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Research article

Certain exponential type m-convexity inequalities for fractional integrals with exponential kernels

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Abstract: By applying exponential type m-convexity, the Hölder inequality and the power mean inequality, this paper is devoted to conclude explicit bounds for the fractional integrals with exponential kernels inequalities, such as right-side Hadamard type, midpoint type, trapezoid type and Dragomir-Agarwal type inequalities. The results of this study are obtained for mappings ω where ω and $|\omega'|$ (or $|\omega'|^q$ with $q \geq 1$) are exponential type m-convex. Also, the results presented in this article provide generalizations of those given in earlier works.

Keywords: Hermite-Hadamard type inequality; fractional integrals; exponential type m-convex mappings
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1. Introduction

The concept of h-convexity related to the nonnegative real mapping has been intorduced by Varošanec in [22], a generalization of s-convex mappings in the second sense and non-negative convex mappings and P-convex mappings and Godunova-Levin mappings, as follows.

Definition 1. *Let I, J be intervals in $\mathbb{R}$, and let $h : (0, 1) \subseteq J \to \mathbb{R}$ be a non-negative mapping with $h \not\equiv 0$. A non-negative mapping $\omega : I \to \mathbb{R}$ is named h-convex if the inequality*

$$\omega(s\tau_1 + (1 - s)\tau_2) \leq h(s)\omega(\tau_1) + h(1 - s)\omega(\tau_2), \tag{1.1}$$

holds for all $\tau_1, \tau_2 \in I$ and $s \in (0, 1)$.

Many scholars used h-convexity and other special inequalities (Hölder inequality, power mean inequality, and so on) to deflate the equation to obtain various types of inequalities. For example, Bombardelli and Varošanec [4] proved the Hermite-Hadamard-Fejér inequalities for h-convex

mappings. Tunç [17] gave some Ostrowski-type inequalities via h-convex mappings. Wang et al. [23] presented certain k-fractional integral trapezium-like inequalities through (h, m)-convex mappings. Delavar and Dargomir [6] established a new trapezoid form of Fejér inequality which the absolute value of considered function is h-convex. For more information associated with h-convex mappings see reference in [2, 11, 15, 23].

Using generalized convexity to construct fractional integral inequalities has become a hot research direction. In [13], Raina defined the following results connected with the general class of fractional integral operators.

$$F_{\rho,\lambda}^{\sigma}(x) = F_{\rho,\lambda}^{\sigma(0),\sigma(1),\cdots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k (\rho, \lambda; |x| < R), \tag{1.2}$$

where the coefficient $\sigma(k)(k \in \mathbb{N} \cup 0)$ is a bounded sequence of positive real numbers and $\mathbb{R}$ is the set of real numbers.

Raina defined the following left-side and right-sided fractional integral operators, respectively, based on (1.2) in [13].

$$J_{\rho,\lambda,a^+;\omega}^{\sigma} f(x) = \int_a^x (x - t)^{\lambda-1} F_{\rho,\lambda}^{\sigma}[\omega(x - t)^{\rho}] f(t) \mathrm{d}t, \, x > a, \tag{1.3}$$

and

$$J_{\rho,\lambda,b^-;\omega}^{\sigma} f(x) = \int_x^b (t - x)^{\lambda-1} F_{\rho,\lambda}^{\sigma}[\omega(t - x)^{\rho}] f(t) \mathrm{d}t, \, x < b, \tag{1.4}$$

where $\lambda > 0, \rho > 0, \omega \in \mathbb{R}$ and $f(t)$ is such that the integrals on the right side exists.

Based on the above-mentioned generalized fractional integral operator and s-convexity, Usta et al. [19] gave a number of refinements inequalities for the Hermite-Hadamard's type inequality and conclude explicit bounds for the trapezoid inequalities. Chebyshev type inequalities for the generalized fractional integral operators were studied for two synchronous functions in [21]. In [20], the authors obtained some generalized Montgomery identities via above-mentioned generalized fractional integral operator, and established some inequalities of Ostrowski type for mapping whose derivatives are bounded, based on obtained identities. For more results for fractional order with kernels, please see [10, 27, 28] and the references cited therein.

Therefore, this paper intends to establish some general fractional integral inequalities.

2. Preliminaries

Recently, Kadakal and İşcan [8] introduced a new generalized convex mapping, which is named exponential type convex mapping, as follows.

Definition 2. *The non-negative mapping $\omega : I \to \mathbb{R}$ is called exponential type convex mapping, if the following inequality*

$$\omega(s\tau_1 + (1 - s)\tau_2) \le (e^s - 1)\omega(\tau_1) + (e^{1-s} - 1)\omega(\tau_2), \tag{2.1}$$

holds for all $\tau_1, \tau_2 \in I$ and $s \in [0, 1]$.

It is obvious that every exponential type convex mapping is an h-convex mapping with $h(s) = e^s - 1$. In [8], the authors also obtained Hermite-Hadamard type inequality and refinements of the Hermite-Hadamard type inequality for the exponential type convex mappings as follows.

Theorem 1. *Let $\omega : [\tau_1, \tau_2] \to \mathbb{R}$ be an exponential type convex mapping with $\tau_1 \leq \tau_2$. If $\omega \in L^1[\tau_1, \tau_2]$, then the following inequalities exist:*

$$\frac{1}{2(\sqrt{e} - 1)}\omega\Big(\frac{\tau_1 + \tau_2}{2}\Big) \leq \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \omega(s)\mathrm{d}s \leq (\sqrt{e} - 2)[\omega(\tau_1) + \omega(\tau_2)]. \tag{2.2}$$

Theorem 2. *Let $\omega : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I and $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$, such that $\omega' \in L^1[\tau_1, \tau_2]$. If $|\omega'|$ is exponential type convex, then the following inequality*

$$\left|\frac{\omega(\tau_1) + \omega(\tau_2)}{2} - \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \omega(s)\mathrm{d}s\right| \leq (\tau_2 - \tau_1)\Big(4\sqrt{e} - e - \frac{7}{2}\Big)\frac{|\omega'(\tau_1)| + |\omega'(\tau_2)|}{2}, \tag{2.3}$$

holds for all $s \in [0, 1]$.

This method of constructing generalized convex functions had inspired some researchers. For example, Butt et al. [3] introduced a kind of extension mapping of exponential type convex mapping, which is called n-polynomial (s,m)-exponential-type convex mapping, and proved some Hermite-Hadamard type inequalities for such mappings. Gao et al. [7] gave and studied n-polynomial harmonically exponential type convexity. Kashuri et al. [9] obtained several k-fractional integral inequalities for (s, m)-exponential type convex mappings.

Next, we restate some concepts and known results associated with fractional integral operators with exponential kernels.

In [1], Ahmad et al. defined a new fractional integral operators with an exponential kernel and gave new versions of Hermite-Hadamard inequality based on this fractional integral operators as follows.

Definition 3. *[1] Let $\omega \in L^1([\tau_1, \gamma_2])$. The fractional integrals $I_{\tau_1^+}^{\mu}\omega$ and $I_{\tau_2^-}^{\mu}\omega$ of order $\mu \in (0, 1)$ are defined respectively by*

$$I_{\tau_1^+}^{\mu}\omega(x) = \frac{1}{\mu} \int_{\tau_1}^{x} e^{-\frac{1-\mu}{\mu}(x-s)}\omega(s)\mathrm{d}s, \qquad x \geq \tau_1,$$

and

$$I_{\tau_2^-}^{\mu}\omega(x) = \frac{1}{\mu} \int_{x}^{\tau_2} e^{-\frac{1-\mu}{\mu}(s-x)}\omega(s)\mathrm{d}s, \qquad x \leq \tau_2.$$

Note that

$$\lim_{\mu \to 1} I_{\tau_1^+}^{\mu}\omega(x) = \int_{\tau_1}^{x} \omega(s)\mathrm{d}s \quad and \quad \lim_{\mu \to 1} I_{\tau_2^-}^{\mu}\omega(x) = \int_{x}^{\tau_2} \omega(s)\mathrm{d}s.$$

Theorem 3. *[1] Let $\omega : [\tau_1, \tau_2] \to \mathbb{R}$ be a non-negative convex mapping and $0 \leq \tau_1 < \tau_2 < \infty$ with $\omega \in L^1([\tau_1, \tau_2])$, then the following double inequalities for fractional integrals hold:*

$$\omega\Big(\frac{\tau_1 + \tau_2}{2}\Big) \leq \frac{1-\mu}{2(1 - e^{-\rho})}\Big[I_{\tau_1^+}^{\mu}\omega(\tau_2) + I_{\tau_2^-}^{\mu}\omega(\tau_1)\Big] \leq \frac{\omega(\tau_1) + \omega(\tau_2)}{2}, \tag{2.4}$$

where $\rho = \frac{1-\mu}{\mu}(\tau_2 - \tau_1)$.

