2.** The Caputo fractional-order derivative via Laplace transformation  ${}^{LC}D_{\tau}^{\mathfrak{J}}\{g(\tau)\}$  is defined as

$$\mathbb{L}\{{}^{LC}D_{\tau}^{\mathfrak{J}}\{g(\tau)\}\}(\omega) = \frac{1}{\omega^{n-\mathfrak{J}}} \left[ \omega^n \mathbb{L}\{g(\xi, \tau)\}(\omega) - \omega^{n-1} g(\xi, 0) - \dots - g^{n-1}(\xi, 0) \right]. \quad (2.2)$$

**Definition 2.3.** The Atangana-Baleanu derivative in Caputo manner is given as

$${}^{ABC}D_{\tau}^{\mathfrak{J}}\{g(\tau)\} = \frac{A(\mathfrak{J})}{1 - \mathfrak{J}} \int_a^{\tau} g'(k) E_{\mathfrak{J}} \left[ -\frac{\mathfrak{J}}{1 - \mathfrak{J}} (1 - k)^{\mathfrak{J}} \right] dk, \quad (2.3)$$

where  $A(\gamma)$  is a normalization function such that  $A(0) = A(1) = 1$ ,  $g \in H^1(a, b)$ ,  $b > a$ ,  $\mathfrak{J} \in [0, 1]$  and  $E_{\gamma}$  represent the Mittag-Leffler function.

**Definition 2.4.** The Atangana-Baleanu derivative in Riemann-Liouville manner is given as

$${}^{ABC}D_{\tau}^{\mathfrak{J}}\{g(\tau)\} = \frac{A(\mathfrak{J})}{1 - \mathfrak{J}} \frac{d}{d\tau} \int_a^{\tau} g(k) E_{\mathfrak{J}} \left[ -\frac{\gamma}{1 - \mathfrak{J}} (1 - k)^{\mathfrak{J}} \right] dk. \quad (2.4)$$

**Definition 2.5.** The Laplace transform connected with the Atangana-Baleanu operator is define as

$${}^{AB}D_{\tau}^{\mathfrak{J}}\{g(\tau)\}(\omega) = \frac{A(\gamma)\omega^{\mathfrak{J}}\mathbb{L}\{g(\tau)\}(\omega) - \omega^{\mathfrak{J}-1}g(0)}{(1 - \mathfrak{J})\left(\omega^{\mathfrak{J}} + \frac{\mathfrak{J}}{1-\gamma}\right)}. \quad (2.5)$$

**Definition 2.6.** Consider  $0 < \mathfrak{J} < 1$ , and  $g$  is a function of  $\mathfrak{J}$ , then the fractional-order integral operator of  $\mathfrak{J}$  is given as

$${}^{ABC}I_{\tau}^{\mathfrak{J}}\{g(\tau)\} = \frac{1 - \mathfrak{J}}{A(\mathfrak{J})} g(\tau) + \frac{\mathfrak{J}}{A(\mathfrak{J})\Gamma(\mathfrak{J})} \int_a^{\tau} g(k)(\tau - k)^{\mathfrak{J}-1} dk. \quad (2.6)$$

## 3. The idea of LTDM

The solution by LTDM for partial differential equations having fractional-order is described in this section.

$$D_{\tau}^{\mathfrak{J}}\mu(\xi, \tau) + \bar{\mathcal{G}}_1(\xi, \tau) + \mathcal{N}_1(\xi, \tau) = \mathcal{F}(\xi, \tau), \quad 0 < \mathfrak{J} \leq 1, \quad (3.1)$$

with some initial sources

$$\mu(\xi, 0) = \xi(\xi) \quad \text{and} \quad \frac{\partial}{\partial \tau} \mu(\xi, 0) = \zeta(\xi),$$

where  $D_\tau^\mathfrak{J} = \frac{\partial^\mathfrak{J}}{\partial \tau^\mathfrak{J}}$  is the fractional-order  $AB$  operator having order  $\mathfrak{J}$ ,  $\bar{\mathcal{G}}_1$  is linear operator and  $\mathcal{N}_1$  is non-linear and  $\mathcal{F}(\xi, \tau)$  indicates the source term.

Employing the Laplace transform to (3.1), and we acquire

$$L[D_\tau^\mathfrak{J}\mu(\xi, \tau) + \bar{\mathcal{G}}_1(\xi, \tau) + \mathcal{N}_1(\xi, \tau)] = L[\mathcal{F}(\xi, \tau)]. \quad (3.2)$$

By the virtue of Laplace differentiation property, we have

$$L[\mu(\xi, \tau)] = \Theta(\xi, \omega) - \frac{\omega^\mathfrak{J} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^\mathfrak{J}} L[\bar{\mathcal{G}}_1(\xi, \tau) + \mathcal{N}_1(\xi, \tau)], \quad (3.3)$$

where

$$\Theta(\xi, \omega) = \frac{1}{\omega^{\mathfrak{J}+1}} [\omega^\mathfrak{J} g_0(\xi) + \omega^{\mathfrak{J}-1} g_1(\xi) + \cdots + g_1(\xi)] + \frac{\omega^\mathfrak{J} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^\mathfrak{J}} \mathcal{F}(\xi, \tau).$$

Now, applying inverse Laplace transform yields (3.3) into

$$\mu(\xi, \tau) = \Theta(\xi, \omega) - L^{-1} \left\{ \frac{\omega^\mathfrak{J} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^\mathfrak{J}} L[\bar{\mathcal{G}}_1(\xi, \tau) + \mathcal{N}_1(\xi, \tau)] \right\}, \quad (3.4)$$

where  $\Theta(\xi, \omega)$  demonstrates the terms occurring from source factor. LTDM determines the solution of the infinite sequence of  $\mu(\xi, \tau)$

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau). \quad (3.5)$$

and decomposing the nonlinear operator  $\mathcal{N}_1$  as

$$\mathcal{N}_1(\xi, \tau) = \sum_{m=0}^{\infty} \mathcal{A}_m, \quad (3.6)$$

where  $\mathcal{A}_m$  are Adomian polynomials given as

$$\mathcal{A}_m = \frac{1}{m!} \left[ \frac{\partial^m}{\partial \ell^m} \left\{ \mathcal{N}_1 \left( \sum_{k=0}^{\infty} \ell^k \xi_k, \sum_{k=0}^{\infty} \ell^k \tau_k \right) \right\} \right]_{\ell=0}. \quad (3.7)$$

Putting (3.5) and (3.7) into (3.4), gives

$$\sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \Theta(\xi, \omega) - L^{-1} \left\{ \frac{\omega^\mathfrak{J} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^\mathfrak{J}} L[\bar{\mathcal{G}}_1(\sum_{m=0}^{\infty} \xi_m, \sum_{m=0}^{\infty} \tau_m) + \sum_{m=0}^{\infty} \mathcal{A}_m] \right\}. \quad (3.8)$$

The following terms are described:

$$\mu_0(\xi, \tau) = \Theta(\xi, \omega) \quad (3.9)$$

$$\mu_1(\xi, \tau) = L^{-1} \left\{ \frac{\omega^\mathfrak{J} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^\mathfrak{J}} L[\bar{\mathcal{G}}_1(\xi_0, \tau_0) + \mathcal{A}_0] \right\}. \quad (3.10)$$

Thus all components for  $m \geq 1$  are calculated as

$$\mu_{m+1}(\xi, \tau) = L^{-1} \left\{ \frac{\omega^\mathfrak{J} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^\mathfrak{J}} L[\bar{\mathcal{G}}_1(\xi_m, \tau_m) + \mathcal{A}_m] \right\}. \quad (3.11)$$

#### 4. VITM formulation

The VITM solution for FPDEs is defined in this section.

$$D_{\tau}^{\mathfrak{J}}\mu(\xi, \tau) + \mathcal{M}\mu(\xi, \tau) + \mathcal{N}\mu(\xi, \tau) - \mathcal{P}(\xi, \tau) = 0, \quad m - 1 < \mathfrak{J} \leq m, \quad (4.1)$$

with initial source

$$\mu(\xi, 0) = g_1(\xi), \quad (4.2)$$

where  $D_{\tau}^{\mathfrak{J}} = \frac{\partial^{\mathfrak{J}}}{\partial \tau^{\mathfrak{J}}}$  is stand for fractional-order *AB* operator,  $\mathcal{M}$  is a linear operator and  $\mathcal{N}$  is nonlinear term and  $\mathcal{P}$  indicates the source term.

