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Research article

Threshold dynamics of a time-periodic two-strain SIRS epidemic model with distributed delay

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Abstract: In this paper, a two-strain SIRS epidemic model with distributed delay and spatiotemporal heterogeneity is proposed and investigated. We first introduce the basic reproduction number R_0^i and the invasion number $\hat{R}_0^i$ $(i = 1, 2)$ for each strain i. Then the threshold dynamics of the model is established in terms of R_0^i and $\hat{R}_0^i$ by using the theory of chain transitive sets and persistence. It is shown that if $\hat{R}_0^i > 1$ $(i = 1, 2)$, then the disease in two strains is persist uniformly; if $R_0^i > 1 \geq R_0^j$ $(i \neq j, i, j = 1, 2)$, then the disease in i-th strain is uniformly persist, but the disease in j-th strain will disappear; if $R_0^i < 1$ or $R_0^i = 1$ $(i = 1, 2)$ and $\beta_i(x, t) > 0$, then the disease in two strains will disappear.

Keywords: SIRS epidemic model; spatiotemporal heterogeneity; distributed delay; threshold dynamics
Mathematics Subject Classification: 35B35, 35B40, 35K57, 92D30

1. Introduction

Nowadays there are always various communicable diseases, such as malaria, dengue fever, HIV/AIDS, Zika virus, and COVID-19, which impair the health of people around the globe [2]. Especially, as of now, COVID-19 has killed more than 4 million people and is still prevailing in many countries over the world. Since Covid-19 was first identified in January 2020, thousands of mutations have been detected [34]. Moreover, it has been reported that various new strains of COVID-19 are considered as more dangerous than the original virus. In fact, the variation of pathogens is very common in epidemiology, we can refer to [4] for the instance of the mutation of influenza virus. Besides, Dengue fever is one of the most typical vector-borne infectious disease prevailing in the tropical and subtropical areas. Usually, the fever is caused by five different serotypes (DEN I-IV) and the corresponding fatality rates of these serotypes are dramatically different. This means that a person living in an endemic area might be facing the risk of infection from five distinct serotypes, and a individual who recovered form one of the serotypes could get permanent immunity to itself and only temporary cross-immune against the others. In recent years, mathematical model increasingly become a effective tool in the investigation of the spread of epidemics. With the aid of proper analysis for the mathematical models, we can better understand the transmission mechanism of infectious diseases and then take appropriate prevention and control measures to combat the diseases. In fact, the researches of epidemic dynamics models involving multi-strain interactions have attracted considerable attention of many scholars. Baba et al. [4] studied a two-strain model containing vaccination for both strains. Cai et al. [6] studied a two-strain model including vaccination, and analyzed the interaction between the strains under the vaccination theme. A class of multi-chain models with discrete time delays, moreover, is considered in the case of temporary immunity and multiple cross immunity by Bauer et al. [5]. For more literatures corresponding to pathogens with multiple strains, we can refer to [1, 8, 24, 30, 33, 39] and the references.

In reality, accumulating empirical evidence shows that seasonal factors can affect the host-pathogen interactions [3], and the incidence of many infectious diseases fluctuates over time, often with a cyclical pattern(see, e.g., [16, 31, 37]). In addition, Yang, et al [35] found that temperature and relative humidity were mainly the driving factors on COVID-19 transmission. It is therefore necessary to consider infectious disease models with time-dependent parameters. Martcheva et al. [24] considered a class of multi-chain models with time-periodic coefficients. Precisely speaking, they presented sufficient conditions to guarantee the coexistence of the two-strain, and further proved that competitive exclusion would occur only when the transmission rates on each chain are linearly correlated.

At the same time, it is noticed that the resources, humidity and temperature are not uniformly distributed in space, then spatial heterogeneity should not be ignored a practical epidemiological model. From the point of view of model's rationalization, the main parameters, such as infection rate and recovery rate, should be intrinsically spatially dependent. Taking into consideration both the spatial heterogeneity of the environment and the impact of individual movement on disease transmission, Tuncer et al. [33] proposed the following two-strain model:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_S \Delta S(x,t) - \dfrac{(\beta_1(x) I_1(x,t) + \beta_2(x) I_2(x,t))\, S(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t)} \\ \qquad\qquad + r_1(x,t) I_1(x,t) + r_2(x,t) I_2(x,t), \\ \dfrac{\partial I_1(x,t)}{\partial t} = d_1 \Delta I_1(x,t) + \dfrac{\beta_1(x) S(x,t) I_1(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t)} - r_1(x,t) I_1(x,t), \\ \dfrac{\partial I_2(x,t)}{\partial t} = d_2 \Delta I_2(x,t) + \dfrac{\beta_2(x) S(x,t) I_2(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t)} - r_2(x,t) I_2(x,t). \end{cases} \tag{1.1}$$

Furthermore, Acklehd et al. [1] studied a model with bilinear incidence, the results showed that the spatial heterogeneity facilitated the coexistence of strains. Taking into account alternation of seasons, Peng et al. [29] studied a reaction-diffusion SIS model, in which the disease transmission rate and recovery rate are all spatial-dependent and temporally periodic. The results show that temporal heterogeneity have little effect on the extinction and persistence of the diseases, nevertheless, the combination of temporal and spatial heterogeneity would increase the duration of the disease.

It is well known that the incubation period exist commonly in most infectious diseases, and the length of the incubation periods corresponding to different diseases are often different. We can refer Leung [19] for more information about the difference of the incubation period of COVID-19 between various different variants. During the incubation period, random movements of individuals can give rise to nonlocal effects, precisely speaking, the rate of gaining infectious individuals at current position at the present time actually depends on the infections at all possible locations and all possible previous times. This nonlocal interaction will affect the global dynamic behavior of the solutions [7, 13], traveling wave phenomena [14], etc. Guo et al. [12] studied the threshold dynamics of a reaction-diffusion model with nonlocal effects. In particular, Zhao et al. [38] considered the threshold dynamics of a model with fixed latent period on the basis of model (1.1). In particular, when $R_0^i = 1$ and the infection rate is assumed to be strictly positive, they studied the threshold dynamics of the model by constructing the upper control system.

Due to the individual difference in age, nutrition, lifestyle and health status, there are significant difference in the immunity among different individuals [27]. This further lead to the difference of incubation periods in different individuals. As McAloon, et al. [27] points out, it is critically important to understand the variation in the distribution within the population. Thus, the fixed incubation period is not always an ideal description for most diseases. Takeuchi et al. [32] considered a vector-borne SIR infectious disease model with distributed time delay. Zhao et al. [39] studied a two-group reaction-diffusion model with distributed delay. In [39], the recovered individuals are assumed to be lifelong immune to the disease. However, this assumption is not suitable for all epidemics. Then it is very necessary to establish and analysis a SIRS model involving aforementioned various factors, and thus to further improve the existing relevant research. The purpose of this paper is to investigate the threshold dynamics of a two-strain SIRS epidemic model with distributed delay and spatiotemporal heterogeneity.

The remainder of this paper is organized as follows. In the next section, we derive the model and show its well-posedness. In section 3, we established the threshold dynamics for the system in term of the basic reproduction number R_0^i and the invasion number $\hat{R}_0^i$ $(i = 1, 2)$ for each strain i. At the end of the current paper, a brief but necessary discussion is presented to show some epidemiological implications of this study.

2. Model formulation and well-posedness

In this section, we propose a time-periodic two-strain SIRS model with distributed delay and spatiotemporal heterogeneity, and further analyze some useful properties of the solutions of the model.

2.1. Model formulation

Let $\Omega \in R^n$ denote the spatial habitat with smooth boundary $\partial\Omega$. We suppose that only one mutant can appear in a pathogen, and a susceptible individual can only be infected by one virus strain. Denote the densities of the two different infectious classes with infection age $a \geq 0$ and at position x, and time t by $E_1(x, a, t)$ and $E_2(x, a, t)$, respectively. By a standard argument on structured population and spatial diffusion (see e.g., [28]), we obtain

$$\begin{cases} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)E_i(x,a,t)=D_i\Delta E_i-(\delta_i(x,a,t)+r_i(x,a,t)+d(x,t))\,E_i(x,a,t),\ x\in\Omega,\ a>0,\ t>0\\ \frac{\partial E_i(x,a,t)}{\partial n}=0,\ x\in\partial\Omega,\ a>0,\ t>0, i=1,2, \end{cases} \tag{2.1}$$

where $d(x,t)$ is the natural death rate at location x and time t; $r_i(x,a,t)$ and $\delta_i(x,a,t)$ represent the recovery rates and mortality rates induced by the disease of the i-th infectious classes with infection age $a \geq 0$ at position x and time t; the constants D_i denote the diffusion rates of the i-th infectious class for $i = 1,2$. We divide the population into six compartments: the susceptible group $S(x,t)$, two latent groups $L_i(x,t)$, two infective groups $I_i(x,t)$, and the recovered group $R(x,t)$, $i = 1,2$. Let $N(x,t) = S(x,t) + \sum_{i=1,2}(L_i(x,t) + I_i(x,t)) + R(x,t)$. We assume that only a portion of recovered individuals would be permanently immune to the virus. Let $\alpha(x,a,t)$ be the loss of immunity rate with infection age $a \geq 0$ at position x and time t. In order to simplify the model reasonably, we further suppose that

$$\delta_i(x,a,t)=\delta_i(x,t),\ r_i(x,a,t)=r_i(x,t),\ \alpha(x,a,t)=\alpha(x,t),\ \forall x\in\Omega,\ a,t\geq 0,\ i=1,2.$$

On account of the individual differences of the incubation period among the different individuals, infections individuals of the i-th population be capable of infecting others until after a possible infection age $a \in (0,\tau_i]$, where the positive constant τ_i is the maximum incubation period of i-th strain, $i = 1,2$. Let $f_i(r)dr$ denote the probability of becoming into the individuals who are capable of infecting others between the infection age r and $r+dr$, then $F_i(a) = \int_0^a f_i(r)dr$ represents the probability of turning into the individuals with infecting others before the infection age a for $i = 1,2$. It is clear that $F_i(a) \geq 0$ for $a \in (0,\tau_i)$, $F_i(a) \equiv 1$ for $a \in [\tau_i,+\infty)$, $i = 1,2$, and

$$\begin{aligned} L_i(x,t) &= \int_0^{\tau_i}(1-F_i(a))\,E_i(x,a,t)da,\\ I_i(x,t) &= \int_0^{\tau_i}F_i(a)E_i(x,a,t)da+\int_{\tau_i}^{+\infty}E_i(x,a,t)da,\ i=1,2. \end{aligned} \tag{2.2}$$

Let

$$I_{i£1}(x,t)=\int_0^{\tau_i}F_i(a)E_i(x,a,t)da,\ I_{i£2}(x,t)=\int_{\tau_i}^{+\infty}E_i(x,a,t)da.$$