Taking $\mu \to 1$ i.e. $\rho = \frac{1-\mu}{\mu}(\tau_2 - \tau_1) \to 0$ in Theorem 3, we can recapture classical Hermite-Hadamard inequality for a convex function ω on $[\tau_1, \tau_2]$:

$$\omega\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \frac{1}{\tau_2 - \tau_1} \int_0^1 \omega(s) \mathrm{d}s \leq \frac{\omega(\tau_1) + \omega(\tau_2)}{2}. \tag{2.5}$$

These results attract attention for many authors, some well-known integral inequalities by the approach of this fractional calculus have been carried out by many researchers. For example, Wu et al. [26] constructed three fundamental integral identities to establish some Hermite-Hadamard type inequalities via fractional integrals with exponential kernels. Zhou et al. [29] derived some parameterized fractional integrals with an exponential kernel inequalities for convex mappings. Rashid et al. [14] applied the mappings having the harmonically convexity property and the fractional integral operators with exponential kernels to established several Hermite-Hadamard, Hermite-Hadamard- Fejér and Pachpatte-type integral inequalities. For other works involving fractional integrals with exponential kernels, we refer an interseted reader to [5, 12, 16, 18, 24].

Motivated by the above results mentioned, our principal goal is to establish some new fractional integrals with exponential kernels inequalities for exponential type m-convex mappings. For this, we take some different exponential kernels to establish three right-side Hadamard-type inequalities. We suppose that the absolute value of the derivative of the considered mapping is exponential type m-convex to derive some new midpoint-type, trapezoid-type and Dragomir-Agarwal-type inequalities for fractional integrals with exponential kernels.

3. Certain Hadamard-type inequalities

In this section, we state the definition of exponential type m-convexity and examine how to obtain Hadamard-type inequalities for such mappings.

Definition 4. *The mapping $\omega : [0, \beta] \to \mathbb{R}$ with $\beta > 0$ is named exponential type m-convex mapping, if*

$$\omega(s\tau_1 + m(1 - s)\tau_2) \leq (e^s - 1)\omega(\tau_1) + m(e^{1-s} - 1)\omega(\tau_2), \tag{3.1}$$

holds for all $\tau_1, \tau_2 \in [0, \beta]$ and $s \in [0, 1]$ with some fixed $m \in [0, 1]$

It is obvious that exponential type m-convex mappings are special (h, m)-convex mappings with $h(s) = e^s - 1$.

Remark 1. *If we take $m = 1$ in definition 4, then the exponential type m-convexity reduces to exponential type convexity. If we take $m = 0$ in definition 4, we have $\omega(s\tau_1) \leq (e^s - 1)\omega(\tau_1)$ for all $\tau_1, \tau_2 \in [0, \beta]$ and $s \in [0, 1]$. Here, we call ω is an exponential type starshaped mappings.*

Example 1. *The mapping $\omega(s) = \frac{1}{12}s^4 - \frac{5}{12}s^3 + \frac{3}{4}s^2 - \frac{5}{12}s$, $s \in (0, \infty)$ is an exponential type $\frac{16}{17}$-convex mapping.*

Theorem 4. *Let $\omega : [0, \infty) \to \mathbb{R}$ be an exponential type m-convex mapping with $m \in (0, 1]$. If $\omega \in L^1[\tau_1, \tau_2]$ and $0 \leq \tau_1 < m\tau_2$, then we have*

$$\frac{\mu}{m\tau_2 - \tau_1}\left[I^\mu_{\tau_1^+}\omega(m\tau_2) + I^\mu_{m\tau_2^-}\omega(\tau_1)\right] \leq \left(\frac{e^{1-\delta} - 1}{1 - \delta} + \frac{e - e^{-\delta}}{1 + \delta} - \frac{2(1 - e^{-\delta})}{\delta}\right)[\omega(\tau_1) + m\omega(\tau_2)], \tag{3.2}$$

where $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$.

Proof. By means of exponential type m-convexity of ω, we have

$$\omega(s\tau_1 + m(1-s)\tau_2) \leq (e^s - 1)\omega(\tau_1) + m(e^{1-s} - 1)\omega(\tau_2) \tag{3.3}$$

and

$$\omega((1-s)\tau_1 + ms\tau_2) \leq (e^{1-s} - 1)\omega(\tau_1) + m(e^s - 1)\omega(\tau_2). \tag{3.4}$$

By adding inequality (3.3) and inequality (3.4) together, we deduce

$$\omega(s\tau_1 + m(1-s)\tau_2) + \omega((1-s)\tau_1 + ms\tau_2) \leq \left[e^s + e^{1-s} - 2\right]\left(\omega(\tau_1) + m\omega(\tau_2)\right). \tag{3.5}$$

We can obtain the desired inequality by multiplying (3.5) with $e^{-\delta s}$ and then integrating over $[0, 1]$ with respect to $\mathrm{d}s$. Since

$$\int_0^1 e^{-\delta s}\left[\omega(s\tau_1 + m(1-s)\tau_2) + \omega((1-s)\tau_1 + ms\tau_2)\right]\mathrm{d}s$$

$$= \int_0^1 e^{-\delta s}\omega(s\tau_1 + m(1-s)\tau_2)\mathrm{d}s + \int_0^1 e^{-\delta s}\omega((1-s)\tau_1 + ms\tau_2)\mathrm{d}s$$

$$= \frac{1}{m\tau_2 - \tau_1}\int_{\tau_1}^{m\tau_2} e^{-\frac{1-\mu}{\mu}(m\tau_2 - \tau_1)\frac{m\tau_2 - x}{m\tau_2 - \tau_1}}\omega(x)\mathrm{d}x + \frac{1}{m\tau_2 - \tau_1}\int_{\tau_1}^{m\tau_2} e^{-\frac{1-\mu}{\mu}(m\tau_2 - \tau_1)\frac{x - \tau_1}{m\tau_2 - \tau_1}}\omega(x)\mathrm{d}x$$

$$= \frac{\mu}{m\tau_2 - \tau_1}\left[I_{\tau_1^+}^{\mu}\omega(m\tau_2) + I_{m\tau_2^-}^{\mu}\omega(\tau_1)\right]$$

and

$$\int_0^1 e^{-\delta s}\left[e^s + e^{1-s} - 2\right]\left(\omega(\tau_1) + m\omega(\tau_2)\right)\mathrm{d}s = \left(\frac{e^{1-\delta} - 1}{1 - \delta} + \frac{e - e^{-\delta}}{1 + \delta} - \frac{2(1 - e^{-\delta})}{\delta}\right)\left[\omega(\tau_1) + m\omega(\tau_2)\right].$$

This ends the proof.

Corollary 1. *If we consider $\mu \to 1$ i.e. $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1) \to 0$ in Theorem 4, then we have*

$$\frac{1}{m\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \omega(\tau)\mathrm{d}\tau \leq (e - 2)\left[\omega(\tau_1) + m\omega(\tau_2)\right]. \tag{3.6}$$

Theorem 5. *Let $\omega : [0, \infty) \to \mathbb{R}$ be an exponential type m-convex mapping with $m \in (0, 1]$. If $\omega \in L^1[\tau_1, \tau_2]$ and $0 \leq \tau_1 < \tau_2$, then the resulting expression holds:*

$$\frac{\mu}{\tau_2 - \tau_1}\left[I_{\tau_2^-}^{\mu}\omega(\tau_1) + I_{\tau_1^+}^{\mu}\omega(\tau_2)\right]$$

$$\leq \left[\frac{e^{1-\kappa} - 1}{1 - \kappa} + \frac{e^{-\kappa} - 1}{\kappa}\right]\left(\omega(\tau_1) + \omega(\tau_2)\right) + m\left[\frac{e - e^{-\kappa}}{1 + \kappa} + \frac{e^{-\kappa} - 1}{\kappa}\right]\left(\omega\left(\frac{\tau_2}{m}\right) + \omega\left(\frac{\tau_1}{m}\right)\right), \tag{3.7}$$

where $\kappa = \frac{\mu}{1-\mu}(\tau_2 - \tau_1)$.