The Laplace transform is applied to Eq (4.1), we have

$$L[D_{\tau}^{\mathfrak{J}}\mu(\xi, \tau)] + L[\mathcal{M}\mu(\xi, \tau) + \mathcal{N}\mu(\xi, \tau) - \mathcal{P}(\xi, \tau)] = 0. \quad (4.3)$$

By the property of *LT* differentiation, we get

$$L[\mu(\xi, \tau)] = \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \mathfrak{J})} L[\mathcal{M}\mu(\xi, \tau) + \mathcal{N}\mu(\xi, \tau) - \mathcal{P}(\xi, \tau)]. \quad (4.4)$$

The iteration technique for (4.4) as

$$\mu_{m+1}(\xi, \tau) = \mu_m(\xi, \tau) + \mathfrak{J}(s) \left[ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \mathfrak{J})} L[\mathcal{M}\mu(\xi, \tau) + \mathcal{N}\mu(\xi, \tau) - \mathcal{P}(\xi, \tau)] \right], \quad (4.5)$$

where  $\mathfrak{J}(s)$  is Lagrange multiplier and

$$\mathfrak{J}(s) = -\frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \mathfrak{J})}{\omega^{\mathfrak{J}}}, \quad (4.6)$$

with the application of inverse Laplace transform, (4.5) series form solution is given by

$$\begin{aligned} \mu_0(\xi, \tau) &= \mu(0) + L^{-1} [\mathfrak{J}(s)L[-\mathcal{P}(\xi, \tau)]] \\ \mu_1(\xi, \tau) &= L^{-1} [\mathfrak{J}(s)L[\mathcal{M}\mu(\xi, \tau) + \mathcal{N}\mu(\xi, \tau)]] \\ &\vdots \\ \mu_{n+1}(\xi, \tau) &= L^{-1} [\mathfrak{J}(s)L[\mathcal{M}[\mu_0(\xi, \tau) + \mu_1(\xi, \tau) + \cdots + \mu_n(\xi, \tau)]] \\ &\quad + \mathcal{N}[\mu_0(\xi, \tau) + \mu_1(\xi, \tau) + \cdots + \mu_n(\xi, \tau)]]. \end{aligned}$$

#### 5. Convergence analysis

Here we discuss uniqueness and convergence analysis.

**Theorem 5.1.** *The result of (3.1) is unique for  $LTDM_{ABC}$ , when*

$$0 < (\theta_1 + \theta_2) \left( 1 - \mathfrak{J} + \mathfrak{J} \frac{\tau^{\mu}}{\Gamma(\mu + 1)} \right) < 1.$$

*Proof.* Let  $H = (C[J], \|\cdot\|)$  with the norm  $\|\phi(\tau)\| = \max_{\tau \in J} |\phi(\tau)|$  be the Banach space, for all continuous function on  $J$ . Let  $I : H \rightarrow H$  is a non-linear mapping, where

$$\mu_{l+1}^C = \mu_0^C + L^{-1}[p(\mathfrak{I}, \nu, \omega)L[\mathcal{L}(\mu_l(\xi, \tau)) + \mathbb{N}(\mu_l(\xi, \tau))]], \quad l \geq 0.$$

Suppose that  $|\mathcal{L}(\mu) - \mathcal{L}(\mu^*)| < \theta_1|\mu - \mu^*|$  and  $|\mathbb{N}(\mu) - \mathbb{N}(\mu^*)| < \theta_2|\mu - \mu^*|$ , where  $\mu := \mu(\xi, \tau)$  and  $\mu^* := \mu^*(\xi, \tau)$  are two different function values and  $\theta_1, \theta_2$  are Lipschitz constants.

$$\begin{aligned} \|I\mu - I\mu^*\| &\leq \max_{t \in J} |L^{-1}[q(\mathfrak{I}, \nu, \omega)L[\mathcal{L}(\mu) - \mathcal{L}(\mu^*)] + q(\mathfrak{I}, \nu, \omega)L[\mathbb{N}(\mu) - \mathbb{N}(\mu^*)]]| \\ &\leq \max_{t \in J} \left[ \theta_1 L^{-1}[q(\mathfrak{I}, \nu, \omega)L[|\mu - \mu^*|]] + \theta_2 L^{-1}[q(\mathfrak{I}, \nu, \omega)L[|\mu - \mu^*|]] \right] \\ &\leq \max_{t \in J} (\theta_1 + \theta_2) \left[ L^{-1}[q(\mathfrak{I}, \nu, \omega)L|\mu - \mu^*|] \right] \\ &\leq (\mathfrak{I}_1 + \theta_2) \left[ L^{-1}[q(\mathfrak{I}, \nu, \omega)L|\mu - \mu^*|] \right] \\ &= (\theta_1 + \theta_2) \left( 1 - \mathfrak{I} + \mathfrak{I} \frac{\tau^{\mathfrak{I}}}{\Gamma(\mathfrak{I} + 1)} \right) \|\mu - \mu^*\|, \end{aligned} \quad (5.1)$$

where  $I$  is contraction as

$$0 < (\theta_1 + \theta_2) \left( 1 - \mathfrak{I} + \mathfrak{I} \frac{\tau^{\mathfrak{I}}}{\Gamma(\mathfrak{I} + 1)} \right) < 1.$$

From Banach fixed point theorem, the result of (3.1) is unique.  $\square$

**Theorem 5.2.** *The LTDM<sub>ABC</sub> result of (3.1) is convergent.*

*Proof.* Let  $\mu_m = \sum_{r=0}^m \mu_r(\xi, \tau)$ . To show that  $\mu_m$  is a Cauchy sequence in  $H$ . For  $n \in \mathbb{N}$ , let

$$\begin{aligned} \|\mu_m - \mu_n\| &= \max_{\tau \in J} \left| \sum_{r=n+1}^m \mu_r \right| \\ &\leq \max_{\tau \in J} \left| L^{-1} \left[ q(\mathfrak{I}, \nu, \omega) L \left[ \sum_{r=n+1}^m (\mathcal{L}(\mu_{r-1}) + \mathbb{N}(\mu_{r-1})) \right] \right] \right| \\ &= \max_{\tau \in J} \left| L^{-1} \left[ q(\mathfrak{I}, \nu, \omega) L \left[ \sum_{r=n+1}^{m-1} (\mathcal{L}(\mu_r) + \mathbb{N}(\mu_r)) \right] \right] \right| \\ &\leq \max_{\tau \in J} \left| L^{-1} [q(\mathfrak{I}, \nu, \omega) L[(\mathcal{L}(\mu_{m-1}) - \mathcal{L}(\mu_{n-1})) + (\mathbb{N}(\mu_{m-1}) - \mathbb{N}(\mu_{n-1}))]] \right| \\ &\leq \theta_1 \max_{\tau \in J} \left| L^{-1} [q(\mathfrak{I}, \nu, \omega) L[(\mathcal{L}(\mu_{m-1}) - \mathcal{L}(\mu_{n-1}))]] \right| \\ &\quad + \theta_2 \max_{\tau \in J} \left| L^{-1} [p(\mathfrak{I}, \nu, \omega) L[(\mathbb{N}(\mu_{m-1}) - \mathbb{N}(\mu_{n-1}))]] \right| \\ &= (\theta_1 + \theta_2) \left( 1 - \mathfrak{I} + \mathfrak{I} \frac{\tau^{\mathfrak{I}}}{\Gamma(\mathfrak{I} + 1)} \right) \|\mu_{m-1} - \mu_{n-1}\|. \end{aligned} \quad (5.2)$$