It then follows that

$$\begin{aligned} \frac{\partial L_i(x,t)}{\partial t} =& D_i\Delta L_i(x,t)-(\delta_i(x,t)+r_i(x,t)+d(x,t))\,L_i(x,t)\\ &-\int_0^{\tau_i}f_i(a)E_i(x,a,t)da+E_i(x,0,t), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial I_{i,1}(x,t)}{\partial t} =& D_i\Delta I_{i,1}(x,t)-(\delta_i(x,t)+r_i(x,t)+d(x,t))\,I_{i,1}(x,t)\\ &+\int_0^{\tau_i}f_i(a)E_i(x,a,t)da-E_i(x,\tau_i,t), \end{aligned}$$

$$\frac{\partial I_{i,2}(x,t)}{\partial t}=D_i\Delta I_{i,2}(x,t)-(\delta_i(x,t)+r_i(x,t)+d(x,t))\,I_{i,2}(x,t)+E_i(x,\tau_i,t)-E_i(x,\infty,t),$$

where $i = 1, 2$. Biologically, we assume $E_i(x, \infty, t) = 0$ $(i = 1, 2)$. Then we have

$$\frac{\partial I_i(x,t)}{\partial t} = D_i \Delta I_i(x,t) - (\delta_i(x,t) + r_i(x,t) + d(x,t))\, I_i(x,t) + \int_0^{\tau_i} f_i(a) E_i(x,a,t)da.$$

Denote the infection rate by $\beta_i(x, t) \geq 0$. Due to the fact that the contact of susceptible and infectious individuals yields the new infected individuals, we take $E_i(x, 0, t)$ as follows:

$$E_i(x,0,t) = \frac{\beta_i(x,t)S(x,t)I_i(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t) + R(x,t)},\ i = 1, 2.$$

In the absence of disease, moreover, we suppose that the evolution of the population density follows the following equation:

$$\frac{\partial N(x,t)}{\partial t} = D_N \Delta N(x,t) + \mu(x,t) - d(x,t)N(x,t),$$

where $d(x, t)$ is the natural death rate, $\mu(x, t)$ is the recruiting rate, and D_N denotes the diffusion rate. In conclusion, the disease dynamics is expressed by the following system:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = D_S \Delta S(x,t) + \mu(x,t) - d(x,t)S(x,t) + \alpha(x,t)R(x,t) \\ \qquad - \dfrac{\beta_1(x,t)S(x,t)I_1(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t) + R(x,t)} - \dfrac{\beta_2(x,t)S(x,t)I_2(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t) + R(x,t)}, \\ \dfrac{\partial L_i(x,t)}{\partial t} = D_i \Delta L_i(x,t) - (\delta_i(x,t) + r_i(x,t) + d(x,t))\, L_i(x,t) \\ \qquad + \dfrac{\beta_i(x,t)S(x,t)I_i(x,t)}{S(x,t) + I_1(x,t) + I_2(x,t) + R(x,t)} - \displaystyle\int_0^{\tau_i} f_i(a)E_i(x,a,t)da, \\ \dfrac{\partial I_i(x,t)}{\partial t} = D_i \Delta I_i(x,t) - (\delta_i(x,t) + r_i(x,t) + d(x,t))\, I_i(x,t) + \displaystyle\int_0^{\tau_i} f_i(a)E_i(x,a,t)da, \\ \dfrac{\partial R(x,t)}{\partial t} = D_R \Delta R(x,t) + r_1(x,t)(L_1(x,t) + I_1(x,t)) + r_2(x,t)(L_2(x,t) + I_2(x,t)) \\ \qquad - d(x,t)R(x,t) - \alpha(x,t)R(x,t),\ i = 1, 2. \end{cases} \tag{2.3}$$

We make the following basic assumption:

(H) $D_S, D_i, D_R > 0,\ i = 1, 2$, the functions $d(x, t), \mu(x, t), \alpha(x, t), \beta_i(x, t), \delta_i(x, t), r_i(x, t)$ are Hölder continuous and nonnegative nontrivial on $\bar{\Omega} \times R$, and periodic in time t with the same period $T > 0$. Moreover, $d(x, t) > 0$, $x \in \partial\Omega$, $t > 0$.

For the sake of simplicity, we let $h_i(x, t) = \delta_i(x, t) + r_i(x, t) + d(x, t),\ i = 1, 2$. In order to determine $E_i(x, a, t)$, let $V_i(x, a, \xi) = E_i(x, a, a + \xi)$, $\forall \xi \geq 0$, $i = 1, 2$. By a similar idea as that in [36], we have

$$\begin{cases} \frac{\partial V_i(x,a,\xi)}{\partial a} = D_i \Delta V_i(x,a,\xi) - h_i(x,t)V_i(x,a,\xi), \\ V_i(x,0,\xi) = E_i(x,0,\xi) = \frac{\beta_i(x,\xi)S(x,\xi)I_i(x,\xi)}{S(x,\xi)+I_1(x,\xi)+I_2(x,\xi)+R(x,\xi)},\ i = 1, 2. \end{cases}$$

Let $\Gamma_i(x, y, t, s)$ with $x, y \in \Omega$ and $t > s \geq 0$ be the fundamental solution associated with the partial differential operator $\partial t - D_i \Delta - h_i(x, t)(i = 1, 2)$. Then we have

$$V_i(x,a,\xi) = \int_\Omega \Gamma_i(x,y,\xi + a,\xi) \frac{\beta_i(y,\xi)S(y,\xi)I_i(y,\xi)}{S(y,\xi) + I_1(y,\xi) + I_2(y,\xi) + R(y,\xi)} dy. \tag{2.4}$$

According to the periodicity of h_i and β_i, $\Gamma_i(x,y,t,s)$ is periodic, that is, $\Gamma_i(x,y,t+T,s+T) = \Gamma_i(x,y,t,s),\ \forall x,y\in\Omega,\ t>s\geq 0,\ i=1,2$. It follows from $E_i(x,a,t)=V_i(x,a,t-a)$ that

$$E_i(x,a,t) = \int_\Omega \Gamma_i(x,y,t,t-a)\frac{\beta_i(y,t-a)S(y,t-a)I_i(y,t-a)}{S(y,t-a)+I_1(y,t-a)+I_2(y,t-a)+R(y,t-a)}dy. \tag{2.5}$$

Substituting (2.5) into (2.3), and dropping the L_i equations from (2.3) (since they are decoupled from the other equations), we obtain the following system:

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t} = D_S\Delta S(x,t)+\mu(x,t)-d(x,t)S(x,t)+\alpha(x,t)R(x,t)\\ \qquad -\dfrac{\beta_1(x,t)S(x,t)I_1(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)+R(x,t)}-\dfrac{\beta_2(x,t)S(x,t)I_2(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)+R(x,t)},\\ \dfrac{\partial I_i(x,t)}{\partial t} = D_i\Delta I_i(x,t)-(\delta_i(x,t)+r_i(x,t)+d(x,t))\,I_i(x,t)+\displaystyle\int_0^{\tau_i} f_i(a)\\ \qquad \cdot\displaystyle\int_\Omega \Gamma_i(x,y,t,t-a)\frac{\beta_i(y,t-a)S(y,t-a)I_i(y,t-a)}{S(y,t-a)+I_1(y,t-a)+I_2(y,t-a)+R(y,t-a)}dyda,\\ \dfrac{\partial R(x,t)}{\partial t} = D_R\Delta R(x,t)+r_1(x,t)I_1(x,t)+r_2(x,t)I_2(x,t)-d(x,t)R(x,t)\\ \qquad -\alpha(x,t)R(x,t),\ i=1,2.\end{cases} \tag{2.6}$$

Set $\tau=\max\{\tau_1,\tau_2\}>0$. Let $X := C\left(\bar\Omega,R^4\right)$ be the Banach space with a supremum norm $\|\cdot\|_X$. Let $C_\tau := C([-\tau,0],X)$ be a Banach space with the norm $\|\phi\| = \max\limits_{\theta\in[-\tau,0]}\|\phi(\theta)\|_X,\ \forall\phi\in C_\tau$. Define $X^+ := C\left(\bar\Omega,R^4_+\right)$, $C^+_\tau := C([-\tau,0],X^+)$, the (X,X^+) and (C_τ,C^+_τ) are strongly ordered spaces. For $\sigma>0$ and a given function $u(t):[-\tau,\sigma]\to X$, we denote $u_t\in C_\tau$ by

$$u_t(\theta)=u(t+\theta),\ \forall\theta\in[-\tau,0].$$

Similarly, define $Y=C\left(\bar\Omega,R\right)$ and $Y^+=C\left(\bar\Omega,R^+\right)$. Furthermore, we consider the following system:

$$\begin{cases}\frac{\partial\omega(x,t)}{\partial t} = D_S\Delta\omega(x,t)-d(x,t)\omega(x,t),\ x\in\Omega,\ t>0,\\ \frac{\partial\omega(x,t)}{\partial t}=0,\ \ x\in\partial\Omega,\ t>0,\\ \omega(x,0)=\phi_S(x),\ x\in\Omega,\ \phi_S\in Y^+.\end{cases} \tag{2.7}$$

By the arguments in [15], Eq (2.7) exists an evolution operator $V_S(t,s): Y_+ \longrightarrow Y_+$ for $0\leq s\leq t$, which satisfies $V_S(t,t)=I,\ V_S(t,s)V_S(s,\rho)=V_S(t,\rho),\ 0\leq\rho\leq s\leq t,\ V_S(t,0)\phi_S=\omega(x,t;\phi_S),\ x\in\Omega,\ t\geq 0,\ \phi_S\in Y_+$, where $\omega(x,t;\phi_S)$ is the solution of (2.7).

Consider the following periodic system:

$$\begin{cases}\frac{\partial\bar\omega_i(x,t)}{\partial t} = D_i\Delta\bar\omega_i(x,t)-h_i(x,t)\bar\omega_i(x,t)), & x\in\Omega,\ t>0,\\ \frac{\partial\bar\omega_i(x,t)}{\partial n}=0, & x\in\partial\Omega,\ t>0,\\ \bar\omega_i(x,0)=\phi_i(x), & x\in\Omega,\ \phi_i\in Y^+.\end{cases} \tag{2.8}$$

and

$$\begin{cases}\frac{\partial\tilde\omega_R(x,t)}{\partial t} = D_R\Delta\tilde\omega_R(x,t)-k(x,t)\tilde\omega_R(x,t)), & x\in\Omega,\ t>0,\\ \frac{\partial\tilde\omega_R(x,t)}{\partial t}=0, & x\in\partial\Omega,\ t>0,\\ \tilde\omega_R(x,0)=\phi_R(x), & x\in\Omega,\ \phi_R\in Y^+,\end{cases} \tag{2.9}$$

where $k(x,t)=\alpha(x,t)+d(x,t)$. Let $V_i(t,s)$, $i=1,2$, and $V_R(t,s)$ be the evolution operators determined by (2.8) and (2.9), respectively. The periodicity hypothesis (**H**) combining with [9, Lemma 6.1] yield that $V_S(t+T,s+T)=V_S(t,s)$, $V_i(t+T,s+T)=V_i(t,s)$ and $V_R(t+T,s+T)=V_R(t,s)$, $t\geq s\geq 0$. In addition, for any $t,s\in R$ and $s<t$. $V_S(t,s)$, $V_i(t,s)$ and $V_R(t,s)$ are compact, analytic and strongly positive operators on Y_+. It then follows from [9, Theorem 6.6] that there exist constants $Q\geq 1$, $Q_i\geq 1$ and $c_0, c_i\in R$ $(i=1,2)$ such that

$$\|V_S(t,s)\|, \|V_R(t,s)\|\leq Qe^{-c_0(t-s)},\ \|V_i(t,s)\|\leq Q_ie^{-c_i(t-s)},\ \forall t\geq s,\ i=1,2.$$

Let $c_i^*:=\bar{\omega}(V_i)$, where

$$\bar{\omega}(V_i)=\inf\{\omega|\exists M\geq 1:\ \forall s\in R,\ t\geq 0, \|V_i(t+s,s)\|\leq M\cdot e^{\omega t}\}$$

is the exponent growth bound of the evolution operator $V_i(t,s)$. It is clear that $c_i^*<0$.