Proof. Since ω is exponential type m-convexity, we deduce

$$\omega(s\tau_1 + (1-s)\tau_2) \le (e^s - 1)\omega(\tau_1) + m(e^{1-s} - 1)\omega\left(\frac{\tau_2}{m}\right)$$

and

$$\omega(s\tau_2 + (1-s)\tau_1) \le (e^s - 1)\omega(\tau_2) + m(e^{1-s} - 1)\omega\left(\frac{\tau_1}{m}\right).$$

Multiplying above-mentioned inequalities with $e^{-\kappa s}$ and then integrating over $[0, 1]$ with respect to ds, we get

$$
\begin{aligned}
\int_0^1 e^{-\kappa s}\omega(s\tau_1 + (1-s)\tau_2)\mathrm{d}s &= \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} e^{-\frac{1-\mu}{\mu}(\tau_2 - x)}\omega(x)\mathrm{d}x \\
&= \frac{\mu}{\tau_2 - \tau_1}I_{\tau_2^-}^\mu \omega(\tau_1) \\
&\le \int_0^1 e^{-\kappa s}\left[(e^s - 1)\omega(\tau_1) + m(e^{1-s} - 1)\omega\left(\frac{\tau_2}{m}\right)\right]\mathrm{d}s \\
&= \left[\frac{e^{1-\kappa} - 1}{1 - \kappa} + \frac{e^{-\kappa} - 1}{\kappa}\right]\omega(\tau_1) + m\left[\frac{e - e^{-\kappa}}{1 + \kappa} + \frac{e^{-\kappa} - 1}{\kappa}\right]\omega\left(\frac{\tau_2}{m}\right)
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\int_0^1 e^{-\kappa s}\omega(s\tau_2 + (1-s)\tau_1)\mathrm{d}s &= \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} e^{-\frac{1-\mu}{\mu}(x - \tau_1)}\omega(x)\mathrm{d}x \\
&= \frac{\mu}{\tau_2 - \tau_1}I_{\tau_1^+}^\mu \omega(\tau_2) \\
&\le \int_0^1 e^{-\kappa s}\left[(e^s - 1)\omega(\tau_2) + m(e^{1-s} - 1)\omega\left(\frac{\tau_1}{m}\right)\right]\mathrm{d}s \\
&= \left[\frac{e^{1-\kappa} - 1}{1 - \kappa} + \frac{e^{-\kappa} - 1}{\kappa}\right]\omega(\tau_2) + m\left[\frac{e - e^{-\kappa}}{1 + \kappa} + \frac{e^{-\kappa} - 1}{\kappa}\right]\omega\left(\frac{\tau_1}{m}\right).
\end{aligned}
\tag{3.9}
$$

By adding inequality (3.8) and inequality (3.9) together, we can get desired inequality (3.7). This ends the proof.

Corollary 2. *If we consider* $\mu \to 1$ *i.e.* $\kappa = \frac{1-\mu}{\mu}(\tau_2 - \tau_1) \to 0$ *in Theorem 5, then we have*

$$\frac{2}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2}\omega(s)\mathrm{d}s \le (e - 2)\left[\omega(\tau_1) + \omega(\tau_2) + m\left(\omega\left(\frac{\tau_1}{m}\right) + \omega\left(\frac{\tau_2}{m}\right)\right)\right]. \tag{3.10}$$

Theorem 6. *Under the assumptions of Theorem 5, if we take* $\theta = \frac{1-\mu}{\mu}\frac{\tau_2 - \tau_1}{2}$, *then the following inequality exists:*

$$
\begin{aligned}
&\frac{2\mu}{\tau_2 - \tau_1}\left[I_{\tau_1^+}^\mu \omega\left(\frac{\tau_1 + \tau_2}{2}\right) + I_{\tau_2^-}^\mu \omega\left(\frac{\tau_1 + \tau_2}{2}\right)\right] \\
&\le \left[\frac{2\sqrt{e} - 2e^{1-\theta}}{2\theta - 1} - \frac{1 - e^{-\theta}}{\theta}\right](\omega(\tau_1) + \omega(\tau_2)) + m\left[\frac{2\sqrt{e} - 2e^{-\theta}}{2\theta + 1} - \frac{1 - e^{-\theta}}{\theta}\right]\left(\omega\left(\frac{\tau_1}{m}\right) + \omega\left(\frac{\tau_2}{m}\right)\right).
\end{aligned}
\tag{3.11}
$$

Proof. Using the exponential type m-convexity of ω, we obtain

$$\omega\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}\tau_2\right) \le \left(e^{\frac{1+s}{2}} - 1\right)\omega(\tau_1) + m\left(e^{\frac{1-s}{2}} - 1\right)\omega\left(\frac{\tau_2}{m}\right) \tag{3.12}$$

and

$$\omega\left(\frac{1+s}{2}\tau_2 + \frac{1-s}{2}\tau_1\right) \le \left(e^{\frac{1+s}{2}} - 1\right)\omega(\tau_2) + m\left(e^{\frac{1-s}{2}} - 1\right)\omega\left(\frac{\tau_1}{m}\right). \tag{3.13}$$

Adding inequality(3.12) and inequality(3.13) together and then multiplying by $e^{-\theta s}$, we have

$$e^{-\theta s}\left[\omega\left(\frac{1+s}{2}\tau_2 + \frac{1-s}{2}\tau_1\right) + \omega\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}\tau_2\right)\right]$$
$$\le e^{-\theta s}\left(\left(e^{\frac{1+s}{2}} - 1\right)[\omega(\tau_1) + \omega(\tau_2)] + m\left(e^{\frac{1-s}{2}} - 1\right)\left[\omega\left(\frac{\tau_1}{m}\right) + \omega\left(\frac{\tau_2}{m}\right)\right]\right). \tag{3.14}$$

Integrating on both sides of inequality (3.14) respect to s over $[0,1]$, we have completed the proof. Since

$$\int_0^1 e^{-\theta s}\left[\omega\left(\frac{1+s}{2}\tau_2 + \frac{1-s}{2}\tau_1\right) + \omega\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}\tau_2\right)\right]\mathrm{d}s$$
$$= \int_0^1 e^{-\theta s}\omega\left(\frac{1+s}{2}\tau_2 + \frac{1-s}{2}\tau_1\right)\mathrm{d}s + \int_0^1 e^{-\theta s}\omega\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}\tau_2\right)\mathrm{d}s$$
$$= \frac{2}{\tau_2 - \tau_1}\int_{\tau_1}^{\frac{\tau_1+\tau_2}{2}} e^{-\frac{1-\mu}{\mu}\frac{\tau_2-\tau_1}{2}\frac{(\tau_1+\tau_2)-2x}{\tau_2-\tau_1}}\omega(x)\mathrm{d}x + \frac{2}{\tau_2 - \tau_1}\int_{\frac{\tau_2+\tau_1}{2}}^{\tau_2} e^{-\frac{1-\mu}{\mu}\frac{\tau_2-\tau_1}{2}\frac{2x-(\tau_1+\tau_2)}{\tau_2-\tau_1}}\omega(x)\mathrm{d}x$$
$$= \frac{2\mu}{\tau_2 - \tau_1}\left[I_{\tau_1^+}^{\mu}\omega\left(\frac{\tau_1+\tau_2}{2}\right) + I_{\tau_2^-}^{\mu}\omega\left(\frac{\tau_1+\tau_2}{2}\right)\right]$$

and

$$\int_0^1 e^{-\theta s}\left(\left(e^{\frac{1+s}{2}} - 1\right)[\omega(\tau_1) + \omega(\tau_2)] + m\left(e^{\frac{1-s}{2}} - 1\right)\left[\omega\left(\frac{\tau_1}{m}\right) + \omega\left(\frac{\tau_2}{m}\right)\right]\right)$$
$$= \left[\frac{2\sqrt{e} - 2e^{1-\theta}}{2\theta - 1} - \frac{1 - e^{-\theta}}{\theta}\right]\left(\omega(\tau_1) + \omega(\tau_2)\right) + m\left[\frac{2\sqrt{e} - 2e^{-\theta}}{2\theta + 1} - \frac{1 - e^{-\theta}}{\theta}\right]\left(\omega\left(\frac{\tau_1}{m}\right) + \omega\left(\frac{\tau_2}{m}\right)\right).$$

Corollary 3. *If we consider $\mu \to 1$ i.e. $\theta = \frac{1-\mu}{\mu}\frac{\tau_2-\tau_1}{2} \to 0$ in Theorem 6, then we have*

$$\frac{2}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2}\omega(s)\mathrm{d}s \le (2e - 2\sqrt{e} - 1)[\omega(\tau_1) + \omega(\tau_2)] + m(2\sqrt{e} - 3)\left[\left(\omega\left(\frac{\tau_1}{m}\right) + \omega\left(\frac{\tau_2}{m}\right)\right)\right]. \tag{3.15}$$

4. Mid-point type and trapezoid type inequality

In this section, we investigate how to establish mid-point type inequalities and trapezoid type inequalities for exponential type m-convex mappings.