Let  $m = n + 1$ , then

$$\|\mu_{n+1} - \mu_n\| \leq \theta \|\mu_n - \mu_{n-1}\| \leq \theta^2 \|\mu_{n-1} - \mu_{n-2}\| \leq \cdots \leq \theta^n \|\mu_1 - \mu_0\|, \quad (5.3)$$

where

$$\theta = (\theta_1 + \theta_2) \left( 1 - \mathfrak{J} + \mathfrak{J} \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} \right).$$

Similarly, we have

$$\begin{aligned} \|\mu_m - \mu_n\| &\leq \|\mu_{n+1} - \mu_n\| + \|\mu_{n+2} - \mu_{n+1}\| + \cdots + \|\mu_m - \mu_{m-1}\| \\ &\leq (\theta^n + \theta^{n+1} + \cdots + \theta^{m-1}) \|\mu_1 - \mu_0\| \\ &\leq \theta^n \left( \frac{1 - \theta^{m-n}}{1 - \theta} \right) \|\mu_1\|. \end{aligned} \quad (5.4)$$

As  $0 < \theta < 1$ , we get  $1 - \theta^{m-n} < 1$ . Therefore, we have

$$\|\mu_m - \mu_n\| \leq \frac{\theta^n}{1 - \theta} \max_{t \in J} \|\mu_1\|. \quad (5.5)$$

Since  $\|\mu_1\| < \infty$  and  $\|\mu_m - \mu_n\| \rightarrow 0$ , when  $n \rightarrow \infty$ . As a result,  $\mu_m$  is a Cauchy sequence in  $H$ , implying that the series  $\mu_m$  is convergent.  $\square$

## 6. Applications

Four cases of nonlinear NWSEs are presented to demonstrate the suggested technique's capability and reliability.

**Example 1.** The NWSE has given in (1.1) for  $g = 2, h = 3, k = 1$  and  $r = 2$  becomes

$$D_{\tau}^{\mathfrak{J}} \mu(\xi, \tau) = D_{\xi}^2 \mu(\xi, \tau) + 2\mu(\xi, \tau) - 3\mu^2(\xi, \tau), \quad 0 < \theta \leq 1, \quad (6.1)$$

with initial source  $\mu(\xi, 0) = \Upsilon$ .

Applying Laplace transform to (6.1), we get

$$\frac{\omega^{\mathfrak{J}} L[\mu(\xi, \tau)] - \omega^{-1} \mu(\xi, 0)}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} = L \left[ D_{\xi}^2 \mu(\xi, \tau) + 2\mu(\xi, \tau) - 3\mu^2(\xi, \tau) \right]. \quad (6.2)$$

On taking Laplace inverse transform, we get

$$\mu(\xi, \tau) = \Upsilon + L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left[ D_{\xi}^2 \mu(\xi, \tau) + 2\mu(\xi, \tau) - 3\mu^2(\xi, \tau) \right] \right]. \quad (6.3)$$

Assume that the solution,  $\mu(\xi, \tau)$  in the form of infinite series given by

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau), \quad (6.4)$$

where  $\mu^2 = \sum_{m=0}^{\infty} \mathcal{A}_m$  are the so-called Adomian polynomials that represent the nonlinear terms, thus (6.3) having certain terms are rewritten as

$$\sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \Upsilon + L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left[ D_{\xi}^2 \mu(\xi, \tau) + 2\mu(\xi, \tau) - 3 \sum_{m=0}^{\infty} \mathcal{A}_m \right] \right]. \quad (6.5)$$

According to (3.7), the decomposition of nonlinear terms by Adomian polynomials is defined as

$$\mathcal{A}_0 = \mu_0^2, \quad \mathcal{A}_1 = 2\mu_0\mu_1, \quad \mathcal{A}_2 = 2\mu_0\mu_2 + (\mu_1)^2. \quad (6.6)$$

Thus, on comparing both sides of (6.5)

$$\mu_0(\xi, \tau) = \Upsilon.$$

For  $m = 0$ , we have

$$\mu_1(\xi, \tau) = \Upsilon(2 - 3\Upsilon) \left[ \frac{\mathfrak{Y}\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y}) \right].$$

For  $m = 1$ , we have

$$\mu_2(\xi, \tau) = 2\Upsilon(2 - 3\Upsilon)(1 - 3\Upsilon) \left[ \frac{\mathfrak{Y}^2\tau^{2\mathfrak{Y}}}{\Gamma(2\mathfrak{Y} + 1)} + 2\mathfrak{Y}(1 - \mathfrak{Y})\frac{\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y})^2 \right].$$

The approximate series solution is expressed as

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \mu_0(\xi, \tau) + \mu_1(\xi, \tau) + \mu_2(\xi, \tau) + \dots.$$

Therefore, we have

$$\begin{aligned} \mu(\xi, \tau) = & \Upsilon + \Upsilon(2 - 3\Upsilon) \left[ \frac{\mathfrak{Y}\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y}) \right] \\ & + 2\Upsilon(2 - 3\Upsilon)(1 - 3\Upsilon) \left[ \frac{\mathfrak{Y}^2\tau^{2\mathfrak{Y}}}{\Gamma(2\mathfrak{Y} + 1)} + 2\mathfrak{Y}(1 - \mathfrak{Y})\frac{\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y})^2 \right] + \dots. \end{aligned}$$

Particularly, putting  $\mathfrak{Y} = 1$ , we get the exact solution

$$\mu(\mu, \tau) = \frac{-\frac{2}{3}\Upsilon \exp(2\tau)}{-\frac{2}{3} + \Upsilon - \Upsilon \exp(2\tau)}. \quad (6.7)$$

The analytical results by VITM:

The iteration formulas for (6.1), we have

$$\mu_{m+1}(\xi, \tau) = \mu_m(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})}{\omega^{\mathfrak{Y}}} L \left\{ \frac{\omega^{\mathfrak{Y}}}{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})} D_{\xi}^2 \mu_m(\xi, \tau) + 2\mu_m(\xi, \tau) - 3\mu_m^2(\xi, \tau) \right\} \right], \quad (6.8)$$

where  $\mu_0(\xi, \tau) = \Upsilon$ .

For  $m = 0, 1, 2, \dots$ , we have

$$\begin{aligned} \mu_1(\xi, \tau) &= \mu_0(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})}{\omega^{\mathfrak{Y}}} L \left\{ \frac{\omega^{\mathfrak{Y}}}{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})} D_{\xi}^2 \mu_0(\xi, \tau) + 2\mu_0(\xi, \tau) - 3\mu_0^2(\xi, \tau) \right\} \right] \\ \mu_1(\xi, \tau) &= \Upsilon(2 - 3\Upsilon) \left[ \frac{\mathfrak{Y}\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y}) \right], \quad (6.9) \\ \mu_2(\xi, \tau) &= \mu_1(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})}{\omega^{\mathfrak{Y}}} L \left\{ \frac{\omega^{\mathfrak{Y}}}{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})} D_{\xi}^2 \mu_0(\xi, \tau) + 2\mu_0(\xi, \tau) - 3\mu_0^2(\xi, \tau) \right\} \right] \end{aligned}$$



$$\mu_2(\xi, \tau) = 2\Upsilon(2 - 3\Upsilon)(1 - 3\Upsilon) \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right], \quad (6.10)$$