Define functions $F_S, F_i, F_R:[0,\infty)\longrightarrow Y$ respectively by

$$\begin{aligned}
F_S(t,\phi)&=\mu(\cdot,t)+\alpha(\cdot,t)\phi_S(\cdot,0)-\sum_{i=1}^{2}\frac{\beta_i(\cdot,0)\phi_S(\cdot,0)\phi_i(\cdot,0)}{\phi_S(\cdot,0)+\phi_1(\cdot,0)+\phi_2(\cdot,0)+\phi_R(\cdot,0)},\\
F_i(t,\phi)&=\int_0^{\tau_i}f_i(a)\int_\Omega\Gamma_i(x,y,t,t-a)\frac{\beta_i(y,t-a)\phi_S(y,-a)\phi_i(y,-a)}{\phi_S(y,-a)+\phi_1(y,-a)+\phi_2(y,-a)+\phi_R(y,-a)}dyda,\\
F_R(t,\phi)&=r_1(\cdot,0)\phi_1(\cdot,0)+r_2(\cdot,0)\phi_2(\cdot,0).
\end{aligned}$$

Let $F=(F_S,F_1,F_2,F_R)$, it is clear that F is a function from $[0,\infty)$ to X. Define

$$U(t,s):=\begin{pmatrix}V_S(t,s)&0&0&0\\0&V_1(t,s)&0&0\\0&0&V_2(t,s)&0\\0&0&0&V_R(t,s)\end{pmatrix}.$$

Then $U(t,s)$ is an evolution operator from X to X. Note that V_S, V_i $(i=1,2)$ and V_R are analytic operators, it follows that $U(t,s)$ is an analytic operator for $(t,s)\in R^2$ with $t\geq s\geq 0$. Let

$$\begin{aligned}
&D(A_S(t))=\left\{\psi\in C^2(\bar{\Omega})\mid\ \frac{\partial}{\partial n}\psi=0 \text{ on } \partial\Omega\right\};\\
&[A_S(t)\psi](x)=D_S\Delta\psi(x)-d(x,t)\psi(x),\ \forall\psi\in D(A_S(t));\\
&D(A_i(t))=\left\{\psi\in C^2(\bar{\Omega})\mid\ \frac{\partial}{\partial n}\psi=0 \text{ on } \partial\Omega\right\};\\
&[A_i(t)\psi](x)=D_i\Delta\psi(x)-h_i(x,t)\psi(x),\ \forall\psi\in D(A_i(t)),
\end{aligned}$$

and

$$\begin{aligned}
&D(A_R(t))=\left\{\psi\in C^2(\bar{\Omega})\mid\ \frac{\partial}{\partial n}\psi=0 \text{ on } \partial\Omega\right\};\\
&[A_R(t)\psi](x)=D_R\Delta\psi(x)-k(x,t)\psi(x),\ \forall\psi\in D(A_R(t)).
\end{aligned}$$

Moreover, we let

$$A(t) := \begin{pmatrix} A_S(t) & 0 & 0 & 0 \\ 0 & A_1(t) & 0 & 0 \\ 0 & 0 & A_2(t) & 0 \\ 0 & 0 & 0 & A_R(t) \end{pmatrix}.$$

Then (2.3) can be rewritten as the following Cauchy problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = A(t)u(x,t) + F(t,u_t), \ x \in \Omega, \ t > 0, \\ u(x,\zeta) = \phi(x,\zeta), \ x \in \Omega, \ \zeta \in [-\tau, 0], \end{cases} \tag{2.10}$$

where $u(x,t) = (S(x,t), I_1(x,t), I_2(x,t), R(x,t))^{\mathrm{T}}$. Furthermore, it can be rewritten as the following integral equation

$$u(t,\phi) = U(t,0)\phi(0) + \int_0^t U(t,s)F(t,u_s)ds, \ t \geq 0, \ \phi \in C_\tau^+.$$

Then the solution of above integral equation is called a mild solution of (2.10).

2.2. Well-posedness

Theorem 2.1. *For each $\phi \in C_\tau^+$, system* (2.6) *admits a unique solution $u(t,\phi)$ on* $[0,+\infty)$ *with $u_0 = \phi$, and $u(t,\phi)$ is globally bounded.*

Proof. By the definition of $F(t,\phi)$ and the assumption (**H**), $F(t,\phi)$ is locally Lipschitz continuous on $R_+ \times C_\tau^+$. We first show

$$\lim_{\theta\to 0^+} \mathrm{dist}(\phi(0) + \theta F(t,\phi), X^+) = 0, \ \forall (t,\phi) \in R_+ \times C_\tau^+. \tag{2.11}$$

Set

$$\bar{\beta} = \max\{\max_{x\in\bar{\Omega},t\in[0,\tau]} \beta_1(x,t), \max_{x\in\bar{\Omega},t\in[0,\tau]} \beta_2(x,t)\};$$
$$m_i(x,t) = \frac{\beta_i(x,t)\phi_S(x,0)\phi_i(x,0)}{\phi_S(x,0) + \phi_1(x,0) + \phi_2(x,0) + \phi_R(x,0)};$$
$$n_i(x,t) = \frac{\beta_i(x,t)\phi_S(x,t)\phi_i(x,t)}{\phi_S(x,t) + \phi_1(x,t) + \phi_2(x,t) + \phi_R(x,t)}.$$

For any $t \geq 0$, $\theta \geq 0$ and $x \in \bar{\Omega}$, $\phi \in C_\tau^+$, we have

$$\begin{aligned} \phi(x,0) + \theta F(t,\phi)(x) &= \begin{pmatrix} \phi_S(x,0) + \theta[\mu(x,t) + \alpha(x,t)\phi_R(x,0) - \sum_{i=1}^{2} m_i(x,t)] \\ \phi_1(x,0) + \theta \int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(x,y,t,t-a)\phi_1(y,t-a)dyda \\ \phi_2(x,0) + \theta \int_0^{\tau_2} f_1(a) \int_\Omega \Gamma_2(x,y,t,t-a)\phi_2(y,t-a)dyda \\ \phi_R(x,0) + r_1(x,t)\phi_1(x,0) + r_2(x,t)\phi_2(x,0) \end{pmatrix} \\ &\geq \begin{pmatrix} \phi_S(x,0)\left(1 - \theta \sum_{i=1}^{2} \frac{\beta_i(x,t)\phi_i(x,0)}{\phi_S(x,0)+\phi_1(x,0)+\phi_2(x,0)+\phi_R(x,0)}\right) \\ \phi_1(x,0) \\ \phi_2(x,0) \\ \phi_R(x,0) \end{pmatrix} \end{aligned}$$

$$\geq \begin{pmatrix} \phi_S(x,0)\left(1-\theta\bar{\beta}\sum\limits_{i=1}^{2}\frac{\phi_i(x,0)}{\phi_S(x,0)+\phi_1(x,0)+\phi_2(x,0)+\phi_R(x,0)}\right) \\ \phi_1(x,0) \\ \phi_2(x,0) \\ \phi_R(x,0) \end{pmatrix}.$$

The above inequality implies that (2.11) holds when θ is small enough. Consequently, by [25, Corollary 4] with $K = X^+$ and $S(t,s) = U(t,s)$, system (2.6) admits a unique mild solution $u(x,t;\phi)$ with $u_0(\cdot,\cdot;\phi) = \phi,\ t \in [0,t_\phi]$. Since $U(t,s)$ is an analytic operator on X for any $t,s \in R,\ s < t$, it follows that $u(x,t;\phi)$ is a classical solution for $t > \tau$. Set

$$P(t) = \int_\Omega \left(S(x,t) + \sum_{i=1}^{2}(L_i(x,t) + I_i(x,t)) + R(x,t)\right)dx,$$

$$\mu_{\max} = \sup_{(x,t)\in\bar{\Omega}\times[0,T]} \mu(x,t),\ \bar{\mu}_{\max} = \mu_{\max}\cdot|\Omega|,\ d_{\min} = \inf_{(x,t)\in\bar{\Omega}\times[0,T]} d(x,t).$$

Then

$$\begin{aligned}
\frac{dP(t)}{dt} &= \int_\Omega \mu(x,t) - d(x,t)\left(S(x,t) + \sum_{i=1}^{2}(L_i(x,t) + I_i(x,t)) + R(x,t)\right) \\
&\quad - \sum_{i=1}^{2}\delta_i(x,t)(L_i(x,t) + I_i(x,t)) - \sum_{i=1}^{2} r_i(x,t)L_i(x,t)dx \\
&\leq \int_\Omega \mu(x,t)dx - \int_\Omega d(x,t)\left(S(x,t) + \sum_{i=1}^{2}(L_i(x,t) + I_i(x,t)) + R(x,t)\right)dx \\
&\leq \bar{\mu}_{\max} - d_{\min}P(t),\ t > 0.
\end{aligned}$$

We obtain that there are $l := l_\phi$ large enough and $M = \frac{\bar{\mu}_{\max}}{d_{\min}} + 1 > 0$, so that for each $\phi \in C_\tau^+$, one has

$$P(t) \leq M,\ \forall t \geq lT + \tau.$$

Then $\int_\Omega I_i(x,t)dx \leq M,\ \forall t \geq lT + \tau$. According to [11] and assumption **(H)**, we obtain that $\Gamma_i(x,y,t,t-a)$ and $\beta_i(x,t)$ are uniformly bounded functions for any $x, y \in \Omega,\ t \in [a, a+T]$. Set $B_i = \sup\limits_{x,y\in\Omega,t\in[a,a+T]} \Gamma_i(x,y,t,t-a)\beta_i(y,t-a)$, then we obtain

$$\begin{aligned}
\frac{\partial I_i}{\partial t} &\leq D_i\Delta I_i - h_i(x,t)I_i(x,t) + \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a)I_i(y,t-a)dyda \\
&\leq D_i\Delta I_i - h_i(x,t)I_i(x,t) + B_i\int_0^{\tau_i} f_i(a)\int_\Omega I_i(y,t-a)dyda \\
&\leq D_i\Delta I_i - h_i(x,t)I_i(x,t) + B_iMF_i(\tau_i) \\
&= D_i\Delta I_i - h_i(x,t)I_i(x,t) + B_iM,\ x \in \Omega,\ t \geq lT + \tau.
\end{aligned} \tag{2.12}$$

Consider the following equation:

$$\begin{cases} \frac{\partial\omega_i(x,t)}{\partial t} = D_i\Delta\omega_i(x,t) - h_i(x,t)\omega_i(x,t) + B_iM, & x \in \Omega,\ t > lT + \tau, \\ \frac{\partial\omega_i(x,t)}{\partial n} = 0, & x \in \partial\Omega,\ t > lT + \tau. \end{cases} \tag{2.13}$$