Lemma 1. *Assuming $\omega : [\tau_1, m\tau_2] \to \mathbb{R}$ is a differentiable mapping, such that $\omega' \in L^1([\tau_1, m\tau_2])$ with $0 \leq \tau_1 < m\tau_2 < \infty$, then the following identity holds:*

$$
\begin{aligned}
&\Psi(\tau_1, m\tau_2, \mu, \delta) \\
&= \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})} \left\{ \int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s - \int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{2 - s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\mathrm{d}s \right\},
\end{aligned} \tag{4.1}
$$

where $0 < m \leq 1$, $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$ and

$$
\Psi(\tau_1, \tau_2, \mu, \delta) := -\frac{1 - \mu}{2(1 - e^{-\delta})}\left[I^\mu_{\frac{m\tau_2 + \tau_1}{2}^-}\omega(\tau_1) + I^\mu_{\frac{m\tau_2 + \tau_1}{2}^+}\omega(m\tau_2)\right] + \omega\left(\frac{\tau_1 + m\tau_2}{2}\right). \tag{4.2}
$$

Proof. It suffices to note that

$$
\begin{aligned}
\xi &= \int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s - \int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{2 - s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\mathrm{d}s \\
&= \left[\int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s\right] - \left[\int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{2 - s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\mathrm{d}s\right] \\
&= \xi_1 - \xi_2.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
\xi_1 &:= \int_0^1 [e^{-\delta s} - 1]\omega'\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s \\
&= \int_0^1 e^{-\delta s}\omega'\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s - \int_0^1 \omega'\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s \\
&= -\frac{2}{m\tau_2 - \tau_1}\int_0^1 e^{-\delta s}\mathrm{d}\left(\omega\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\right) + \frac{2}{m\tau_2 - \tau_1}\left[\omega\left(\frac{m\tau_2 + \tau_1}{2}\right) - \omega(m\tau_2)\right] \\
&= -\frac{2}{m\tau_2 - \tau_1}\left[e^{-\delta s}\omega\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\Big|_0^1 - \int_0^1 \omega\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}\left(e^{-\delta s}\right) - \omega\left(\frac{m\tau_2 + \tau_1}{2}\right) + \omega(m\tau_2)\right] \\
&= -\frac{2}{m\tau_2 - \tau_1}\left[e^{-\delta}\omega\left(\frac{\tau_1 + m\tau_2}{2}\right) - \omega(m\tau_2) + \delta\int_0^1 e^{-\delta s}\omega\left(\frac{s}{2}\tau_1 + \frac{2 - s}{2}m\tau_2\right)\mathrm{d}s - \omega\left(\frac{m\tau_2 + \tau_1}{2}\right) + \omega(m\tau_2)\right] \\
&= \frac{2}{m\tau_2 - \tau_1}\left[(1 - e^{-\delta})\omega\left(\frac{\tau_1 + m\tau_2}{2}\right) - \frac{2\mu\delta}{m\tau_2 - \tau_1}I^\mu_{\frac{m\tau_2 + \tau_1}{2}^+}\omega(m\tau_2)\right] \\
&= \frac{2}{m\tau_2 - \tau_1}\left[(1 - e^{-\delta})\omega\left(\frac{\tau_1 + m\tau_2}{2}\right) - (1 - \mu)I^\mu_{\frac{m\tau_2 + \tau_1}{2}^+}\omega(m\tau_2)\right],
\end{aligned}
$$

$$\tag{4.3}$$

and

$$\xi_2 := \int_0^1 \left[e^{-\delta s} - 1 \right] \omega' \left(\frac{2-s}{2} \tau_1 + \frac{s}{2} m\tau_2 \right) ds$$

$$= \int_0^1 e^{-\delta s} \omega' \left(\frac{2-s}{2} \tau_1 + \frac{s}{2} m\tau_2 \right) ds - \int_0^1 \omega' \left(\frac{2-s}{2} \tau_1 + \frac{s}{2} m\tau_2 \right) ds$$

$$= \frac{2}{m\tau_2 - \tau_1} \int_0^1 e^{-\delta s} d \left(\frac{2-s}{2} \tau_1 + \frac{s}{2} m\tau_2 \right) - \left[\omega \left(\frac{m\tau_2 + \tau_1}{2} \right) - \omega(\tau_1) \right]$$

$$= \frac{2}{m\tau_2 - \tau_1} \left[e^{-\delta s} \omega \left(\frac{2-s}{2} \tau_1 + \frac{s}{2} m\tau_2 \right) \Big|_0^1 - \int_0^1 \omega \left(\frac{2-\tau}{2} \tau_1 + \frac{\tau}{2} m\tau_2 \right) d \left(e^{-\delta s} \right) - \omega \left(\frac{m\tau_2 + \tau_1}{2} \right) + \omega(\tau_1) \right]$$

$$= \frac{2}{m\tau_2 - \tau_1} \left[e^{-\delta} \omega \left(\frac{\tau_1 + m\tau_2}{2} \right) - \omega(\tau_1) - \omega \left(\frac{m\tau_2 + \tau_1}{2} \right) + \omega(\tau_1) + \delta \int_0^1 e^{-\delta s} \omega \left(\frac{2-s}{2} \tau_1 + \frac{s}{2} m\tau_2 \right) ds \right]$$

$$= \frac{2}{m\tau_2 - \tau_1} \left[e^{-\delta} \omega \left(\frac{\tau_1 + m\tau_2}{2} \right) - \omega \left(\frac{m\tau_2 + \tau_1}{2} \right) + \frac{2\mu\delta}{m\tau_2 + \tau_1} I^\mu_{\frac{m\tau_2+\tau_1}{2}-} \omega(\tau_1) \right]$$

$$= \frac{2}{m\tau_2 - \tau_1} \left[(e^{-\delta} - 1) \omega \left(\frac{\tau_1 + m\tau_2}{2} \right) + (1-\mu) I^\mu_{\frac{m\tau_2+\tau_1}{2}-} \omega(\tau_1) \right].$$

$$(4.4)$$

From ξ_1 and ξ_2, we deduce

$$\frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})} [\xi_1 - \xi_2] = -\frac{1-\mu}{2(1 - e^{-\delta})} \left[I^\mu_{\frac{m\tau_2+\tau_1}{2}-} \omega(\tau_1) + I^\mu_{\frac{m\tau_2+\tau_1}{2}+} \omega(m\tau_2) \right] + \omega \left(\frac{\tau_1 + m\tau_2}{2} \right),$$

which completes the proof.