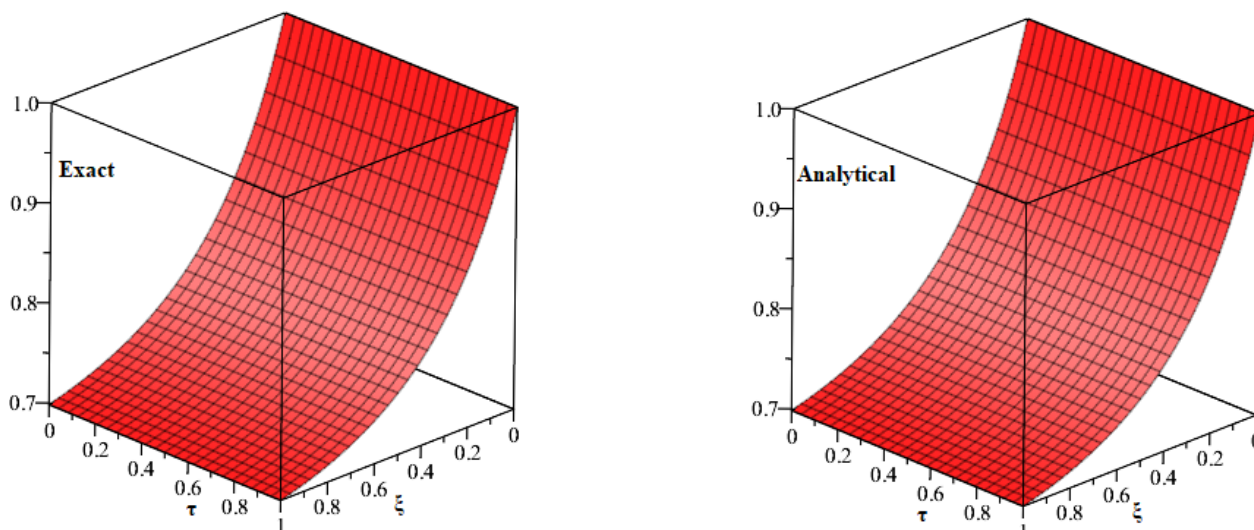
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Therefore, we obtain

$$\begin{aligned} \mu(\xi, \tau) &= \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \Upsilon + \Upsilon(2 - 3\Upsilon) \left[ \frac{\mathfrak{J} \tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right] \\ &+ 2\Upsilon(2 - 3\Upsilon)(1 - 3\Upsilon) \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right] + \dots \end{aligned} \quad (6.11)$$

Particularly, putting  $\mathfrak{J} = 1$ , we get the exact solution, see Figure 1 and Table 1.

$$\mu(\xi, \tau) = \frac{-\frac{2}{3}\Upsilon \exp(2\tau)}{-\frac{2}{3} + \Upsilon - \Upsilon \exp(2\tau)}. \quad (6.12)$$



**Figure 1.** The solution graph of Example 1, the exact solution and the analytical solution at  $\mathfrak{J} = 1$ .

**Table 1.**  $\mu(\xi, \tau)$  Comparison of the exact solution, our methods solution and absolute error (AE) of Example 1.

$\tau = 0.0001$	Exact solution	Our methods solution	AE of our methods	AE of our methods	AE of our methods
$\Upsilon$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 0.9$	$\mathfrak{J} = 0.8$
0	0.0000000000000000	0.0000000000000000	0.0000000000E+00	0.0000000000E+00	0.0000000000E+00
0.1	0.1000001700000000	0.1000001000000000	7.0000000000E-08	2.4390000000E-07	1.5317000000E-06
0.2	0.2000002800000000	0.2000002000000000	8.0000000000E-08	5.4790000000E-07	3.1233000000E-06
0.3	0.3000003300000000	0.3000003000000000	3.0000000000E-08	9.1180000000E-07	4.7750000000E-06
0.4	0.4000003200000000	0.4000004000000000	8.0000000000E-08	1.3357000000E-06	6.4866000000E-06
0.5	0.5000002500000000	0.5000005000000000	2.5000000000E-07	1.8197000000E-06	8.2582000000E-06
0.6	0.6000001200000000	0.6000006000000000	4.8000000000E-07	2.3636000000E-06	1.0089700000E-05
0.7	0.6999999300000000	0.7000007000000000	7.7000000000E-07	2.9675000000E-06	1.1981300000E-05
0.8	0.7999996800000000	0.8000008000000000	1.1200000000E-06	3.6315000000E-06	1.3932800000E-05
0.9	0.8999993700000000	0.9000009000000000	1.5300000000E-06	4.3554000000E-06	1.5944400000E-05
1.0	0.9999990000000000	1.0000010000000000	2.0000000000E-06	5.1390000000E-06	1.8016000000E-05

**Example 2.** The NWSE has given in (1.1) for  $g = 1, h = 1, k = 1$  and  $r = 2$  becomes

$$D_{\tau}^{\mathfrak{J}}\mu(\xi, \tau) = D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau)(1 - \mu(\xi, \tau)), \quad 0 < \mathfrak{J} \leq 1, \quad (6.13)$$

with initial source

$$\mu(\xi, 0) = \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^2}.$$

Applying Laplace transform to (6.13), we get

$$\frac{\omega^{\mathfrak{J}}L[\mu(\xi, \tau)] - \omega^{-1}\mu(\xi, 0)}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} = L\left[D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau)(1 - \mu(\xi, \tau))\right]. \quad (6.14)$$

On taking Laplace inverse transform, we get

$$\mu(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^2} + L^{-1}\left[\frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}}\right]L\left[D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau)(1 - \mu(\xi, \tau))\right]. \quad (6.15)$$

Assume that the solution,  $\mu(\xi, \tau)$  in the form of infinite series given by

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau), \quad (6.16)$$

where  $\mu^2 = \sum_{m=0}^{\infty} \mathcal{A}_m$  are the so-called Adomian polynomials that represent the nonlinear terms, thus (6.15) having certain terms are rewritten as

$$\sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^2} + L^{-1}\left[\frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}}\right]L\left[D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau) - \sum_{m=0}^{\infty} \mathcal{A}_m\right]. \quad (6.17)$$

Applying the proposed analytical approach and the nonlinear terms can be obtained with the aid of Adomian's polynomials stated in (3.7), we acquire

$$\mu_0(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^2}.$$

For  $m = 0$ , we have

$$\mu_1(\xi, \tau) = \frac{5}{3} \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^3} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right].$$

For  $m = 1$ , we have

$$\mu_2(\xi, \tau) = \frac{25}{18} \left( \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)(-1 + 2\exp\left(\frac{\xi}{\sqrt{6}}\right))}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^4} \right) \left[ \frac{\mathfrak{J}^2\tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J})\frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right].$$

The approximate series solution is expressed as

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \mu_0(\xi, \tau) + \mu_1(\xi, \tau) + \mu_2(\xi, \tau) + \dots$$

Therefore, we obtain

$$\begin{aligned} \mu(\xi, \tau) = & \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^2} + \frac{5}{3} \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^3} \left[ \frac{\mathfrak{Y}\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y}) \right] + \frac{25}{18} \left( \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)(-1 + 2 \exp\left(\frac{\xi}{\sqrt{6}}\right))}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^4} \right) \\ & \times \left[ \frac{\mathfrak{Y}^2\tau^{2\mathfrak{Y}}}{\Gamma(2\mathfrak{Y} + 1)} + 2\mathfrak{Y}(1 - \mathfrak{Y})\frac{\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y})^2 \right] + \dots \end{aligned}$$

Particularly, putting  $\mathfrak{Y} = 1$ , we get the exact solution

$$\mu(\mu, \tau) = \left( \frac{1}{1 + \exp\left(\frac{\xi}{\sqrt{6}} - \frac{5}{6}\tau\right)} \right)^2. \quad (6.18)$$

The analytical results by VITM:

The iteration formulas for (6.13), we have

$$\mu_{m+1}(\xi, \tau) = \mu_m(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})}{\omega^{\mathfrak{Y}}} L \left\{ \frac{\omega^{\mathfrak{Y}}}{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})} D_{\xi}^2 \mu_m(\xi, \tau) + \mu_m(\xi, \tau)(1 - \mu_m(\xi, \tau)) \right\} \right], \quad (6.19)$$

where

$$\mu_0(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)}.$$

For  $m = 0, 1, 2, \dots$ , we have

$$\begin{aligned} \mu_1(\xi, \tau) = & \mu_0(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})}{\omega^{\mathfrak{Y}}} L \left\{ \frac{\omega^{\mathfrak{Y}}}{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})} D_{\xi}^2 \mu_0(\xi, \tau) + \mu_0(\xi, \tau)(1 - \mu_0(\xi, \tau)) \right\} \right] \\ \mu_1(\xi, \tau) = & \frac{5}{3} \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^3} \left[ \frac{\mathfrak{Y}\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y}) \right] \end{aligned} \quad (6.20)$$

$$\begin{aligned} \mu_2(\xi, \tau) = & \mu_1(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})}{\omega^{\mathfrak{Y}}} L \left\{ \frac{\omega^{\mathfrak{Y}}}{\omega^{\mathfrak{Y}} + \mathfrak{Y}(1 - \omega^{\mathfrak{Y}})} D_{\xi}^2 \mu_1(\xi, \tau) + \mu_1(\xi, \tau)(1 - \mu_1(\xi, \tau)) \right\} \right] \\ \mu_2(\xi, \tau) = & \frac{25}{18} \left( \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)(-1 + 2 \exp\left(\frac{\xi}{\sqrt{6}}\right))}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^4} \right) \left[ \frac{\mathfrak{Y}^2\tau^{2\mathfrak{Y}}}{\Gamma(2\mathfrak{Y} + 1)} + 2\mathfrak{Y}(1 - \mathfrak{Y})\frac{\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y})^2 \right] \end{aligned} \quad (6.21)$$