It is evident that system (2.13) admits a strictly positive periodic solution with the period $T > 0$, which is globally attractive. According to (2.12), the first equation of system (2.6) can be dominated by (2.13) for any $t > lT+\tau$. So there exists $B_1 > 0$ such that for each $\phi \in C_\tau^+$, we can find a $l_i = l_i(\phi) \gg l(\phi)$ satisfying $I_i(x,t;\phi) \le B_1 (i = 1,2)$ for $x \in \bar{\Omega}$ and $t \ge l_i T + \tau$. Thus

$$\begin{cases} \frac{\partial R(x,t)}{\partial t} \le D_R \Delta R(x,t) - k(x,t)R(x,t) + B_1(r_1(x,t) + r_2(x,t)), & x \in \Omega,\ t > l_i T + \tau, \\ \frac{\partial R(x,t)}{\partial n} = 0, & x \in \partial\Omega,\ t > l_i T + \tau. \end{cases} \tag{2.14}$$

Similarly, there exists $B_2 > 0$ such that for each $\phi \in C_\tau^+$, there exists $l_R = l_R(\phi) \gg l_i$ satisfying $R(x,t;\phi) \le B_2$ for $x \in \bar{\Omega}$ and $t \ge l_R T + \tau$. Then we have

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} \le D_S \Delta S(x,t) + \mu(x,t) - d(x,t)S(x,t) + B_2\alpha(x,t), & x \in \Omega,\ t > l_R T + \tau, \\ \frac{\partial S(x,t)}{\partial n} = 0, & x \in \partial\Omega,\ t > l_R T + \tau. \end{cases} \tag{2.15}$$

Hence, there are $B_3 > 0$ and $l_S = l_S(\phi) \gg l_R$ such that for each $\phi \in C_\tau^+$, $S(x,t;\phi) \le B_3 (i = 1,2)$ for $x \in \bar{\Omega}$ and $t \ge l_S T + \tau$, and hence, $t_\phi = +\infty$. □

Theorem 2.2. *System* (2.6) *generates a T-periodic semi-flow* $\Phi_t := u_t(\cdot) : C_\tau^+ \to C_\tau^+$, *namely* $\Phi_t(\phi)(x,s) = u_t(\phi)(x,s) = u(x,t+s;\phi)$ *for any* $\phi \in C_\tau^+,\ t \ge 0$ *and* $s \in [-\tau, 0]$. *In addition,* Φ_T *admits a global compact attractor on* C_τ^+, *where* $u(x,t;\phi)$ *is a solution of system* (2.6).

Proof. By a similar argument as the proof of [26, Theorem 8.5.2], one can show that $\Phi_t(\phi)$ is continuous for any $\phi \in C_\tau^+$ and $t \ge 0$. In addition, similarly as the proof of [36, Lemma 2.1], we can further verify that Φ_t is a T-periodic semi-flow on C_τ^+. According to Theorem 2.1, we obtain that Φ_t is dissipative. Moreover, by the arguments similar to those in the proof of [15, Proposition 21.2], we get that there exists $n_0 \ge 1$ such that $\Phi_T^{n_0} = u_{n_0 T}$ is compact on C_τ^+ for $n_0 T \ge \tau$. Following from [23, Theorem 2.9], we have that $\Phi_T : C_\tau^+ \to C_\tau^+$ admits a global compact attractor. □

3. Threshold dynamics

In this section, we first analyze the threshold dynamics of a single-strain model with the help of the basic reproduction number, and then study the threshold dynamics of model (2.6).

3.1. Threshold dynamics of single-strain SIRS model

Let $I_j(x,t) \equiv 0, \forall x \in \Omega,\ t > 0,\ j = 1,2$, and $i \ne j$. Then system (2.6) reduces to the following single-strain model:

$$\begin{cases} \dfrac{\partial S}{\partial t} = D_S \Delta S + \mu(x,t) - d(x,t)S + \alpha(x,t)R(x,t) - \dfrac{\beta_i(x,t)S(x,t)I_i(x,t)}{S(x,t)+I_i(x,t)+R(x,t)}, \\ \dfrac{\partial I_i}{\partial t} = D_i \Delta I_i(x,t) - h_i(x,t)I_i(x,t) \\ \qquad + \displaystyle\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a) \dfrac{\beta_i(y,t-a)S(y,t-a)I_i(y,t-a)}{S(y,t-a)+I_i(y,t-a)+R(y,t-a)} dyda, \\ \dfrac{\partial R}{\partial t} = D_R \Delta R(x,t) + r_i(x,t)I_i(x,t) - d(x,t)R(x,t) - \alpha(x,t)R(x,t). \end{cases} \tag{3.1}$$

Consider the following linear equation:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = D_S \Delta S(x,t) + \mu(x,t) - d(x,t)S(x,t), & x \in \Omega,\ t > 0, \\ \frac{\partial S(x,t)}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.2}$$

According to [36, Lemma 2.1], there is an unique T-periodic solution $S^*(x,t)$ of (3.2). Linearizing the I_i-equation of system (3.1) at the disease-free periodic solution $(S^*, 0, 0)$, we have

$$\begin{cases} \dfrac{\partial \omega_i(x,t)}{\partial t} = D_i \Delta \omega_i(x,t) - h_i(x,t)\omega_i(x,t) \\ \qquad\qquad + \displaystyle\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a)\omega_i(y,t-a)dyda,\ x \in \Omega,\ t > 0, \\ \dfrac{\partial \omega_i(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.3}$$

Let

$$C_T(\bar{\Omega} \times R, R) := \{u | u \in C(\bar{\Omega} \times R, R), u(x, t+T) = u(x,t), (x,t) \in \Omega \times R, T > 0\},$$

with the supremum norm, and define C_T^+ as the positive cone of $C_T(\bar{\Omega} \times R, R)$, namely,

$$C_T^+ := \{u \in C_T :\ u(t)(x) \geq 0,\ \forall t \in R,\ x \in \bar{\Omega}\}.$$

Let $\psi_i(x,t) \in C_T(\bar{\Omega} \times R, R)$ be the initial distribution of infected individuals of the i-strain at the spatial position $x \in \bar{\Omega}$ and time $t \in R$, then $V_i(t-a, s)\psi_i(s)(s < t-a)$ is the density of those infective individuals at location x who were infective at time s and retain infective at time $t-a$ when time evolved from s to $t-a$. Furthermore, $\int_{-\infty}^{t-a} V_i(t-a, s)\psi_i(s)ds$ is the density distribution of the accumulative infective individuals at positive x and time $t-a$ for all previous time $s < t-a$. Hence the density of new infected individuals at time t and location x can be written as

$$\begin{aligned} &\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a) \int_{-\infty}^{t-a} \left(V_i(t-a,s)\psi_i(s)\right)(x)dsdyda \\ = &\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a) \int_a^{+\infty} \left(V_i(t-a,t-s)\psi_i(t-s)\right)(x)dsdyda. \end{aligned}$$

Defining operator $C_i : C_T(\bar{\Omega} \times R, R) \longrightarrow C_T(\bar{\Omega} \times R, R)$ by

$$\left(C_i\psi_i\right)(x,t) = \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a)\psi_i(y,t-a)dyda.$$

Set

$$A_i(\psi_i)(x,t) = (C_i\psi_i)(x,t),\ B_i(\psi_i)(x,t) = \int_a^{+\infty} \left(V_i(t,t-s+a)\psi_i(t-s+a)\right)(x)ds.$$

Defining other operators $L_i, \hat{L}_i : C_T \longrightarrow C_T$ by

$$\begin{aligned} &(L_i\psi_i) := \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a) \int_a^{+\infty} V_i(t-a,t-s)\psi_i(t-s)(x)dsdyda, \\ &\left(\hat{L}_i\psi_i\right)(x,t) := \int_0^{+\infty} V_i(t,t-s)\left(C_i\psi_i\right)(t-s)(x)ds,\ t \in R,\ s \geq 0. \end{aligned}$$

Clearly, $L_i = A_iB_i$, $\hat{L}_i = B_iA_i$, L_i and $\hat{L}_i$ are compact, bounded and positive operators. Let $r(L_i)$ and $r(\hat{L}_i)$ are the spectral radius of L_i and $\hat{L}_i$ respectively, then $r(L_i) = r(\hat{L}_i)$. Similar to [18, 20], we define the basic reproduction number for system (3.1), that is, $R_0^i = r(L_i) = r(\hat{L}_i)$.

Define $Q := C([-\tau, 0], Y)$, and let $\|\phi\|_Q := \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_Y$ for any $\phi \in Q$. Denote $Q^+ := C([-\tau, 0], Y^+)$ as the positive cone of Q. Then (Q, Q^+) is strongly ordered Banach space. Let $P := C\left(\bar{\Omega}, R^3\right)$ be the Banach space with supremum norm $\|\cdot\|_P$. For $\tau > 0$, let $D_\tau := C([-\tau, 0], P)$ be the Banach space with $\|\phi\| = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_P$ for all $\phi \in D_\tau$. Define $P^+ := C\left(\bar{\Omega}, R^3_+\right)$ and $D^+_\tau := C([-\tau, 0], P^+)$, then both (P, P^+) and (D_τ, D^+_τ) are strongly ordered space. By the arguments in [21, 39], we have the following observation:

Theorem 3.1. *The signs of* $R_0^i - 1$ *and* $r^i - 1$ *are same.*

Consider the following equation

$$
\begin{cases}
\dfrac{\partial\omega_i(x,t)}{\partial t} = D_i\Delta\omega_i(x,t) - h_i(x,t)\omega_i(x,t) \\
\qquad\qquad + \displaystyle\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(x,y,t,t-a)\frac{B_3\beta_i(y,t-a)}{B_3+\omega_i(y,t-a)}\omega_i(y,t-a)dyda, \\
\qquad\qquad x\in\Omega,\ t>0, \\
\partial\omega_i(x,s) = \psi_i(x,s),\ \psi_i\in Q^+,\ x\in\Omega,\ s\in[-\tau_i,0], \\
\dfrac{\partial\omega_i(x,t)}{\partial n} = 0,\ x\in\partial\Omega,\ t>0,
\end{cases}
\tag{3.4}
$$

where B_3 is the constant in the proof of Theorem 2.1.