Corollary 4. *If we take $m = 1$ in Lemma 1, then we have Lemma 2.1 in [25].*

Theorem 7. *Let ω be defined as in Lemma 1. If the function $|\omega'|^q$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ is an exponential type m-convex mapping on $[\tau_1, \tau_2]$, then the following inequality exists:*

$$\left| \Psi(\tau_1, m\tau_2, \mu, \delta) \right| \leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})} \left[\frac{\delta p + e^{-\delta p} - 1}{\delta p} \right]^{\frac{1}{p}} \left[\left((2\sqrt{e} - 3) \left| \omega'(\tau_1) \right|^q + m(2e - 2\sqrt{e} - 1) \left| \omega'(\tau_2) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left((2e - 2\sqrt{e} - 1) \left| \omega'(\tau_1) \right|^q + m(2\sqrt{e} - 3) \left| \omega'(\tau_2) \right|^q \right)^{\frac{1}{q}} \right],$$

$$(4.5)$$

where $m \in (0, 1]$, $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$ and

$$\Psi(\tau_1, \tau_2, \mu, \delta) := -\frac{1-\mu}{2(1 - e^{-\delta})} \left[I^\mu_{\frac{m\tau_2+\tau_1}{2}-} \omega(\tau_1) + I^\mu_{\frac{m\tau_2+\tau_1}{2}+} \omega(m\tau_2) \right] + \omega \left(\frac{\tau_1 + m\tau_2}{2} \right).$$

Proof. Applying Lemma 1, Hölder inequality and the exponential type m-convexity of $|\omega'|^q$, we deduce

$$\left|\Psi(\tau_1, m\tau_2, \mu, \delta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left\{\int_0^1 \left|e^{-\delta s} - 1\right|\left|\omega'\left(\frac{s}{2}\tau_1 + \frac{2-s}{2}m\tau_2\right)\right|ds + \int_0^1 \left|e^{-\delta s} - 1\right|\left|\omega'\left(\frac{2-s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\right|ds\right\}$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left\{\left(\int_0^1 \left|e^{-\delta s} - 1\right|^p ds\right)^{\frac{1}{p}}\left(\int_0^1 \left|\omega'\left(\frac{s}{2}\tau_1 + \frac{2-s}{2}m\tau_2\right)\right|^q ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1 \left|e^{-\delta s} - 1\right|^p ds\right)^{\frac{1}{p}}\left(\int_0^1 \left|\omega'\left(\frac{2-s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\right|^q ds\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left[\left(\int_0^1 \left|e^{-\delta s} - 1\right|^p ds\right)^{\frac{1}{p}}\left(\int_0^1 (e^{\frac{s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{2-s}{2}} - 1)\left|\omega'(\tau_2)\right|^q ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1 \left|e^{-\delta s} - 1\right|^p ds\right)^{\frac{1}{p}}\left(\int_0^1 (e^{\frac{2-s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{s}{2}} - 1)\left|\omega'(\tau_2)\right|^q ds\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left[\left(\int_0^1 1 - e^{-p\delta s} ds\right)^{\frac{1}{p}}\left(\int_0^1 (e^{\frac{s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{2-s}{2}} - 1)\left|\omega'(\tau_2)\right|^q ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1 1 - e^{-p\delta s} ds\right)^{\frac{1}{p}}\left(\int_0^1 (e^{\frac{2-s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{s}{2}} - 1)\left|\omega'(\tau_2)\right|^q ds\right)^{\frac{1}{q}}\right]$$

$$= \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left[\frac{\delta p + e^{-\delta p} - 1}{\delta p}\right]^{\frac{1}{p}}\left[\left((2\sqrt{e} - 3)\left|\omega'(\tau_1)\right|^q + m(2e - 2\sqrt{e} - 1)\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left. + \left((2e - 2\sqrt{e} - 1)\left|\omega'(\tau_1)\right|^q + m(2\sqrt{e} - 3)\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}}\right].$$

Here, we use the fact that $(x - y)^q \leq x^q - y^q$ for any $x \geq y \geq 0$ and $q \geq 1$. This ends the proof. $\blacksquare$

Theorem 8. *Let ω be defined as in Lemma 1. If the function $|\omega'|^q$ for $q \geq 1$ is exponential type m-convex on $[\tau_1, \tau_2]$ with some fixed $m \in (0, 1]$, then the following inequality for fractional integrals with exponential kernels holds:*

$$\left|\Psi(\tau_1, m\tau_2, \mu, \delta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left[\frac{\delta + e^{-\delta} - 1}{\delta}\right]^{1-\frac{1}{q}}\left\{\left(\Delta_1\left|\omega'(\tau_1)\right|^q + m\Delta_2\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}} + \left(\Delta_2\left|\omega'(\tau_1)\right|^q + m\Delta_2\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}}\right\} \quad (4.6)$$

where $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$, $\Delta_1 := \frac{2 - 2e^{\frac{1}{2}-\delta}}{1 - 2\delta} + \frac{1 - e^{-\delta}}{\delta} + 2\sqrt{e} - 3$, $\Delta_2 := -\frac{2e - 2e^{\frac{1}{2}-\delta}}{1 + 2\delta} + \frac{1 - e^{-\delta}}{\delta} + 2e - 2\sqrt{e} - 1$
and

$$\Psi(\tau_1, \tau_2, \mu, \delta) := -\frac{1 - \mu}{2(1 - e^{-\delta})}\left[I^\mu_{\frac{m\tau_2 + \tau_1}{2}^-}\omega(\tau_1) + I^\mu_{\frac{m\tau_2 + \tau_1}{2}^+}\omega(m\tau_2)\right] + \omega\left(\frac{\tau_1 + m\tau_2}{2}\right).$$

Proof. Applying Lemma 1, power-mean inequality and the exponential type m-convexity of $|\omega'|^q$, we

deduce

$$\left|\Psi(\tau_1, m\tau_2, \mu, \delta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left\{ \int_0^1 \left|e^{-\delta s} - 1\right|\left|\omega'\left(\frac{s}{2}\tau_1 + \frac{2-s}{2}m\tau_2\right)\right|ds + \int_0^1 \left|e^{-\delta s} - 1\right|\left|\omega'\left(\frac{2-s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\right|ds\right\}$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left\{ \left(\int_0^1 \left|e^{-\delta s} - 1\right|ds\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|e^{-\delta s} - 1\right|\left|\omega'\left(\frac{s}{2}\tau_1 + \frac{2-s}{2}m\tau_2\right)\right|^q ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1 \left|e^{-\delta s} - 1\right|ds\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|e^{-\delta s} - 1\right|\left|\omega'\left(\frac{2-s}{2}\tau_1 + \frac{s}{2}m\tau_2\right)\right|^q ds\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left[\frac{\delta + e^{-\delta} - 1}{\delta}\right]^{1-\frac{1}{q}}\left\{\left(\int_0^1 \left|e^{-\delta s} - 1\right|\left[(e^{\frac{s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{2-s}{2}} - 1)\left|\omega'(\tau_2)\right|^q\right]ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1 \left|e^{-\delta s} - 1\right|\left[(e^{\frac{2-s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{s}{2}} - 1)\left|\omega'(\tau_2)\right|^q\right]ds\right)^{\frac{1}{q}}\right\}.$$

By calculation, we have

$$\Delta_1 := \int_0^1 \left(e^{\frac{s}{2}} - 1\right)(1 - e^{-\delta s})ds = \frac{2 - 2e^{\frac{1}{2}-\delta}}{1 - 2\delta} + \frac{1 - e^{-\delta}}{\delta} + 2\sqrt{e} - 3$$

and

$$\Delta_2 := \int_0^1 \left(e^{\frac{2-s}{2}} - 1\right)(1 - e^{-\delta s})ds = -\frac{2e - 2e^{\frac{1}{2}-\delta}}{1 + 2\delta} + \frac{1 - e^{-\delta}}{\delta} + 2e - 2\sqrt{e} - 1.$$

After suitable arrangements, we obtain

$$\left|\Psi(\tau_1, m\tau_2, \mu, \delta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\left[\frac{\delta + e^{-\delta} - 1}{\delta}\right]^{1-\frac{1}{q}}\left\{\left(\Delta_1\left|\omega'(\tau_1)\right|^q + m\Delta_2\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}} + \left(\Delta_2\left|\omega'(\tau_1)\right|^q + m\Delta_2\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}}\right\}.$$

The proof is completed.

The following lemma is used to prove the trapezoid type inequalities for generalized fractional integral operators.