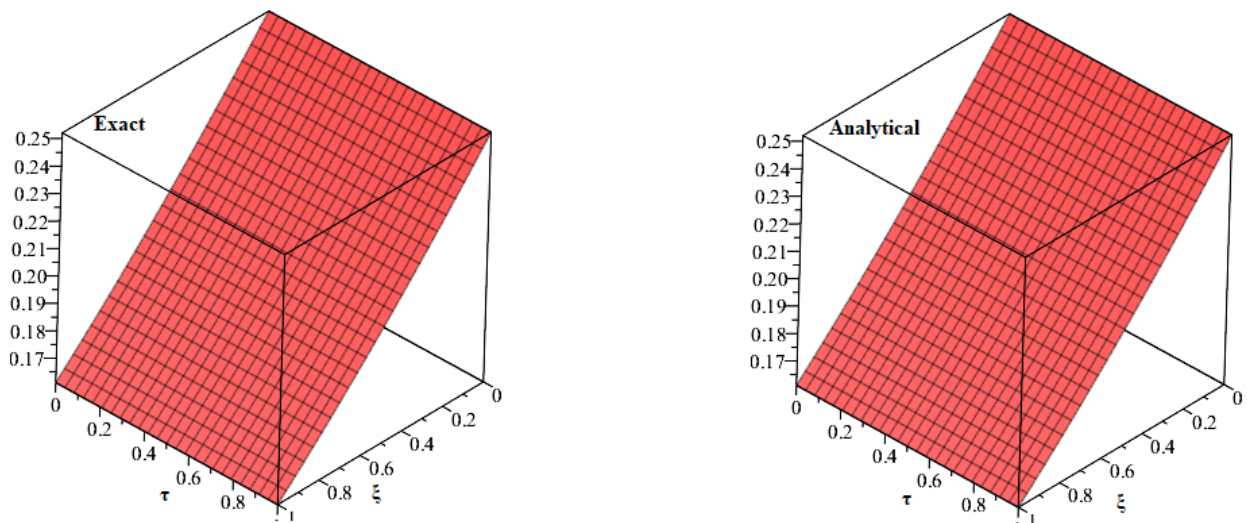
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Therefore, we obtain

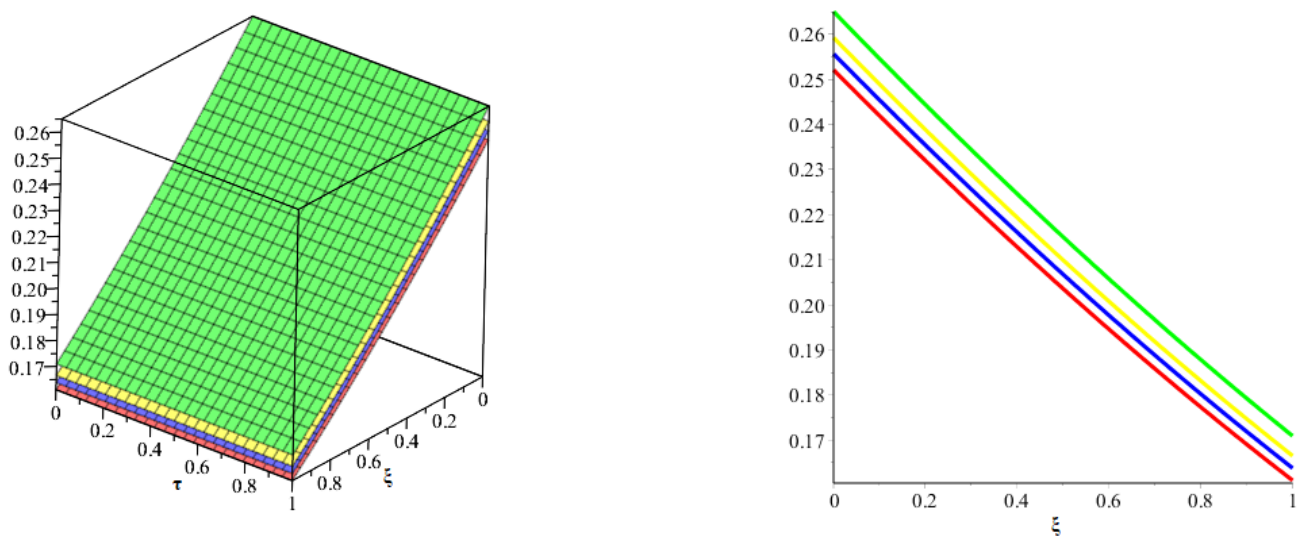
$$\begin{aligned} \mu(\xi, \tau) = & \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^2} + \frac{5}{3} \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^3} \left[ \frac{\mathfrak{Y}\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y}) \right] \\ & + \frac{25}{18} \left( \frac{\exp\left(\frac{\xi}{\sqrt{6}}\right)(-1 + 2 \exp\left(\frac{\xi}{\sqrt{6}}\right))}{\left(1 + \exp\left(\frac{\xi}{\sqrt{6}}\right)\right)^4} \right) \left[ \frac{\mathfrak{Y}^2\tau^{2\mathfrak{Y}}}{\Gamma(2\mathfrak{Y} + 1)} + 2\mathfrak{Y}(1 - \mathfrak{Y})\frac{\tau^{\mathfrak{Y}}}{\Gamma(\mathfrak{Y} + 1)} + (1 - \mathfrak{Y})^2 \right] + \dots \end{aligned} \quad (6.22)$$

Particularly, putting  $\mathfrak{J} = 1$ , we get the exact solution, see Figures 2,3 and Table 2.

$$\mu(\xi, \tau) = \left( \frac{1}{1 + \exp\left(\frac{\xi}{\sqrt{6}} - \frac{5}{6}\tau\right)} \right)^2. \quad (6.23)$$



**Figure 2.** The exact solution and analytical solution of Example 2 at  $\mathfrak{J} = 1$ .



**Figure 3.** The two and three dimensional different fractional order graphs of  $\mathfrak{J}$  example 2.

**Table 2.**  $\mu(\xi, \tau)$  Comparison of exact solution, our methods solution and Absolute Error (AE) of Example 2.

$\tau = 0.0001$	Exact solution	Our methods solution	AE of our methods	AE of our methods	AE of our methods
$\xi$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 0.9$	$\mathfrak{J} = 0.8$
0	0.250020833800000	0.250020833700000	1.0000000000E-10	3.3580800000E-05	1.2032390000E-04
0.1	0.239919747900000	0.239919747800000	5.0941566800E-11	3.2881658260E-05	1.1781851670E-04
0.2	0.230035072700000	0.230035072600000	8.1583004800E-11	3.2156851240E-05	1.1522127190E-04
0.3	0.220374241800000	0.220374241700000	1.3295068340E-10	3.1408781520E-05	1.1254074080E-04
0.4	0.210943948200000	0.210943948100000	1.7637132220E-10	3.0639941880E-05	1.0978578190E-04
0.5	0.201750129200000	0.201750129000000	1.5669454130E-10	2.9852894210E-05	1.0696543270E-04
0.6	0.192797956100000	0.192797956000000	1.0301314750E-10	2.9050141560E-05	1.0408873760E-04
0.7	0.184091829000000	0.184091828700000	2.3924337590E-10	2.8234000470E-05	1.0116457590E-04
0.8	0.175635376500000	0.175635376200000	2.9445508530E-10	2.7407274000E-05	9.8202291510E-05
0.9	0.167431461100000	0.167431460800000	3.1297190840E-10	2.6572425380E-05	9.5210825130E-05
1.0	0.159482188800000	0.159482188600000	2.6399135400E-10	2.5731952810E-05	9.2199048510E-05