Theorem 3.2. *Assume that* $\omega_i(x, t; \psi_i)$ *is the solution of* (3.4) *with an initial value of* $\psi_i \in Q$. *If* $R_0^i = 1$ *and* $\beta_i(x, t) > 0$ *for all* $x \in \bar{\Omega},\ t > 0$, *then* $\omega_i(x, t) \equiv 0$ *is globally attractive.*

Proof. By a straightforward computation, one has that (3.4) is dominated by (3.3). Define the map $P_i^{n_o} : Q \to Q$ by $P_i^{n_o}(\psi_i) = \omega_{i,T}^{n_0}$ with $\omega_{i,T}^{n_0} = \omega_i(x, n_0T + s; \psi_i)$, where $\omega_i(x, t; \psi_i)$ is the solution of (3.3). Similar to the argument in [18], $P_i^{n_o}$ is strongly positive on Q^+ when $\beta_i(x, t) > 0, \forall x \in \bar{\Omega},\ t > 0$. It follows from [20, Lemma 3.1] that $P_i^{n_o}$ admits a positive and simple eigenvalue r^i, and a strongly positive eigenfunction defined by ψ_i, that is $P_i(\psi_i) = r^i\psi_i$. It follows from the strong positivity of ψ_i that $\omega_i(x, t; \psi_i) \gg 0$. According to Theorem 3.2, we have $r^i = 1$, and hence, $\mu^i = 0$. By similar arguments as the proof of [18, Lemma 3.2], we can show that there is a positive T-periodic function $v_i^*(x, t) = e^{-\mu^i\cdot 0}\omega_i(x, t; \psi_i) = \omega_i(x, t; \psi_i)$ such that $v_i^*(x, t)$ is a solution of (3.3). Then for each initial value $\psi_i(x, s) \in Q$, there exists a constant $k > 0$ such that $\psi_i(x, s) \le kv_i^*(x, s)$ for all $x \in \Omega,\ t > 0$. Moreover, by the comparison principle, one has $\omega_i(x, t; \psi_i) \le kv_i^*(x, t)$ for all $x \in \Omega,\ t > 0$. Let

$$[0, kv_i^*]_Q = \{u \in Q : 0 \le u(x, s) \le kv_i^*(x, s),\ \forall x \in \bar{\Omega},\ s \in [-\tau_i, 0]\},$$

then

$$S_i^{n_0}(\psi_i) := \omega_i(x, n_oT\omega + S; \psi_I) \subseteq [0, kv_i^*]_Q,\ \ \forall x \in \bar{\Omega},\ s \in [-\tau_i, 0].$$

Hence the positive orbit $\gamma_+(\psi_i) := \{S_i^{kn_0}(\psi_i) : \forall k \in N\}$ of $S_i^{n_0}(\cdot)$ is precompact, and $S_i^{n_0}$ maps $[0, kv_i^*]_Q$ into $[0, kv_i^*]_Q$, Due to comparison principle, we get $S_i^{n_0}(\cdot)$ is monotone. According to [40, Theorem 2.2.2], we obtain that $\omega_i(x, t) \equiv 0$ is globally attractive. □

Theorem 3.3. *Suppose that* $\bar{S}(x,t;\psi) = (S(x,t;\psi), I_i(x,t;\psi), R(x,t;\psi))$ *is the solution of* (3.1) *with the initial data* ψ. *If* $I_i(x,t_0;\psi) \not\equiv 0$ *for some* $t_0 \geq 0$, *then* $I_i(x,t;\psi) > 0,\ \forall x \in \bar{\Omega},\ t > t_0$.

Proof. Obviously, for the secondly equation of (3.1), we get

$$\begin{cases} \frac{\partial I_i(x,t)}{\partial t} \geq D_i \Delta I_i(x,t) - h_i(x,t) I_i(x,t),\ x \in \Omega,\ t > 0, \\ \frac{\partial I_i(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0,\ i = 1,2, \end{cases}$$

and $I_i(x,t_0;\psi) \not\equiv 0,\ t_0 \geq 0,\ i = 1,2$. It follows from [15, Proposition 13.1] that $I_i(x,t;\psi) > 0$ for all $x \in \bar{\Omega}$ and $t > t_0$. □

Theorem 3.4. *Suppose that* $\bar{S}(x,t;\psi) = (S(x,t;\psi), I_i(x,t;\psi), R(x,t;\psi))$ *be the solution of* (3.1) *with the initial data* $\psi = (\psi_S, \psi_i, \psi_R) \in D_\tau,\ i = 1,2$. *Then one has*

(1) *If* $R_0^i = 1$ *and* $\beta_i(x,t) > 0$ *for all* $x \in \Omega$ *and* $t > 0$, *then* $(S^*,0,0)$ *is globally attractive;*
(2) *If* $R_0^i < 1$, *then* $(S^*,0,0)$ *is globally attractive;*
(3) *If* $R_0^i > 1$, *then there is a* $M > 0$ *such that for any* $\psi \in D_\tau^+$, *one has*

$$\liminf_{t\to\infty} S(x,t;\psi) > M,\ \liminf_{t\to\infty} I_i(x,t;\psi) > M,\ \liminf_{t\to\infty} R(x,t;\psi) > M$$

uniformly for $x \in \bar{\Omega}$.

Proof. (1) According to the proof of Theorem 2.1, for $t > l_s T + \tau$, we have $S(x,t;\phi) \leq B_3,\ \forall x \in \bar{\Omega},\ \phi \in C_\tau^+$. Thus, when $t > l_s T + \tau$, the second equation of (3.1) is dominated by (3.4) for $x \in \bar{\Omega}$. In addition, one has $I_i(x,t;\psi) \leq \omega_i(x,t)$ for $x \in \bar{\Omega}$ and $t > l_s T + \tau$. Since $R_0^i = 1$ and $\beta_i(x,t) > 0$ for $x \in \bar{\Omega},\ t > 0$. It follows from Theorem 3.5 that $\lim_{t\to\infty} \omega_i(x,t) = 0$ for all $x \in \bar{\Omega}$. In addition, one has $\lim_{t\to\infty} I_i(x,t;\psi) = 0$ for all $x \in \bar{\Omega}$, and $\lim_{t\to\infty} R(x,t;\psi) = 0$ for all $x \in \bar{\Omega}$. Hence the first equation of (3.1) is asymptotic to (3.2). It follows from [36, Lemma 2.1] that system (3.2) admits an unique positive T-periodic solution $S^*(x,t)$,which is globally attractive.

Let $P = \Phi_T,\ J = \bar{\omega}(\psi)$ denotes the omega limit set for P. That is

$$J = \{(\phi_S^*, \phi_i^*, \phi_R^*) \in C_\tau^+ : \exists \{k_i\} \to \infty\ s.t.\ \lim_{i\to\infty} P^{k_i}(\phi_S, \phi_i, \phi_R) = (\phi_S^*, \phi_i^*, \phi_R^*)\}.$$

It follows fron [17, Lemma 2.1] that J is an internally chain transitive sets for P. Since $\lim_{t\to\infty} I_i(x,t;\psi) = 0$ and $\lim_{t\to\infty} R(x,t;\psi) = 0$ for all $x \in \bar{\Omega}$, then $J = J_1 \times \{\hat{0}\} \times \{\hat{0}\}$. According to Theorem 3.5, one has $\hat{0} \notin J_1$. Let $\omega(x,t;\psi_S(\cdot,0))$ be the solution of (3.2) with the initial value $\omega(x,0) = \psi_S(x,0)$, where $\psi_S \in Q^+$. Define

$$\omega_t(x,\theta;\psi_S) = \begin{cases} \omega(x,\theta+t;\psi_S(0)) & t+\theta > 0,\ t > 0,\ \theta \in [-\tau,0], \\ \psi(x,\theta+t) & t+\theta \leq 0,\ t > 0,\ \theta \in [-\tau,0]. \end{cases}$$

Then we define the solution semiflow ω_t for (3.2).

Let $\bar{P} = \omega_T(\psi_S),\ \bar{\omega}(\psi_S)$ denotes the omega limit set of $\bar{P}$. According to [36, Lemma 2.1], one has $\bar{\omega}(\psi_S) = \{S^*\}$. Since $P(J) = J$ and $I_i(x,t;(\psi_S,\hat{0},\hat{0})) \equiv 0,\ R(x,t;(\psi_S,\hat{0},\hat{0})) \equiv 0,\ P(J) = \bar{P}(J_1) \times \{\hat{0}\} \times \{\hat{0}\}$, then $\bar{P}(J_1) = J_1$. Therefore, J_1 is an internally chain transitive sets for $\bar{P}$. It follows from [36, Lemma 2.1] that $\{S^*\}$ is globally attractive on Q^+. In addition, $J_1 \cap W^S\{S^*\} = J_1 \cap Q^+ = \emptyset$, where $W^S\{S^*\}$ is the stable set of S^*. According to [40, Theorem 1.2.1], one has $J_1 \subseteq \{S^*\}$, then $J_1 = \{S^*\}$. Consequently, $J = \{(S^*, 0, 0)\}$. By the definition of J, we have

$$\lim_{t\to\infty} \| (S(\cdot, t; \psi), I_i(\cdot, t; \psi), R(\cdot, t; \psi)) - (S^*(\cdot, t), 0, 0) \| = 0.$$

(2) Consider equation

$$\begin{cases} \dfrac{\partial \omega_i^*(x,t)}{\partial t} = D_i \Delta \omega_i^*(x,t) - h_i(x,t)\omega_i^*(x,t) \\ \qquad + \displaystyle\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)(\beta_i(y,t-a)+\varepsilon)\omega_i^*(y,t-a)dyda, \\ \qquad x \in \Omega,\ t > 0, \\ \dfrac{\partial \omega_i^*(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.5}$$

Since $R_0^i < 1$, it follows from Theorem 2.1 that $r^i < 1$. Thus there exists a constant $\varepsilon_0 > 0$ such that $r^{i,\varepsilon} < 1$ for $\varepsilon \in [0, \varepsilon_0)$. Then $\mu^{i,\varepsilon} := \frac{\ln r^{i,\varepsilon}}{T} < 0$ for $\varepsilon \in [0, \varepsilon_0)$. Similar to the proof of [18, Lemma 3.2], there is positive T-periodic function $v_i^\varepsilon(x,t)$ such that $\omega_i^\varepsilon(x,t) = e^{\mu^{i,\varepsilon}} v_i^\varepsilon(x,t)$ satisfies (3.5). Since $\mu^{i,\varepsilon} < 0$, $\lim\limits_{t\to\infty} \omega_i^\varepsilon(x,t) = 0$ uniformly for $x \in \Omega$.

For $x \in \Omega,\ t > 0$, one has

$$\begin{cases} \dfrac{\partial I_i(x,t)}{\partial t} \le D_i \Delta I_i(x,t) - h_i(x,t) I_i(x,t) \\ \qquad + \displaystyle\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x,y,t,t-a)(\beta_i(y,t-a)+\varepsilon) I_i(y,t-a)dyda, \\ \qquad x \in \Omega,\ t > 0, \\ \dfrac{\partial I_i(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.6}$$

For any given initial distribution $\psi \in D_\tau^+$, due to the boundedness of $I_i(x,t;\psi)$, there exists $\alpha > 0$ such that $I_i(x,t;\psi) \le \alpha \cdot \omega_i^\varepsilon(x,t)$, $\forall t \in [kT, kT+\tau]$, $x \in \bar{\Omega}$, and hence, $I_i(x,t;\psi) \le \alpha \cdot \omega_i^\varepsilon(x,t)$ for $t \ge kT + \tau$. Then $\lim\limits_{t\to\infty} I_i(x,t;\psi) = 0$ and $\lim\limits_{t\to\infty} R(x,t;\psi) = 0$ for all $x \in \bar{\Omega}$. Furthermore, similar to the proof of (1), we have

$$\lim_{t\to\infty} \| (S(\cdot, t; \psi), I_i(\cdot, t; \psi), R(\cdot, t; \psi)) - (S^*(\cdot, t), 0, 0) \| = 0.$$

(3) Let

$$W_0^i = \{\psi = (\psi_S, \psi_i, \psi_R) \in D_\tau^+ : \psi_i(\cdot, 0) \ne 0\},$$

$$\partial W_0^i := D_\tau^+ \backslash W_0^i = \{\psi = (\psi_S, \psi_i, \psi_R) \in D_\tau^+ : \psi_i(\cdot, 0) \equiv 0\}.$$

Define $\Phi_t : D_\tau^+ \to D_\tau^+$ by $\Phi_t(\psi)(x,s) = (S(x,t+s;\psi), I_i(x,t+s;\psi), R(x,t+s;\psi))$. By Theorem 3.6, we know that $I_i(x,t+s;\psi) > 0$ for any $\psi \in W_i^0$, $x \in \bar{\Omega}$ and $t > 0$. Thus there exists $k \in N$ such that $\Phi_{n_0T}^k(W_0^i) \subseteq W_0^i$. Define

$$M_\partial^i := \{\psi \in \partial W_0^i : \Phi_{n_0T}^k(\psi) \in \partial W_0^i,\ \forall k \in N\}.$$

Let $M := (S^*, 0, 0)$ and $\omega(\psi)$ be the omega limit set of the orbit $\gamma^+ := \{\Phi^k_{n_0T}(\psi) : \forall k \in N\}$. For any given $\psi \in M^i_\partial$, we have $\Phi^k_{n_0T}(\psi) \in \partial W^i_0$. Thus $I_i(x, t; \psi) \equiv 0,\ \forall x \in \bar{\Omega},\ t \geq 0$. Therefore $R(x, t; \psi) \equiv 0$ for any $x \in \bar{\Omega}$ and $t \geq 0$. By similar arguments as the proof of (1), we have

$$\lim_{t\to\infty} \| (S(\cdot, t; \psi), I_i(\cdot, t; \psi), R(\cdot, t; \psi)) - (S^*(\cdot, t), 0, 0) \| = 0.$$

That is $\omega(\psi) = M$ for any $\psi \in M^i_\partial$.