Lemma 2. *Assuming* $\omega : [\tau_1, m\tau_2] \to \mathbb{R}$ *is a differentiable mapping, such that* $\omega \in L^1([\tau_1, m\tau_2])$ *with* $0 \leq \tau_1 < m\tau_2 < \infty$, *then the following identity exists:*

$$\chi(\tau_1, m\tau_2, \mu, \eta)$$

$$= \frac{m\tau_2 - \tau_1}{4(e^{-\eta} - 1)}\left\{ \int_0^1 [e^{-\eta s} - 1]\omega'\left(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\right)ds - \int_0^1 [e^{-\eta s} - 1]\omega'\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\right)ds\right\},$$
$$\tag{4.7}$$

where $m \in (0, 1]$, $\eta = \frac{1-\mu}{\mu}\frac{m\tau_2 - \tau_1}{2}$ *and*

$$\chi(\tau_1, m\tau_2, \mu, \eta) := \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{1-\mu}{2(1 - e^{-\eta})}\left[I_{\tau_2^-}^\mu \omega\left(\frac{\tau_1 + m\tau_2}{2}\right) + I_{\tau_1^+}^\mu \omega\left(\frac{\tau_1 + m\tau_2}{2}\right)\right]. \tag{4.8}$$

Proof. It suffices to note that

$$\zeta = \int_0^1 [e^{-\eta s} - 1]\omega'\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\mathrm{d}s - \int_0^1 [e^{-\eta s} - 1]\omega'\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\mathrm{d}s$$

$$= \Big[\int_0^1 [e^{-\eta s} - 1]\omega'\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\mathrm{d}s\Big] + \Big[-\int_0^1 [e^{-\eta s} - 1]\omega'\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\mathrm{d}s\Big]$$

$$:= \zeta_1 + \zeta_2.$$

Integrating by parts, we have

$$\zeta_1 := \int_0^1 [e^{-\eta s} - 1]\omega'\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\mathrm{d}s$$

$$= \frac{2}{m\tau_2 - \tau_1} \int_0^1 [e^{-\eta s} - 1]\mathrm{d}\Big(\omega\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\Big)$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[\int_0^1 e^{-\eta s}\mathrm{d}\Big(\omega\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\Big) - \omega\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\Big|_0^1\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[e^{-\eta\tau}\omega\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\Big|_0^1 - \int_0^1 \omega\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\mathrm{d}\big(e^{-\eta s}\big)$$

$$\qquad - \omega(m\tau_2) + \omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big)\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[(e^{-\eta} - 1)\omega(m\tau_2) + \eta\int_0^1 e^{-\eta s}\omega\Big(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\Big)\mathrm{d}s\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[(e^{-\eta} - 1)\omega(m\tau_2) + \frac{2}{m\tau_2 - \tau_1}\frac{1-\mu}{\mu}\frac{m\tau_2 - \tau_1}{2}\int_{\frac{\tau_1+m\tau_2}{2}}^{m\tau_2} e^{-\frac{1-\mu}{\mu}\frac{m\tau_2-\tau_1}{2}\frac{2x-(\tau_1+m\tau_2)}{m\tau_2-\tau_1}}\omega(x)\mathrm{d}x\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[(e^{-\eta} - 1)\omega(m\tau_2) + (1-\mu)I_{m\tau_2^-}^{\mu}\omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big)\Big],$$

and

$$\zeta_2 := \int_0^1 -[e^{-\eta s} - 1]\omega'\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}\tau_2\Big)\mathrm{d}s$$

$$= \frac{2}{m\tau_2 - \tau_1} \int_0^1 [e^{-\eta s} - 1]\mathrm{d}\Big(\omega\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\Big)$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[\int_0^1 e^{-\eta s}\mathrm{d}\Big(\omega\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\Big) - \omega\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\Big|_0^1\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[e^{-\eta s}\omega\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\Big|_0^1 - \int_0^1 \omega\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\mathrm{d}\big(e^{-\eta s}\big)$$

$$\qquad - \omega(\tau_1) + \omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big)\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[(e^{-\eta} - 1)\omega(\tau_1) + \eta\int_0^1 e^{-\eta s}\omega\Big(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\Big)\mathrm{d}s\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[(e^{-\eta} - 1)\omega(\tau_1) + \frac{2}{m\tau_2 - \tau_1}\frac{1-\mu}{\mu}\frac{m\tau_2 - \tau_1}{2}\int_{\tau_1}^{\frac{\tau_1+m\tau_2}{2}} e^{-\frac{1-\mu}{\mu}\frac{m\tau_2-\tau_1}{2}\frac{(\tau_1+m\tau_2)-2x}{m\tau_2-\tau_1}}\omega(x)\mathrm{d}x\Big]$$

$$= \frac{2}{m\tau_2 - \tau_1}\Big[(e^{-\eta} - 1)\omega(\tau_1) + (1-\mu)I_{\tau_1^+}^{\mu}\omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big)\Big].$$

From ζ_1 and ζ_2, we deduce

$$\frac{m\tau_2 - \tau_1}{4(e^{-\eta} - 1)}\big[\zeta_1 + \zeta_2\big] = \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{1 - \mu}{2(1 - e^{-\eta})}\Big[I^\mu_{\tau_2^-}\omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big) + I^\mu_{\tau_1^+}\omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big)\Big],$$

which completes the proof.

Corollary 5. *If we take $m = 1$ in Lemma 2, then we have Lemma 3.1 in [25].*

Theorem 9. *Let ω be defined as in Lemma 2. If the function $|\omega'|^q$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ is an exponential type m-convex mapping on $[\tau_1, \tau_2]$, then the following inequality exists:*

$$\big|\chi(\tau_1, \tau_2, \mu, \eta)\big| \le \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\Big[\frac{\eta p + e^{-\eta p} - 1}{\eta p}\Big]^{\frac{1}{p}}\Big[\big((2e^{\frac{1}{2}} - 3)|\omega'(\tau_1)|^q + m(2e - 2e^{\frac{1}{2}} - 1)|\omega'(\tau_2)|^q\big)^{\frac{1}{q}}$$

$$+ \big((2e - 2e^{\frac{1}{2}} - 1)|\omega'(\tau_1)|^q + m(2e^{\frac{1}{2}} - 3)|\omega'(\tau_2)|^q\big)^{\frac{1}{q}}\Big], \tag{4.9}$$

where $m \in (0, 1]$, $\eta = \frac{1 - \mu}{\mu}\frac{m\tau_2 - \tau_1}{2}$ and

$$\chi(\tau_1, m\tau_2, \mu, \eta) := \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{1 - \mu}{2(1 - e^{-\eta})}\Big[I^\mu_{\tau_2^-}\omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big) + I^\mu_{\tau_1^+}\omega\Big(\frac{\tau_1 + m\tau_2}{2}\Big)\Big].$$

Proof. Applying Lemma 2, Hölder inequality and the exponential type m-convexity of $|\omega'|^q$, we deduce

$$\big|\chi(\tau_1, \tau_2, \mu, \eta)\big| \le \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\Big[\int_0^1 \big|e^{-\eta s} - 1\big|\Big|\omega'\Big(\frac{1 - s}{2}\tau_1 + \frac{1 + s}{2}m\tau_2\Big)\Big|ds$$

$$+ \int_0^1 \big|e^{-\eta s} - 1\big|\Big|\omega'\Big(\frac{1 + s}{2}\tau_1 + \frac{1 - s}{2}m\tau_2\Big)\Big|ds\Big]$$

$$\le \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\Big[\Big(\int_0^1 \big|e^{-\eta s} - 1\big|^p ds\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|\omega'\Big(\frac{1 - s}{2}\tau_1 + \frac{1 + s}{2}m\tau_2\Big)\Big|^q ds\Big)^{\frac{1}{q}}$$

$$+ \Big(\int_0^1 \big|e^{-\eta s} - 1\big|^p ds\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|\omega'\Big(\frac{1 + s}{2}\tau_1 + \frac{1 - s}{2}m\tau_2\Big)\Big|^q ds\Big)^{\frac{1}{q}}\Big]$$

$$\le \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\Big[\Big(\int_0^1 \big(1 - e^{-p\eta s}\big)ds\Big)^{\frac{1}{p}}\Big(\int_0^1 (e^{\frac{1-s}{2}} - 1)|\omega'(\tau_1)|^q + m(e^{\frac{1+s}{2}} - 1)|\omega'(\tau_2)|^q ds\Big)^{\frac{1}{q}}$$

$$+ \Big(\int_0^1 \big(1 - e^{-p\eta s}\big)ds\Big)^{\frac{1}{p}}\Big(\int_0^1 (e^{\frac{1+s}{2}} - 1)|\omega'(\tau_1)|^q + m(e^{\frac{1-s}{2}} - 1)|\omega'(\tau_2)|^q ds\Big)^{\frac{1}{q}}\Big]$$

$$= \frac{m\tau_2 - \tau_1}{4(1 - e^{-\delta})}\Big[\frac{\eta p + e^{-\eta p} - 1}{\eta p}\Big]^{\frac{1}{p}}\Big[\big((2e^{\frac{1}{2}} - 3)|\omega'(\tau_1)|^q + m(2e - 2e^{\frac{1}{2}} - 1)|\omega'(\tau_2)|^q\big)^{\frac{1}{q}}$$

$$+ \big((2e - 2e^{\frac{1}{2}} - 1)|\omega'(\tau_1)|^q + m(2e^{\frac{1}{2}} - 3)|\omega'(\tau_2)|^q\big)^{\frac{1}{q}}\Big].$$

Here, we use the fact that $(x - y)^q \le x^q - y^q$ for any $x \ge y \ge 0$ and $q \ge 1$. The proof is completed.