**Example 3.** The NWSE has given in (1.1) for  $g = 1, h = 1, k = 1$  and  $r = 4$  becomes

$$D_{\tau}^{\mathfrak{J}}\mu(\xi, \tau) = D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau) - \mu^4(\xi, \tau), \quad 0 < \mathfrak{J} \leq 1 \quad (6.24)$$

with initial source

$$\mu(\xi, 0) = \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}}.$$

Applying Laplace transform to (6.24), we get

$$\frac{\omega^{\mathfrak{J}}L[\mu(\xi, \tau)] - \omega^{-1}\mu(\xi, 0)}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} = L\left[D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau) - \mu^4(\xi, \tau)\right]. \quad (6.25)$$

On taking Laplace inverse transform, we get

$$\mu(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}} + L^{-1}\left[\frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}}\right]L\left[D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau) - \mu^4(\xi, \tau)\right]. \quad (6.26)$$

Assume that the solution,  $\mu(\xi, \tau)$  in the form of infinite series given by

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau), \quad (6.27)$$

where  $\mu^4 = \sum_{m=0}^{\infty} \mathcal{A}_m$  are the so-called Adomian polynomials that represent the nonlinear terms, thus (6.26) having certain terms are rewritten as

$$\sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}} + L^{-1}\left[\frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}}\right]L\left[D_{\xi}^2\mu(\xi, \tau) + \mu(\xi, \tau) - \sum_{m=0}^{\infty} \mathcal{A}_m\right]. \quad (6.28)$$

According to (3.7), the decomposition of nonlinear terms by Adomian polynomials is defined as

$$\mathcal{A}_0 = \mu_0^4, \quad \mathcal{A}_1 = 4\mu_0^3\mu_1, \quad \mathcal{A}_2 = 4\mu_0^3\mu_2 + 6\mu_0^2\mu_1^2. \quad (6.29)$$

Thus, on comparing both sides of (6.28)

$$\mu_0(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}}.$$

For  $m = 0$ , we have

$$\mu_1(\xi, \tau) = \frac{7}{5} \frac{\exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{5}{3}}} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right].$$

For  $m = 1$ , we have

$$\mu_2(\xi, \tau) = \frac{49}{50} \frac{\left(2 \exp\left(\frac{3\xi}{\sqrt{10}}\right) - 3\right) \exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{8}{3}}} \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right].$$

The approximate series solution is expressed as

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \mu_0(\xi, \tau) + \mu_1(\xi, \tau) + \mu_2(\xi, \tau) + \dots.$$

Therefore, we obtain

$$\begin{aligned} \mu(\xi, \tau) = & \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}} + \frac{7}{5} \frac{\exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{5}{3}}} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right] \\ & + \frac{49}{50} \frac{\left(2 \exp\left(\frac{3\xi}{\sqrt{10}}\right) - 3\right) \exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{8}{3}}} \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right] + \dots \end{aligned}$$

Particularly, putting  $\mathfrak{J} = 1$ , we get the exact solution

$$\mu(\xi, \tau) = \frac{1}{2} \tanh\left(-\frac{3}{2\sqrt{10}}\left(\xi - \frac{7}{\sqrt{10}}\tau\right)\right). \quad (6.30)$$

The analytical results by VITM:

The iteration formulas for (6.24), we have

$$\mu_{m+1}(\xi, \tau) = \mu_m(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left\{ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} D_{\xi}^2 \mu_m(\xi, \tau) + \mu_m(\xi, \tau) - \mu_m^4(\xi, \tau) \right\} \right], \quad (6.31)$$

where

$$\mu_0(\xi, \tau) = \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}}.$$

For  $m = 0, 1, 2, \dots$ , we have

$$\mu_1(\xi, \tau) = \mu_0(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left\{ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} D_{\xi}^2 \mu_0(\xi, \tau) + \mu_0(\xi, \tau) - \mu_0^4(\xi, \tau) \right\} \right],$$

$$\mu_1(\xi, \tau) = \frac{7}{5} \frac{\exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{5}{3}}} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right], \quad (6.32)$$

$$\mu_2(\xi, \tau) = \mu_1(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left\{ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} D_{\xi}^2 \mu_1(\xi, \tau) + \mu_1(\xi, \tau) - \mu_1^2(\xi, \tau) \right\} \right],$$

$$\mu_2(\xi, \tau) = \frac{49}{50} \frac{\left(2 \exp\left(\frac{3\xi}{\sqrt{10}}\right) - 3\right) \exp^{\frac{3}{\sqrt{10}}\xi}}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{8}{3}}} \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right]. \quad (6.33)$$

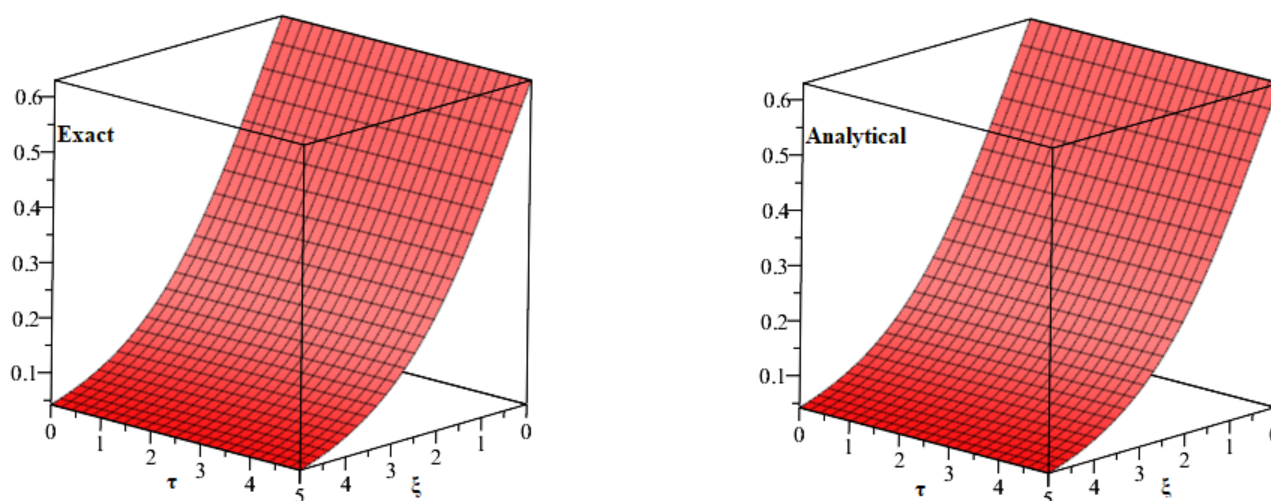
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Therefore, we obtain

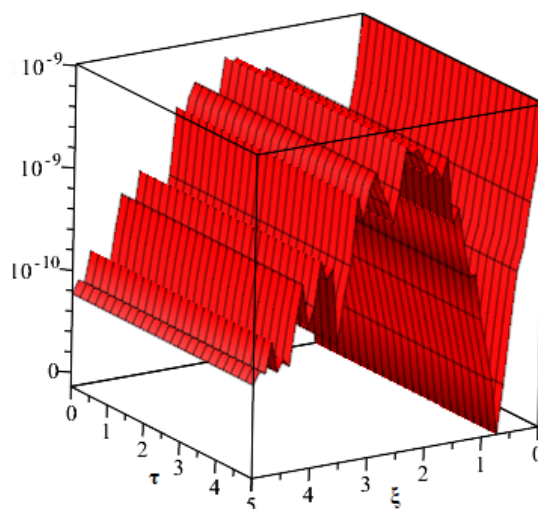
$$\begin{aligned} \mu(\xi, \tau) &= \sum_{m=0}^{\infty} \mu_m(\xi, \tau) \\ &= \frac{1}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{2}{3}}} + \frac{7}{5} \frac{\exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{5}{3}}} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right] \\ &\quad + \frac{49}{50} \frac{\left(2 \exp\left(\frac{3\xi}{\sqrt{10}}\right) - 3\right) \exp\left(\frac{3\xi}{\sqrt{10}}\right)}{\left(1 + \exp\left(\frac{3\xi}{\sqrt{10}}\right)\right)^{\frac{8}{3}}} \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right] + \dots \end{aligned} \quad (6.34)$$