For sufficient small $\bar{\theta} > 0$, consider the following system:

$$\begin{cases} \dfrac{\partial v^\theta_i(x,t)}{\partial t} = D_i \Delta v^\theta_i(x,t) - h_i(x,t) v^\theta_i(x,t) \\ \qquad + \displaystyle\int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(x, y, t, t-a) \frac{\beta_i(y, t-a)(S^*(y, t-a) - \bar{\theta})}{S^*(y, t-a) + \bar{\theta}} v^\theta_i(y, t-a) dy da, \\ \qquad x \in \Omega,\ t > 0, \\ \partial v^\theta_i(x, s) = \psi_i(x, s),\ \psi_i \in Q^+,\ x \in \Omega,\ s \in [-\tau_i, 0], \\ \dfrac{\partial v^\theta_i(x,t)}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.7}$$

Let $v^\theta_i(x, t; \psi_i)$ be the solution of (3.7). Note $v^\theta_{i,n_0T}(\psi_i)(x, s) = v^\theta_i(x, s + n_0T; \psi_i)$ for all $x \in \Omega$ and $s \in [-\tau_i, 0]$. Define the poincar$\acute{e}$ map $(\chi^i_\theta)^{n_0} : Q^+ \to Q^+$ by $(\chi^i_\theta)^{n_0}(\psi_i) = v^\theta_{i,n_0T}(\psi_i)$. It is easy to prove that $(\chi^i_\theta)^{n_0}$ is a compact, strongly positive operator. Let $(r^i_\theta)^{n_0}$ be the spectral radius of $(\chi^i_\theta)^{n_0}$. According to [15, Theorem 7.1], there is a positive eigenvalue $(r^i_\theta)^{n_0}$ and a positive eigenfunction $\tilde{\varphi}_i$ such that $(\chi^i_\theta)^{n_0} = (r^i_\theta)^{n_0}\tilde{\varphi}_i$. Since $R^i_0 > 1$, it follows from Theorem 2.1 that $r^i > 1$. Then there exists a sufficient small number $\theta_1 > 0$ such that $r^i_\theta > 1$ for $\theta \in (0, \theta_1)$.

By the continuous dependence of solutions on initial value, there exists $\theta_0 \in (0, \theta_1)$ such that

$$\|S(x, t; \phi), I_i(x, t; \phi), R(x, t; \phi) - (S^*(x, t), 0, 0)\| < \bar{\theta},\ \forall x \in \bar{\Omega},\ t \in [0, T],$$

if

$$\|(\phi_S(x, s), \phi_i(x, s), \phi_R(x, s)) - (S^*(x, s), 0, 0)\| < \theta_0,\ x \in \bar{\Omega},\ s \in [-\tau_i, 0].$$

Claim. M is a uniformly weak repeller for W^i_0, that is,

$$\limsup_{k\to\infty} \|\Phi^k_{n_0T}(\psi) - M\| \geq \theta_0,\ \forall \psi \in W^i_0.$$

Suppose, by contradiction, there exists $\psi_0 \in W^i_0$ such that

$$\limsup_{k\to\infty} \|\Phi^k_{n_0T}(\psi) - M\| < \theta_0.$$

Then there exist a $k_0 \in N$ such that

$$\begin{aligned} &|S(x, kn_0T + s; \psi_0) - S^*| < \theta_0,\ |I_i(x, kn_0T + s; \psi_0| < \theta_0, \\ &|R(x, kn_0T + s; \psi_0| < \theta_0,\ \forall x \in \bar{\Omega},\ s \in [-\tau_i, 0],\ k \geq k_0. \end{aligned}$$

According to (3.11), for any $t > kn_0T$ and $x \in \bar{\Omega}$,

$$S^* - \bar{\theta} < S(x, t; \psi_0) < S^* + \bar{\theta},\ 0 < I_i(x, t; \psi_0) < \bar{\theta},\ 0 < R(x, t; \psi_0) < \bar{\theta}.$$

Therefore, for I_i-equation of (3.1), we have

$$\begin{aligned}\frac{\partial I_i(x,t)}{\partial t} \geq{}& D_i\Delta I_i(x,t) - h_i(x,t)I_i(x,t)\\ &+ \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(x,y,t,t-a)\frac{\beta_i(y,t-a)(S^*(y,t-a)-\bar{\theta})}{S^*(y,t-a)+\bar{\theta}}I_i(y,t-a)dyda,\\ & x\in\Omega, t>(k_0+1)n_0T.\end{aligned} \tag{3.8}$$

Since

$$I_i(x,t;\psi_0) > 0,\ x\in\bar{\Omega},\ t>(k_0+1)n_0T,$$

there exist some $\kappa > 0$, such that

$$I_i(x,(k_0+1)n_0T+s;\psi_0) \geq \kappa\tilde{\varphi}_i(x,s),\ \forall x\in\bar{\Omega},\ s\in[-\tau_i,0].$$

By (3.12) and the comparison principle, we have

$$I_i(x,t+s;\psi_0) \geq \kappa v_i^\theta(x,t-(k_0+1)n_0T+s;\tilde{\varphi}_i),\ \forall x\in\bar{\Omega},\ t>(k_0+1)n_0T.$$

Therefore, we have

$$I_i(x,kn_0T+s;\psi_0) \geq \kappa v_i^\theta(x,k-(k_0+1)n_0T+s;\tilde{\varphi}_i) = \kappa(r_\theta^i)^{(k-k_0-1)n_0}\tilde{\varphi}_i(x,s), \tag{3.9}$$

where $k\geq k_0+1,\ s\in[-\tau_i,0]$. Since $\tilde{\varphi}_i(x,s)>0$ for $(x,s)\in\bar{\Omega}\times[-\tau_i,0]$, there is $(x_i,s_i)\in\bar{\Omega}\times[-\tau_i,0]$ such that $\hat{\varphi}_i(x_i,s_i)>0$. It follows from $(r_\theta^i)^{n_0}>1$ that $I_i(x_i,kn_0T+s_i;\psi_0)\to+\infty$ as $k\to\infty$, which contradicts to $I_i(x,t;\psi_0)\in(0,\bar{\theta})$.

Let $W^S(M)$ be the stable set of M. In conclusion, $W^S(M)=M_\partial^i$; M is an isolated invariant set for Φ_{n_0T} in W_0^i; $W^S(M)\cap W_0^i = M_\partial^i\cap W_0^i=\varnothing$. According to [40, Theorem 1.3.1] and [40, Remark 1.3.1], one has there is $\bar{\sigma}>o$ such that $\inf d(\omega(\psi),\partial W_0^i)\geq\bar{\sigma}$ for any $\psi\in W_0^i$. That is $\liminf_{t\to\infty} d(\Phi_{n_0T}^k,\partial W_0^i)\geq\bar{\sigma}$ for any $\psi\in W_0^i$. Therefore, $\Phi_{n_0T}: D_\tau^+\to D_\tau^+$ is uniformly persistent with respect to $(W_0^i,\partial W_0^i)$. Similar to Theorem 2.1, it can be proved that the solution $\bar{S}(x,t;\psi)$ of 3.1 is globally bounded for any $\psi\in D_\tau^+$. Therefore, $\Phi_{n_0T}: D_\tau^+\to D_\tau^+$ is point dissipative. It is easy to prove that Φ_{n_0T} is compact on W_0^i for $n_0T>\tau_i$. It follows from [40, Section 1.1] that the compact map Φ_{n_0T} is an $\alpha-$contraction of order 0, and an $\alpha-$contraction of order 0 is $\alpha-$condensing. Then according to [23, Theorem 4.5], $\Phi_{n_0T}: W_0^i\to W_0^i$ admits a compact global attractor Z_0^i.

Similar to the proof of [22, Theroem 4.1], let $P: D_\tau^+\to[0,+\infty)$ by

$$P(\psi) = \min_{x\in\bar{\Omega}}\psi_i(x,0),\ \forall\psi\in D_\tau^+.$$

Since $\Phi_{n_0T}(Z_0^i)=Z_0^i$, we have that $\psi_i(\cdot,0)>0$ for any $\psi\in Z_0^i$. Let $B_i := \bigcup_{t\in[0,n_0T]}\Phi_t(Z_0^i)$, then $B_i\subseteq W_0^i$. In addition, we get $\lim_{t\to\infty} d(\Phi_t(\psi),B_i)=0$ for all $\psi\in W_0^i$. Since B_i is a compact subset of W_0^i, we have $\min_{\psi\in B_i}P(\psi)>0$. Thus, there exists a $\sigma^*>0$ such that $\liminf_{t\to\infty} I_i(\cdot,t;\psi)\geq\sigma^*$. Furthermore, according to Theorem 3.6, there exists $M>0$ such that $\liminf_{t\to\infty} I_i(\cdot,t;\psi)\geq M$. □

3.2. Threshold dynamics of two-strain SIRS model

3.2.1. Coexistence

Consider the following equation:

$$\begin{cases} \dfrac{\partial \bar{S}(x,t)}{\partial t} =& D_{\bar{S}}\Delta \bar{S}(x,t) + \mu(x,t) - d(x,t)\bar{S}(x,t) - \beta_1(x,t)\bar{S}(x,t) \\ & -\beta_2(x,t)\bar{S}(x,t),\ x\in\Omega,\ t>0, \\ \dfrac{\partial \bar{S}(x,t)}{\partial n} =& 0,\ x\in\partial\Omega,\ t>0,\ i=1,2. \end{cases} \tag{3.10}$$

According to [36, Lemma 2.1], equation (3.10) admits a unique positive solution $\bar{S}^*$, which is T-periodic with respect to $t \in R$. Obviously, for the S-equation of (2.6), we have

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} \geq& D_S\Delta S(x,t) + \mu(x,t) - d(x,t)S(x,t) - \beta_1(x,t)S(x,t), \\ & -\beta_2(x,t)S(x,t),\ x\in\Omega,\ t>0, \\ \dfrac{\partial S(x,t)}{\partial n} =& 0,\ x\in\partial\Omega,\ t>0,\ i=1,2. \end{cases} \tag{3.11}$$