Theorem 10. *Let ω be defined as in Lemma 2. If the function $|\omega'|^q$ for $q \ge 1$ is exponential type m-convex on $[\tau_1, \tau_2]$, then the following inequality for fractional integrals with exponential kernels*

holds:

$$\left|\chi(\tau_1, \tau_2, \mu, \eta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\left[\frac{\eta + e^{-\eta} - 1}{\eta}\right]^{1-\frac{1}{q}}\left\{\left(\Upsilon_1\left|\omega'(\tau_1)\right|^q + m\Upsilon_2\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}} + \left(\Upsilon_2\left|\omega'(\tau_1)\right|^q + m\Upsilon_1\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}}\right\},$$

where $m \in (0, 1]$, $\eta = \frac{1-\mu}{\mu}\frac{m\tau_2 - \tau_1}{2}$ *and*

$$\chi(\tau_1, m\tau_2, \mu, \eta) := \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{1-\mu}{2(1-e^{-\eta})}\left[I^{\mu}_{\tau_2^-}\omega\left(\frac{\tau_1 + m\tau_2}{2}\right) + I^{\mu}_{\tau_1^+}\omega\left(\frac{\tau_1 + m\tau_2}{2}\right)\right].$$

Proof. Applying Lemma 2, power-mean inequality and the exponential type m-convex of $|\omega'|^q$, we deduce

$$\left|\chi(\tau_1, \tau_2, \mu, \eta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\left[\int_0^1\left|e^{-\eta s} - 1\right|\left|\omega'\left(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\right)\right|ds\right.$$

$$\left. + \int_0^1\left|e^{-\eta s} - 1\right|\left|\omega'\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\right)\right|ds\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\left[\left(\int_0^1\left|e^{-\eta s} - 1\right|ds\right)^{1-\frac{1}{q}}\left(\int_0^1\left|e^{-\eta s} - 1\right|\left|\omega'\left(\frac{1-s}{2}\tau_1 + \frac{1+s}{2}m\tau_2\right)\right|^q ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1\left|e^{-\eta \tau} - 1\right|ds\right)^{1-\frac{1}{q}}\left(\int_0^1\left|e^{-\eta s} - 1\right|\left|\omega'\left(\frac{1+s}{2}\tau_1 + \frac{1-s}{2}m\tau_2\right)\right|^q ds\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\left[\frac{\eta + e^{-\eta} - 1}{\eta}\right]^{1-\frac{1}{q}}\left\{\left(\int_0^1\left|1 - e^{-\eta s}\right|\left[(e^{\frac{1-s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{1+s}{2}} - 1)\left|\omega'(\tau_2)\right|^q\right]ds\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_0^1\left|1 - e^{-\eta s}\right|\left[(e^{\frac{1+s}{2}} - 1)\left|\omega'(\tau_1)\right|^q + m(e^{\frac{1-s}{2}} - 1)\left|\omega'(\tau_2)\right|^q\right]ds\right)^{\frac{1}{q}}\right\}.$$

By calculation, we have

$$\Upsilon_1 := \int_0^1\left(e^{\frac{1-s}{2}} - 1\right)(1 - e^{-\eta s})ds = \frac{-\frac{4\eta^2}{\sqrt{e}} + 2\eta^2 + \eta + e^{-\eta} - 1}{\eta - 2\eta^2}$$

and

$$\Upsilon_2 := \int_0^1\left(e^{\frac{1+s}{2}} - 1\right)(1 - e^{-\eta s})ds = \frac{4\sqrt{e}\eta}{1 - 2\eta} + \frac{2e^{1-\eta}}{2\eta - 1} - \frac{e^{-\eta}}{\eta} + \frac{1}{\eta} + 2e - 1.$$

After suitable arrangements, we obtain

$$\left|\chi(\tau_1, \tau_2, \mu, \eta)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{4(1 - e^{-\eta})}\left[\frac{\eta + e^{-\eta} - 1}{\eta}\right]^{1-\frac{1}{q}}\left\{\left(\Upsilon_1\left|\omega'(\tau_1)\right|^q + m\Upsilon_2\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}} + \left(\Upsilon_2\left|\omega'(\tau_1)\right|^q + m\Upsilon_1\left|\omega'(\tau_2)\right|^q\right)^{\frac{1}{q}}\right\}.$$

The proof is completed.

5. Dragomir-Agarwal type inequalities

We now use the following lemma, which is presented in [24], to obtain some Dragomir-Agarwal type inequalities for exponential type m-convex mappings.

Lemma 3. *Assuming* $\omega : [\tau_1, m\tau_2] \to \mathbb{R}$ *is a differentiable mapping with* $0 \le \tau_1 < m\tau_2 < \infty$ *and* $0 < m \le 1$. *If* $\omega' \in L^1([\tau_1, m\tau_2])$, *then the following identity holds:*

$$\Omega(\tau_1, \tau_2, \delta, m) = \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})} \Big[\int_0^1 e^{-\delta s} \omega'(s\tau_1 + m(1 - s)\tau_2) \mathrm{d}s - \int_0^1 e^{-\delta(1-s)} \omega'(s\tau_1 + m(1 - \tau)\tau_2) \mathrm{d}s \Big],$$
(5.1)

where $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$ *and*

$$\Omega(\tau_1, \tau_2, \delta, m) := \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{(1 - \mu)}{2(1 - e^{-\delta})} \Big[I_{\tau_1^+}^\mu \omega(m\tau_2) + I_{m\tau_2^-}^\mu \omega(\tau_1) \Big].$$
(5.2)

Theorem 11. *Let* ω *be defined as in Lemma 5.1. If the function* $|\omega'|^q$ *for* $q \ge 1$ *is exponential type* m-convex *on* $[\tau_1, \tau_2]$ *with some fixed* $m \in (0, 1]$, *then the following inequality for fractional integrals with exponential kernels holds:*

$$\Big| \Omega(\gamma_1, \gamma_2, \eta, m) \Big| \le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})} \Big[\frac{e^{-\delta}(e^{\frac{\delta}{2}} - 1)^2}{\delta} \Big]^{1 - \frac{1}{q}} \Big[\Big(\frac{1}{1 + \delta} \Big[e + e^{-\delta} - 2e^{\frac{1-\delta}{2}} \Big] + \frac{2}{\delta} \Big[2e^{-\frac{\delta}{2}} - (1 + e^{-\delta}) \Big]$$

$$+ \frac{1}{1 - \delta} \Big[2e^{\frac{1-\delta}{2}} - (1 + e^{1-\delta}) \Big] \Big) \Big[\big| \omega'(\tau_1) \big|^q + m \big| \omega'(\tau_2) \big|^q \big] \Big]^{\frac{1}{q}},$$
(5.3)

where $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$ *and*

$$\Omega(\tau_1, \tau_2, \delta, m) := \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{(1 - \mu)}{2(1 - e^{-\delta})} \Big[I_{\tau_1^+}^\mu \omega(m\tau_2) + I_{m\tau_2^-}^\mu \omega(\tau_1) \Big].$$
(5.4)