Particularly, putting  $\mathfrak{J} = 1$ , we get the exact solution, see Figures 4,5 and Table 3.

$$\mu(\xi, \tau) = \frac{1}{2} \tanh\left(-\frac{3}{2\sqrt{10}} \left(\xi - \frac{7}{\sqrt{10}}\tau\right)\right). \quad (6.35)$$



**Figure 4.** The exact solution and analytical solution graph of Example 3 at  $\mathfrak{J} = 1$ .



**Figure 5.** The error graph of Example 3 at  $\mathfrak{J} = 1$ .

**Table 3.**  $\mu(\xi, \tau)$  Comparison of exact solution, our methods solution and Absolute Error (AE) of Example 3.

$\tau = 0.0001$	Exact solution	Our methods solution	AE of our methods	AE of our methods	AE of our methods
$\xi$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 0.9$	$\mathfrak{J} = 0.8$
0	0.630004621400000	0.630004623000000	1.6000000000E-09	7.1080200000E-05	2.5467930000E-04
0.1	0.609938405400000	0.609938406700000	1.3721312690E-09	7.2077200130E-05	2.5825129220E-04
0.2	0.589628520300000	0.589628521400000	1.1413782460E-09	7.2815436450E-05	2.6089584380E-04
0.3	0.569148496800000	0.569148497700000	8.7201231790E-10	7.3288425700E-05	2.6259010230E-04
0.4	0.548573070400000	0.548573071000000	6.9266503090E-10	7.3494089020E-05	2.6332630440E-04
0.5	0.527977019100000	0.527977019700000	5.8687018460E-10	7.3434320340E-05	2.6311129590E-04
0.6	0.507434030100000	0.507434030300000	1.9085441870E-10	7.3114619290E-05	2.6196573810E-04
0.7	0.487015628200000	0.487015628200000	1.4619007300E-11	7.2545024110E-05	2.5992429560E-04
0.8	0.466790202200000	0.466790202100000	1.0115621760E-10	7.1738176370E-05	2.5703268760E-04
0.9	0.446822151300000	0.446822151000000	3.7765613140E-10	7.0709457140E-05	2.5334660600E-04
1.0	0.427171171700000	0.427171171200000	5.2356035770E-10	6.9477036140E-05	2.4893040880E-04

**Example 4.** The NWSE has given in (1.1) for  $g = 3, h = 4, k = 1$  and  $r = 3$  becomes

$$D_{\tau}^{\mathfrak{J}}\mu(\xi, \tau) = D_{\xi}^2\mu(\xi, \tau) + 3\mu(\xi, \tau) - 4\mu^3(\xi, \tau), \quad 0 < \mathfrak{J} \leq 1, \quad (6.36)$$

with initial source

$$\mu(\xi, 0) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi)}.$$

Applying Laplace transform to (6.36), we get

$$\frac{\omega^{\mathfrak{J}}L[\mu(\xi, \tau)] - \omega^{-1}\mu(\xi, 0)}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} = L[D_{\xi}^2\mu(\xi, \tau) + 3\mu(\xi, \tau) - 4\mu^3(\xi, \tau)]. \quad (6.37)$$



On taking Laplace inverse transform, we get

$$\mu(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi)} + L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left[ D_{\xi}^2 \mu(\xi, \tau) + 3\mu(\xi, \tau) - 4\mu^3(\xi, \tau) \right] \right]. \quad (6.38)$$

Assume that the solution,  $\mu(\xi, \tau)$  in the form of infinite series given by

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau), \quad (6.39)$$

where  $\mu^3 = \sum_{m=0}^{\infty} \mathcal{A}_m$  are the so-called Adomian polynomials that represent the nonlinear terms, thus (6.38) having certain terms are rewritten as

$$\sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi)} + L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left[ D_{\xi}^2 \mu(\xi, \tau) + 3\mu(\xi, \tau) - 4 \sum_{m=0}^{\infty} \mathcal{A}_m \right] \right]. \quad (6.40)$$

According to (3.7), the decomposition of nonlinear terms by Adomian polynomials is defined as,

$$\mathcal{A}_0 = \mu_0^3, \quad \mathcal{A}_1 = 3\mu_0^2\mu_1, \quad \mathcal{A}_2 = 3\mu_0\mu_1^2 + 3\mu_0\mu_1^2. \quad (6.41)$$

Thus, on comparing both sides of (6.40)

$$\mu_0(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi)}.$$

For  $m = 0$ , we have

$$\mu_1(\xi, \tau) = \frac{9}{2} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp(\frac{\sqrt{6}}{2}\xi)}{(\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi))^2} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right].$$

For  $m = 1$ , we have

$$\begin{aligned} \mu_2(\xi, \tau) &= \frac{81}{4} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp(\frac{\sqrt{6}}{2}\xi) (-\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi))}{(\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi))^3} \\ &\quad \times \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right]. \end{aligned}$$

The approximate series solution is expressed as

$$\mu(\xi, \tau) = \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \mu_0(\xi, \tau) + \mu_1(\xi, \tau) + \mu_2(\xi, \tau) + \dots$$

Therefore, we obtain

$$\mu(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi)} + \frac{9}{2} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp(\frac{\sqrt{6}}{2}\xi)}{(\exp(\sqrt{6}\xi) + \exp(\frac{\sqrt{6}}{2}\xi))^2} \left[ \frac{\mathfrak{J}\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right]$$

$$\begin{aligned}
& + \frac{81}{4} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp\left(\frac{\sqrt{6}}{2}\xi\right) \left(-\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)}{\left(\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)^3} \\
& \times \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right] + \dots .
\end{aligned}$$

Particularly, putting  $\mathfrak{J} = 1$ , we get the exact solution

$$\mu(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi - \frac{9}{2}\tau\right)}. \quad (6.42)$$

The analytical results by VITM:

The iteration formulas for (6.36), we have

$$\mu_{m+1}(\xi, \tau) = \mu_m(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left\{ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} D_{\xi}^2 \mu_m(\xi, \tau) + 3\mu_m(\xi, \tau) - 4\mu_m^3(\xi, \tau) \right\} \right], \quad (6.43)$$

where

$$\mu_0(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)}.$$

For  $m = 0, 1, 2, \dots$ , we have

$$\begin{aligned}
\mu_1(\xi, \tau) &= \mu_0(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left\{ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} D_{\xi}^2 \mu_0(\xi, \tau) + 3\mu_0(\xi, \tau) - 4\mu_0^3(\xi, \tau) \right\} \right], \\
\mu_1(\xi, \tau) &= \frac{9}{2} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp\left(\frac{\sqrt{6}}{2}\xi\right)}{\left(\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)^2} \left[ \frac{\mathfrak{J} \tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right], \quad (6.44)
\end{aligned}$$

$$\begin{aligned}
\mu_2(\xi, \tau) &= \mu_1(\xi, \tau) - L^{-1} \left[ \frac{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})}{\omega^{\mathfrak{J}}} L \left\{ \frac{\omega^{\mathfrak{J}}}{\omega^{\mathfrak{J}} + \mathfrak{J}(1 - \omega^{\mathfrak{J}})} D_{\xi}^2 \mu_1(\xi, \tau) + 3\mu_1(\xi, \tau) - 4\mu_1^3(\xi, \tau) \right\} \right], \\
\mu_2(\xi, \tau) &= \frac{81}{4} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp\left(\frac{\sqrt{6}}{2}\xi\right) \left(-\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)}{\left(\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)^3} \\
& \times \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right]. \quad (6.45)
\end{aligned}$$