It follows from the comparison principle, one has

$$\liminf_{t\to\infty} S(x,t) \geq \bar{S}^*(x,t),\ \forall x\in\bar{\Omega}.$$

According to Theorem 2.1, there exist constants B_1, B_2 and l_R, such that

$$I_i(x,t;\phi) \leq B_1 (i=1,2),\ R(x,t;\phi) \leq B_2$$

for $t \geq l_R T + \tau$. Consider the following equation:

$$\begin{cases} \dfrac{\partial u_i(x,t)}{\partial t} =& D_i\Delta u_i(x,t) - h_i(x,t)u_i(x,t) \\ & + \displaystyle\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a)\frac{\bar{S}^*(x,t)}{\bar{S}^*(x,t)+B_1+B_2}u_i(y,t-a)dyda, \\ & x\in\Omega,\ t>0, \\ \dfrac{\partial u_i(x,t)}{\partial n} =& 0,\ x\in\partial\Omega,\ t>0,\ i=1,2. \end{cases} \tag{3.12}$$

Let $u_i(x,t;\phi_i)$ be the solution of (3.12) for $\phi_i \in Q, (x,s) \in \bar{\Omega}\times[-\tau,0]$. Define $\bar{P}_i : Q \to Q$ by $\bar{P}_i(\phi_i) = u_{i,T}(\phi_i)$ for any $\phi_i \in Q$, where $u_{i,T}(\phi_i)(x,t) = u_i(x,s+T;\phi_i),\ (x,s)\in\bar{\Omega}\times[-\tau,0]$. Let ρ_i^0 be the spectral of $\bar{P}_i$. We define the linear operator $\bar{L}_i : C_T \to C_T$ by:

$$\begin{aligned} \bar{L}_i(\psi_i)(x,t) = & \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(x,y,t,t-a)\beta_i(y,t-a)\frac{S^*(x,t)}{S^*(x,t)+B_1+B_2} \\ & \cdot \int_a^\infty (V_i(t-a,t-s)\psi_i(t-s))(y)dsdyda. \end{aligned}$$

Then the operator $\bar{L}_i$ is positive and bounded on $C_T(\bar{\Omega}\times R, R)$. Let $r(\bar{L}_i)$ denote the spectral radius of $\bar{L}_i$. Similar to [18, 20], define the invasion number $\hat{R}_0^i$ for strain i by $\hat{R}_0^i := r(\bar{L}_i)$, and we have the following result.

Theorem 3.5. *The signs of* $\hat{R}_0^i - 1$ *and* $\rho_i^0 - 1$ *are same.*

By the arguments similar to those in the proof of [38, Proposition 5.10], we further have the following observation.

Theorem 3.6. *If* $\hat{R}_0^i > 1$, *then* $R_0^i > 1,\ i = 1,2$.

Theorem 3.7. *Suppose that* $\hat{R}_0^i > 1\ (i = 1,2)$. *Then for any* $\psi = (\psi_S, \psi_1, \psi_2, \psi_R) \in C_\tau^+,\ \psi_i \not\equiv 0\ (i = 1,2)$, *there is an* $\eta > 0$ *such that*

$$\liminf_{t\to\infty} S(x,t;\psi) \geq \eta,\ \liminf_{t\to\infty} I_i(x,t;\psi) \geq \eta,\ i = 1,2.$$

Proof. According to Theorem 3.6 and $\hat{R}_0^i > 1\ (i = 1,2)$, one has $R_0^i > 1\ (i = 1,2)$. Let

$$Z_0 = \{\psi = (\psi_S, \psi_1, \psi_2, \psi_R) \in C_\tau^+ : \psi_1(\cdot,0) \not\equiv 0\ \ \psi_2(\cdot,0) \not\equiv 0\},$$

$$\partial Z_0 := C_\tau^+ \backslash W_0 = \{\psi = (\psi_S, \psi_1, \psi_2, \psi_R) \in C_\tau^+ : \psi_1(\cdot,0) \equiv 0\ \ \psi_2(\cdot,0) \equiv 0\},$$

and

$$Z_\partial := \{\psi \in \partial Z_0 : \Phi_{n_0T}^k(\psi) \in \partial Z_0,\ \forall k \in N\}.$$

Define $\Phi_t : C_\tau^+ \to C_\tau^+$ by $\Phi_t(\psi)(x,s) = \tilde{S}(x,t+s;\psi)$, $\forall \psi \in C_\tau^+$ and $\Phi_{n_0T}^k(\psi) := \tilde{S}(x, n_0T + s;\psi)$ for $k \in N$ and $(x,s) \in \bar{\Omega} \times [-\tau, 0]$. It is easy to obtain that $\Phi_t(Z_0) \in Z_0$ for $t > 0$. Let

$$E_0 := (\bar{S}^*, 0, 0, 0),\ E_1 := \{(\psi_S, \psi_1, 0, \psi_R)\},\ E_2 := \{(\psi_S, 0, \psi_2, \psi_R)\},$$

and $\bar{\omega}(\psi)$ denotes the omega limit set of the orbit $\gamma^+ := \{\Phi_{n_0T}^k(\psi) : \forall k \in N\}$ for $\psi \in Z_\partial$, we then have the following claims.

Claim 1. $\bigcup_{\psi\in Z_\partial} \bar{\omega}(\psi) = E_0 \cup E_2 \cup E_2$.

For any $\Phi_{n_0T}^k(\psi) \in Z_\partial$, it can be see that $\Phi_{n_0T}^k(\psi) \in Z_\partial$, $\forall k \in N$. Then $I_1(x,t;\psi) \equiv 0$ or $I_2(x,t;\psi) \equiv 0$ for $x \in \bar{\Omega}$ and $t > 0$. Suppose, by contradiction, if there exists $t_i > 0$ such that $I_i(x,t;\psi) \not\equiv 0$ on $x \in \bar{\Omega},\ i = 1,2$. Then the strong positivity of $V_i(t,s)(t > s)$ implies that $I_i(x,t;\psi) > 0$ for all $t > t_i$ and $x \in \bar{\Omega},\ i = 1,2$, which contradicts with the fact $\Phi_{n_0T}^k(\psi) \in Z_\partial$. If $I_1(x,t;\psi) \equiv 0$ on $(x,t) \in \bar{\Omega} \times R^+$, it follows from Theorem 3.7 that $\bar{\omega}(\psi) = E_0 \cup E_2$. If $I_2(x,t;\psi) \equiv 0$ on $(x,t) \in \bar{\Omega} \times R^+$. Similarly, one has $\bar{\omega}(\psi) = E_0 \cup E_1$. Therefore, Claim 1 holds.

Claim 2. E_0 is a uniformly weak repeller for Z_0, in the sense that,

$$\limsup_{k\to\infty} ||\Phi_{n_0T}^k(\psi) - E_0| \geq \varepsilon_0,\ \forall \psi \in Z_0$$

for $\varepsilon_0 > 0$. The proof of Claim 2 is similar to those in Theorem 3.4(3), so we omit it.

Claim 3. E_1 and E_2 is a uniformly weak repeller for Z_0, in the sense that,

$$\limsup_{k\to\infty} ||\Phi_{n_0T}^k(\psi) - E_i| \geq \varepsilon_0,\ \forall \psi \in Z_0,\ i = 1,2$$

for some $\varepsilon_0 > 0$ small enough. We only give the proof for E_1, the proof of E_2 is similar. Due to Theorem 2.1, there are B_1, B_2 and $l_R \gg 0$, such that

$$I_i(x,t;\phi) \leq B_1\ (i = 1,2),\ R(x,t;\phi) \leq B_2$$

for $t \geq l_R T + \tau$. For sufficient small $\varepsilon > 0$, we consider the following system:

$$\begin{cases} \dfrac{\partial \omega_2^\varepsilon}{\partial t} = D_2 \Delta \omega_2^\varepsilon(x,t) - h_2(x,t)\omega_2^\varepsilon(x,t) \\ \qquad + \displaystyle\int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(x,y,t,t-a)\beta_2(y,t-a)\frac{\bar{S}^*(x,t)-\varepsilon}{\bar{S}^*(x,t)+B_1+B_2}\omega_2^\varepsilon(y,t-a)dyda, \\ \qquad x \in \Omega,\ t > 0, \\ \dfrac{\partial \omega_2^\varepsilon}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0,\ i = 1,2, \end{cases} \tag{3.13}$$

where $\bar{S}^*$ is the positive periodic solution of (3.11). Let $\omega_2^\varepsilon(x,t;\psi_2)$ be the solution of (3.13) with initial data $\omega_2^\varepsilon(x,s) = \psi_2(x,s),\ \psi_2 \in Q^+,\ x \in \Omega,\ s \in [-\tau, 0]$. Note $\omega_{2,n_0T}^\varepsilon(\psi_2)(x,s) = \omega_2^\varepsilon(x, s+n_0T;\psi_2)$ for all $x \in \Omega$ and $s \in [-\tau_1, 0]$. Define $(\Psi_2^\varepsilon)^{n_0} : Q^+ \to Q^+$ by $(\Psi_2^\varepsilon)^{n_0}(\psi_2) = \omega_{2,n_0T}^\varepsilon(\psi_2)$. Let $\hat{r}_\varepsilon^2$ and $(\hat{r}_\varepsilon^2)^{n_0}$ be the spectral radius of Ψ_2^ε and $(\Psi_2^\varepsilon)^{n_0}$, respectively. It is easy to prove that $(\Psi_2^\varepsilon)^{n_0}$ is compact, strongly positive operator. According to [15, Theorem 7.1], we get that $(\Psi_2^\varepsilon)^{n_0}$ admits a positive and simple eigenvalue $(\hat{r}_\varepsilon^2)^{n_0}$ and a positive eigenfunction φ_2 satisfying $(\Psi_2^\varepsilon)^{n_0} = (\hat{r}_\varepsilon^2)^{n_0}\varphi_2$. Since $R_0^2 > 1$, it follows from Theorem 3.5 that $\rho_2^0 > 1$, then there exists a sufficient small number $\varepsilon_1 > 0$ such that $r_\varepsilon^2 > 1$ for any $\varepsilon \in (0, \varepsilon_1)$.