Proof. Applying Lemma 1, power-mean inequality and the exponential type m-convexity of $|\omega'|^q$, we deduce

$$\Big| \Omega(\gamma_1, \gamma_2, \delta, m) \Big|$$

$$\le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})} \Big[\int_0^1 \Big| e^{-\delta s} \omega'(s\tau_1 + m(1 - s)\tau_2) - e^{-\delta(1-s)} \omega'(s\tau_1 + m(1 - s)\tau_2) \Big| \mathrm{d}s \Big]$$

$$\le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})} \Big[\int_0^1 \Big| e^{-\delta s} - e^{-\delta(1-s)} \Big| \Big| \omega'(s\tau_1 + m(1 - s)\tau_2) \Big| \mathrm{d}s \Big]$$

$$\le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})} \Big[\int_0^1 \Big| e^{-\delta s} - e^{-\delta(1-s)} \Big| \mathrm{d}s \Big]^{1 - \frac{1}{q}} \Big[\int_0^1 \Big| e^{-\delta s} - e^{-\delta(1-s)} \Big| \Big| \omega'(s\tau_1 + m(1 - s)\tau_2) \Big|^q \mathrm{d}s \Big]^{\frac{1}{q}}$$

$$\le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})} \Big[\frac{e^{-\delta}(e^{\frac{\delta}{2}} - 1)^2}{\delta} \Big]^{1 - \frac{1}{q}} \Big[\int_0^1 \Big| e^{-\delta s} - e^{-\delta(1-s)} \Big| \Big((e^s - 1) \big| \omega'(\tau_1) \big|^q + m(e^{1-s} - 1) \big| \omega'(\tau_2) \big|^q \Big) \mathrm{d}s \Big]^{\frac{1}{q}}.$$

By calculation, we have

$$\int_0^1 (e^s - 1)\left|e^{-\delta s} - e^{-\delta(1-s)}\right|\mathrm{d}s$$

$$= \frac{1}{1-\delta}\left[2e^{\frac{1-\delta}{2}} - (1 + e^{1-\delta})\right] + \frac{2}{\delta}\left[2e^{-\frac{\delta}{2}} - (1 + e^{-\delta})\right] + \frac{1}{1+\delta}\left[e + e^{-\delta} - 2e^{\frac{1-\delta}{2}}\right]$$

and

$$\int_0^1 m(e^{1-s} - 1)\left|e^{-\delta s} - e^{-\delta(1-s)}\right|\mathrm{d}s$$

$$= m\left(\frac{1}{1+\delta}\left[e + e^{-\delta} - 2e^{\frac{1-\delta}{2}}\right] + \frac{2}{\delta}\left[2e^{-\frac{\delta}{2}} - (1 + e^{-\delta})\right] + \frac{1}{1-\delta}\left[2e^{\frac{1-\delta}{2}} - (1 + e^{1-\delta})\right]\right).$$

After suitable arrangements, we obtain

$$\left|\Omega(\gamma_1, \gamma_2, \delta, m)\right| \le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\frac{e^{-\delta}(e^{\frac{\delta}{2}} - 1)^2}{\delta}\right]^{1-\frac{1}{q}}\left[\left(\frac{1}{1+\delta}\left[e + e^{-\delta} - 2e^{\frac{1-\delta}{2}}\right]\right.\right.$$

$$\left.\left. + \frac{2}{\delta}\left[2e^{-\frac{\delta}{2}} - (1 + e^{-\delta})\right] + \frac{1}{1-\delta}\left[2e^{\frac{1-\delta}{2}} - (1 + e^{1-\delta})\right]\right)\left[\left|\omega'(\tau_1)\right|^q + m\left|\omega'(\tau_2)\right|^q\right]\right]^{\frac{1}{q}}.$$

The proof is completed.

Theorem 12. *Let ω be defined as in Lemma 5.1. If the function $|\omega'|^q$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ is exponential type m-convex on $[\tau_1, \tau_2]$ for some fixed $m \in (0, 1]$, then we have the following inequality:*

$$\left|\Omega(\gamma_1, \gamma_2, \delta, m)\right| \le \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\frac{e^{-\delta p}(e^{\frac{\delta p}{2}} - 1)^2}{\delta p}\right]^{\frac{1}{p}}\left((e - 2)\left|\omega'(\tau_1)\right|^q + m(e - 2)\left|\omega'(\tau_2)\right|^q)\mathrm{d}s\right)^{\frac{1}{q}}, \qquad (5.5)$$

where $\delta = \frac{1-\mu}{\mu}(m\tau_2 - \tau_1)$ and

$$\Omega(\tau_1, \tau_2, \delta, m) := \frac{\omega(\tau_1) + \omega(m\tau_2)}{2} - \frac{(1 - \mu)}{2(1 - e^{-\delta})}\left[I_{\tau_1^+}^\mu \omega(m\tau_2) + I_{m\tau_2^-}^\mu \omega(\tau_1)\right]. \qquad (5.6)$$

Proof. Applying Lemma 1, Hölder inequality and the exponential type m-convexity of $|\omega'|^q$, we deduce

$$\left|\Omega(\gamma_1, \gamma_2, \delta, m)\right|$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\int_0^1 \left|e^{-\delta s}\omega'(s\tau_1 + m(1 - s)\tau_2) - e^{-\delta(1-s)}\omega'(s\tau_1 + m(1 - s)\tau_2)\right|\mathrm{d}s\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\int_0^1 \left|e^{-\delta s} - e^{-\delta(1-s)}\right|\left|\omega'(s\tau_1 + m(1 - s)\tau_2)\right|\mathrm{d}s\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\left(\int_0^1 \left|e^{-\delta s} - e^{-\delta(1-s)}\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^1 \left|\omega'(s\tau_1 + m(1 - s)\tau_2)\right|^q\mathrm{d}s\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\left(\int_0^1 \left|e^{-\delta s} - e^{-\delta(1-s)}\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^1 \left((e^s - 1)\left|\omega'(\tau_1)\right|^q + m(e^{1-s} - 1)\left|\omega'(\tau_2)\right|^q\right)\mathrm{d}s\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\left(\int_0^{\frac{1}{2}} \left(e^{-\delta s} - e^{-\delta(1-s)}\right)^p\mathrm{d}s + \int_{\frac{1}{2}}^1 \left(e^{-\delta(1-s)} - e^{-\delta s}\right)^p\mathrm{d}s\right)^{\frac{1}{p}}\right.$$

$$\left.\times \left((e - 2)\left|\omega'(\tau_1)\right|^q + m(e - 2)\left|\omega'(\tau_2)\right|^q\right)\mathrm{d}s\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\left(\int_0^{\frac{1}{2}} \left(e^{-\delta s} - e^{-\delta(1-s)}\right)^p\mathrm{d}s + \int_{\frac{1}{2}}^1 \left(e^{-\delta(1-s)} - e^{-\delta s}\right)^p\mathrm{d}s\right)^{\frac{1}{p}}\right.$$

$$\left.\times \left((e - 2)\left|\omega'(\tau_1)\right|^q + m(e - 2)\left|\omega'(\tau_2)\right|^q\right)\mathrm{d}s\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\left(\int_0^{\frac{1}{2}} \left(e^{-p\delta s} - e^{-p\delta(1-s)}\right)\mathrm{d}s + \int_{\frac{1}{2}}^1 \left(e^{-p\delta(1-s)} - e^{-p\delta s}\right)\mathrm{d}s\right)^{\frac{1}{p}}\right.$$

$$\left.\times \left((e - 2)\left|\omega'(\tau_1)\right|^q + m(e - 2)\left|\omega'(\tau_2)\right|^q\right)\mathrm{d}s\right)^{\frac{1}{q}}\right]$$

$$= \frac{m\tau_2 - \tau_1}{2(1 - e^{-\delta})}\left[\frac{e^{-\delta p}(e^{\frac{\delta p}{2}} - 1)^2}{\delta p}\right]^{\frac{1}{p}}\left((e - 2)\left|\omega'(\tau_1)\right|^q + m(e - 2)\left|\omega'(\tau_2)\right|^q\right)\mathrm{d}s\right)^{\frac{1}{q}}.$$

Here, we use the fact that $(x - y)^q \leq x^q - y^q$ for any $x \geq y \geq 0$ and $q \geq 1$. The proof is completed.

6. Conclusions

In this paper, some new fractional integrals with exponential kernels inequalities of Hadamard type, midpoint type, trapezoid type and Dragomir-Agarwal type for exponential type m-convex mappings are obtained. In view of this, we first present three right-side Hadamard inequalities for exponential type m-convex mappings by choosing different parameters in fractional integrals with exponential kernels. Then, we scale the three established equations by using the exponential type m-convexity and other deflation methods to obtain the boundary estimates of the midpoint-type, trapezoid-type and Dragomir-Agarwal-type inequalities separately. The results presented in this paper would provide generalizations of those given in earlier works. We hope that the equation we have established can help other scholars build new inequality and we will find out the application of our established inequality in other disciplines.

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Conflict of interest

The authors declare no conflict of interest.

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