⋮

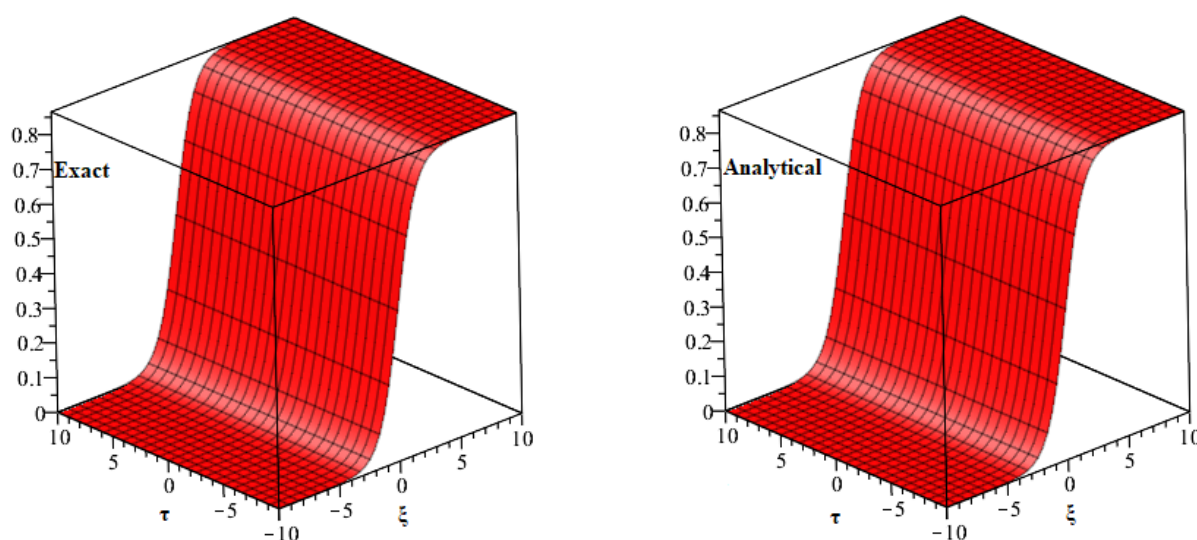
Therefore, we obtain

$$\begin{aligned}
\mu(\xi, \tau) &= \sum_{m=0}^{\infty} \mu_m(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)} \\
& + \frac{9}{2} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp\left(\frac{\sqrt{6}}{2}\xi\right)}{\left(\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)^2} \left[ \frac{\mathfrak{J} \tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J}) \right]
\end{aligned}$$

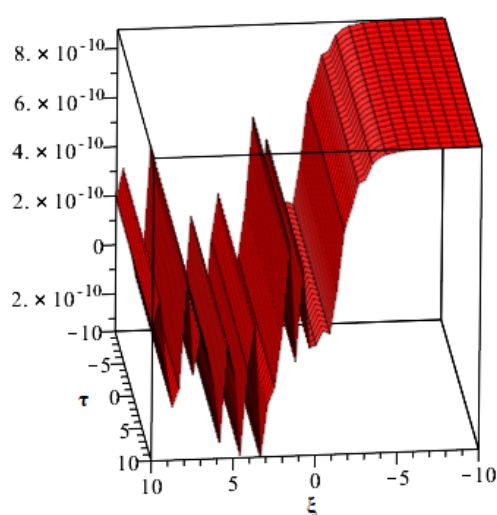
$$\begin{aligned}
& + \frac{81}{4} \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi) \exp\left(\frac{\sqrt{6}}{2}\xi\right) \left(-\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)}{\left(\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi\right)\right)^3} \\
& \times \left[ \frac{\mathfrak{J}^2 \tau^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J} + 1)} + 2\mathfrak{J}(1 - \mathfrak{J}) \frac{\tau^{\mathfrak{J}}}{\Gamma(\mathfrak{J} + 1)} + (1 - \mathfrak{J})^2 \right] + \dots .
\end{aligned} \tag{6.46}$$

Particularly, putting  $\mathfrak{J} = 1$ , we get the exact solution, see Figures 6,7 and Table 4.

$$\mu(\xi, \tau) = \sqrt{\frac{3}{4}} \frac{\exp(\sqrt{6}\xi)}{\exp(\sqrt{6}\xi) + \exp\left(\frac{\sqrt{6}}{2}\xi - \frac{9}{2}\tau\right)}. \tag{6.47}$$



**Figure 6.** The exact solution and analytical solution graph of Example 4 at  $\mathfrak{J} = 1$ .



**Figure 7.** The error graph of Example 4 at  $\mathfrak{J} = 1$ .

**Table 4.**  $\mu(\xi, \tau)$  Comparison of Exact solution, Our methods solution and Absolute Error (AE) of example 4.

$\tau = 0.0001$	Exact solution	Our methods solution	AE of our methods	AE of our methods	AE of our methods
$\xi$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 1$	$\mathfrak{J} = 0.9$	$\mathfrak{J} = 0.8$
0	0.433022444800000	0.433022445000000	1.7320508080E-10	7.9386816730E-06	2.2292879540E-05
0.1	0.459505816600000	0.459505816600000	3.2194421390E-11	7.9089032580E-06	2.2209048130E-05
0.2	0.485791725000000	0.485791725200000	1.6732056440E-10	7.8207979170E-06	2.1961331750E-05
0.3	0.511688623000000	0.511688622800000	2.1377960370E-10	7.6762722350E-06	2.1556125910E-05
0.4	0.537016284000000	0.537016284300000	3.7853826310E-10	7.4809626080E-06	2.1006564010E-05
0.5	0.561610593500000	0.561610593600000	1.3316432970E-10	7.2385076780E-06	2.0326137420E-05
0.6	0.585327390000000	0.585327389900000	4.4316889200E-11	6.9562540670E-06	1.9533837800E-05
0.7	0.608045207500000	0.608045207500000	4.4341099960E-11	6.6414869210E-06	1.8649762960E-05
0.8	0.629666845500000	0.629666845900000	4.6131349780E-10	6.3016518320E-06	1.7694707350E-05
0.9	0.650119804000000	0.650119803700000	7.6816056310E-10	5.9421274000E-06	1.6687344050E-05
1.0	0.669355693000000	0.669355693500000	4.9676049840E-10	5.5740946070E-06	1.5651582380E-05

## 7. Results and discussion

Figure 1, show the behavior of the exact and proposed methods solution at  $\mathfrak{J} = 1$  in (AB fractional derivative) manner of Example 1. The comparison of the exact and analytical solution of Example 2 is shown Figure 2, whereas the graphical view for various fractional orders is demonstrated with the help of figures. In Figure 3, the two and three dimensional different fractional order graphs of Example 2. The figures show that our solution approaches the exact solution as the fractional order goes towards the integer-order. Figure 4, demonstrate the layout of the exact and analytical solution while Figure 5 shows the error comparison of the exact and analytical results of Example 3. The error confirms the efficiency of the suggested techniques. The graphical view of Example 4 for exact and our solution can be seen in Figure 6, however, Figure 7 shows the error comparison of both results. Furthermore, the behavior of the exact and proposed method solution with the aid of absolute error at different orders of  $\mathfrak{J}$  is shown in Tables 1–4. Finally, it is clear from the figures and tables that the proposed methods have a sufficient degree of accuracy and quick convergence towards the exact solution.

## 8. Conclusions

The LTDM and VITM were used for solving time fractional Newell-Whitehead-Segel equation. The solution we obtained is a series that quickly converges to exact solutions. Four cases are studied, which shows that the proposed methods solutions strongly agree with the exact solution. It is found that the suggested techniques are easy to implement and need a small number of calculations. This shows that LTDM and VITM are very efficient, effective, and powerful mathematical tools easily applied in finding approximate analytic solutions for a wide range of real-world problems arising in science and engineering.

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## Conflict of interest

The authors declare that they have no competing interests.

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