By the continuous dependence of solution on initial value, there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that

$$\|\Phi_T^k(\psi) - E_1\| < \bar{\varepsilon},\ \forall x \in \bar{\Omega},\ t \in [0,T], \tag{3.14}$$

if

$$\|\phi(x,s) - E_1\| < \varepsilon_0,\ x \in \bar{\Omega},\ s \in [-\tau, 0].$$

Suppose, by contradiction, there exists $\psi_0 \in W_0$ such that

$$\limsup_{k\to\infty} \|\Phi_{n_0T}^k(\psi) - E_1\| < \varepsilon_0.$$

That is, there is $k_0 \in N$ such that

$$\bar{S}^* - \bar{\varepsilon} < S(x,t;\psi_0) < \bar{S}^* + \bar{\varepsilon};\ 0 < I_1(x,t;\psi_0) < B_1;$$

and

$$0 < I_2(x,t;\psi_0) < \bar{\varepsilon};\ 0 < R(x,t;\psi_0) < B_2$$

for all $k \geq k_0$. Therefore, for I_2-equation of (2.6), we have

$$\begin{aligned} \frac{\partial I_2(x,t)}{\partial t} \geq & D_2 \Delta I_2(x,t) - h_2(x,t)I_2(x,t) \\ & + \int_0^{\tau_2} f_1(a) \int_\Omega \Gamma_2(x,y,t,t-a)\beta_2(y,t-a)\frac{\bar{S}^*(y,t-a)-\bar{\varepsilon}}{\bar{S}^*+B_1+B_2} I_2(y,t-a)dyda \end{aligned} \tag{3.15}$$

for $x \in \Omega$ and $t > (k_0+1)n_0T$. Since

$$I_2(x,t;\psi_0) > 0,\ \forall x \in \bar{\Omega},\ t > (k_0+1)n_0T,$$

there is some $\kappa > 0$, such that

$$I_2(x, (k_0 + 1)n_0T + s; \psi_0) \geq \kappa\varphi_2(x, s), \ \forall x \in \bar{\Omega}, \ s \in [-\tau_2, 0].$$

By (3.15) and the comparison principle, we have

$$I_2(x, t + s; \psi_0) \geq \omega_2^{\varepsilon}(x, t - (k_0 + 1)n_0T + s; \varphi_2), \ \forall x \in \bar{\Omega}, \ t > (k_0 + 1)n_0T.$$

Therefore, we have

$$I_2(x, kn_0T + s; \psi_0) \geq \kappa\omega_2^{\varepsilon}(x, k - (k_0 + 1)n_0T + s; \varphi_2) = \kappa(\hat{r}_{\varepsilon}^2)^{(k-k_0-1)n_0}\varphi_2(x, s), \tag{3.16}$$

where $k \geq k_0 + 1$, $s \in [-\tau_2, 0]$. Since $\varphi_2(x, s) > 0$ for $(x, s) \in \bar{\Omega} \times [-\tau_2, 0]$, there is $(x_2, s_2) \in \bar{\Omega} \times [-\tau_2, 0]$ such that $\varphi_2(x_2, s_2) > 0$. It follows from $(r_{\varepsilon}^2)^{n_0} > 1$ that $I_2(x_2, kn_0T + s_2; \psi_0) \to +\infty$ as $k \to \infty$, which contradicts to $I_2(x, t; \psi_0) \in (0, \bar{\varepsilon})$.

Let $\Theta := E_0 \cup E_1 \cup E_2$, $W^S(\Theta)$ be the stable set of Θ. In conclusion, $W^S(\Theta) = Z_\partial$; Θ is an isolated invariant set for Φ_{n_0T} in Z_0, $W^S(\Theta) \cap Z_0 = Z_\partial \cap Z_0 = \varnothing$. According to [10, Theorem 1.3.1] and [10, Remark 1.3.1], there exists $\bar{\sigma} > 0$ such that $\inf d(\omega(\psi), \partial Z_0) \geq \bar{\sigma}$ for all $\psi \in Z_0$. That is, $\liminf\limits_{t\to\infty} d(\Phi_{n_0T}^k, \partial Z_0) \geq \bar{\sigma}$ for any $\psi \in Z_0$. Therefore, $\Phi_{n_0T} : C_\tau^+ \to C_\tau^+$ is uniformly persistent with respect to $(Z_0, \partial Z_0)$. Similar to Theorem 2.1, it can be proved that the solution $\tilde{S}(x, t; \psi)$ of (2.6) is globally bounded for any $\psi \in D_\tau^+$. Therefore, $\Phi_{n_0T} : C_\tau^+ \to C_\tau^+$is point dissipative. It is easy to prove that Φ_{n_0T} is compact on Z_0 for $n_0T > \tau_1$. It then follows from [40, Section 1.1] that the compact map Φ_{n_0T} is an $\alpha-$contraction of order 0, and an $\alpha-$contraction of order 0 is $\alpha-$condensing. Then according to [23, Theorem 4.5], we obtain that $\Phi_{n_0T} : Z_0 \to Z_0$ admits a compact global attractor N_0.

Similar to the proof of [22, Theroem 4.1], let $P : C_\tau^+ \to [0, +\infty)$ by

$$P(\psi) = \min\{\min_{x\in\bar{\Omega}}\psi_1(x, 0), \min_{x\in\bar{\Omega}}\psi_2(x, 0)\}, \ \forall\psi \in C_\tau^+.$$

Since $\Phi_{n_0T}(N_0) = N_0$, we have $\psi_i(\cdot, 0) > 0$ for any $\psi \in N_0$. Let $B_0 := \bigcup\limits_{t\in[0,n_0T]} \Phi_t(N_0)$, then $B_0 \subseteq Z_0$. In addition, we get $\lim\limits_{t\to\infty} d(\Phi_t(\psi), B_0) = 0$ for all $\psi \in Z_0$. Since B_0 is a compact subset of Z_0. We have $\min\limits_{\psi\in B_0} P(\psi) > 0$. Thus, there exists $\eta > 0$ such that $\liminf\limits_{t\to\infty} I_1(\cdot, t; \psi) \geq \eta$. □

3.2.2. Competitive exclusion

In this subsection, under the condition that the invasion numbers on two strains are greater than 1, it is proved that two strains will always persist uniformly. By the arguments similar to those in the proof of Theorems 3.7 and 3.2, we have the following observations.

Theorem 3.8. *Suppose that $\tilde{S}(x, t; \psi) = (S(x, t; \psi), I_1(x, t; \psi), I_2(x, t; \psi), R(x, t; \psi))$ is the solution of* (2.6) *with initial data $\psi = (\psi_S, \psi_1, \psi_2, \psi_R) \in C_\tau$. If $R_0^1 > 1 > R_0^2$ and $\psi_1(\cdot, 0) \not\equiv 0$, then*

$$\lim_{t\to\infty} I_2(x, t; \psi) = 0,$$

and there is $P > 0$ such that

$$\liminf_{t\to\infty} I_1(x, t; \psi) \geq P, \tag{3.17}$$

uniformly for $x \in \bar{\Omega}$.

Theorem 3.9. *Suppose that $R_0^1 > 1 = R_0^2$ and $\beta_2(x,t) > 0$ on $(x,t) \in \bar{\Omega} \times [0,\infty)$. If C_τ^+ satisfies $\psi_1(\cdot,0) \not\equiv 0$, then we have*

$$\lim_{t\to\infty} I_2(x,t;\psi) = 0,$$

and there is $P > 0$ such that

$$\liminf_{t\to\infty} I_1(x,t;\psi) \geq P,$$

uniformly for $x \in \bar{\Omega}$.

Theorem 3.10. *Suppose that $R_0^2 > 1 > R_0^1$, if $\psi \in C_\tau^+$ satisfies $\psi_2(\cdot,0) \not\equiv 0$, then we have*

$$\lim_{t\to\infty} I_1(x,t;\psi) = 0,$$

and there is $P > 0$ such that

$$\liminf_{t\to\infty} I_2(x,t;\psi) \geq P, \tag{3.18}$$

uniformly for $x \in \bar{\Omega}$.

Theorem 3.11. *Suppose that $R_0^2 > 1 = R_0^1$ and $\beta_1(x,t) > 0$ on $(x,t) \in \bar{\Omega} \times [0,\infty)$. If $\psi \in C_\tau^+$ satisfies $\psi_2(\cdot,0) \not\equiv 0$, then we have*

$$\lim_{t\to\infty} I_1(x,t;\psi) = 0,$$

and there is $P > 0$ such that

$$\liminf_{t\to\infty} I_2(x,t;\psi) \geq P,$$

uniformly for $x \in \bar{\Omega}$.

3.2.3. Global extinction

Finally, we show that the periodic solution $(S^*,0,0,0)$ of (2.6) is globally attractive under some conditions.

Theorem 3.12. *Suppose that $R_0^i < 1$ for $i = 1,2$. Then the periodic $(S^*,0,0,0)$ of (2.6) is globally attractive.*

Proof. Due to $R_0^i < 1,\ i = 1,2$, similar to Theorem 3.4, one has

$$\lim_{t\to\infty} I_i(x,t;\psi) = 0,\ \forall x \in \bar{\Omega},\ \psi \in C_\tau^+,\ i = 1,2.$$

By using the theory of chain transitive sets, we get

$$\lim_{t\to\infty} S(x,t;\psi) = S^*(x,t), \forall x \in \bar{\Omega},\ \psi \in C_\tau^+.$$

Therefore

$$\lim_{t\to\infty} \| (S(\cdot,t;\psi), I_1(\cdot,t;\psi), I_2(\cdot,t;\psi)\pounds R(\cdot,t;\psi)) - (S^*(\cdot,t),0,0,0) \| = 0.$$

That is $(S^*,0,0,0)$ is globally attractive. □

Theorem 3.13. *Suppose that $R_0^i = 1$ and $\beta_i(x,t) > 0$ on $\bar{\Omega} \times [0,\infty)$ for both $i = 1,2$. Then the periodic $(S^*,0,0,0)$ of (2.6) is globally attractive.*

Proof. The proof is similar to Theorem 3.12 by using Theorem 3.2. □

Combining Theorem 3.12 with Theorem 3.13, furthermore, we have the following conclusion.

Theorem 3.14. *If* $R_0^i < 1,\ R_0^j = 1$ *and* $\beta_j(x,t) > 0$ *on* $(x,t) \in \bar{\Omega} \times [0,\infty),\ i,j = 1,2,\ i \neq j$, *then the periodic* $(S^*, 0, 0, 0)$ *of* (2.6) *is globally attractive.*

4. Conclusions

In this paper, we proposed and investigated a two-strain SIRS epidemic model with distributed delay and spatiotemporal heterogeneity. The model is well suitable for simulating the pathogen mutation which is widely founded in variety viral infectious diseases. We have to remark that when the spatiotemporal heterogeneity and distributed delay are incorporated simultaneously, the analysis for the model becomes more difficult. To overcome these difficulties, we used the theory of chain transitive sets and persistence. After introducing the basic reproduction number R_0^i and the invasion number $\hat{R}_0^i$ for each strain i, $i = 1,2$, we established the threshold dynamics for single-strain model and two-strain model, respectively. For the single-strain case, the threshold dynamics results shows that the basic reproduction number R_0^i is a threshold to determine whether the strain i can be persistent. In addition, in such case, we obtained a sufficient condition for the global attraction of the disease free equilibrium when $R_0^i = 1$, $i = 1,2$. Under the condition that two strains is incorporated, we showed that if both of the invasion numbers $\hat{R}_0^i$ are all larger than unit, then the two strains will be persistent uniformly. However, if only one of the reproduction numbers is larger than unit, that is, the other is less than unit, then the strain with larger reproduction number persists, while the strain with the smaller reproduction number dies out. This phenomenon is so called "competitive exclusion"[33]. Further, if both of the two reproduction numbers R_0^i are all less than unit, then the corresponding disease free equilibrium is globally attractive.

Apparently, the dynamical properties of the two-strain model are much more complicated than that of the single-strain case. The most fascinating phenomenon is the appearance of "competitive exclusion" in the two strain model. Generally speaking, the strain with highest basic reproduction number will eliminate the other strain. As is well known, in reality, proper vaccination is a critical for the prevention and control of the most viral infectious disease. Thereby, with the mutating of viruses, the main thing is to ensure the vaccine as safe and effective as possible. However, it is easy to make vaccine administration error. Although some improperly administered vaccines may be valid, sometimes such errors increases the possibility of vaccine recipients being unprotected against viral infection. This paper incorporated the distributed delay, seasonal factor effects and spatial heterogeneity into a two-strain SIRS simultaneously, so the model is more in line with reality. Further, based on these realistic factors, we obtained some valuable results for proper vaccination to viral infection theoretically.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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