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Research article

A Hardy-Hilbert-type inequality involving modified weight coefficients and partial sums

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Abstract: In this article, we construct proper weight coefficients and use them to establish a Hardy-Hilbert-type inequality involving one partial sum. Based on this inequality, the equivalent conditions of the best possible constant factor related to several parameters are discussed. We also consider the equivalent forms and the operator expressions of the obtained inequalities. At the end of the paper, we demonstrate that more new Hardy-Hilbert-type inequalities can be derived from the special cases of the present results.

Keywords: weight coefficient; Euler-Maclaurin summation formula; Hardy-Hilbert-type inequality; partial sums; applications
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1. Introduction

The classical Hardy-Hilbert's inequality asserted that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}(\sum_{m=1}^{\infty}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}b_n^q)^{\frac{1}{q}}, \tag{1}$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geq 0, 0<\sum_{m=1}^{\infty}a_m^p<\infty$ *and* $0<\sum_{n=1}^{\infty}b_n^q<\infty$, $\frac{\pi}{\sin(\pi/p)}$ is the best possible constant factor (cf. [1], Theorem 315).

A sharpened inequality of (1) was included in [1] by Theorem 323, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}(\sum_{m=1}^{\infty}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}b_n^q)^{\frac{1}{q}}. \tag{2}$$

In 2006, Krnić and J. Pečarić [2] provided an extension of (1) by introducing parameters

$\lambda_i \in (0,2]\ (i=1,2), \lambda_1+\lambda_2=\lambda \in (0,4]$, i.e.,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<B(\lambda_1,\lambda_2)[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}, \tag{3}$$

where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\ \ (u,v>0)$$

is the beta function. For $\lambda=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, inequality (3) reduces to (1); for $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, (3) reduces to a generalization of Hilbert's inequality which was proved by Yang in [3].

Recently, by the use of inequality (3), Adiyasuren et al. [4] gave a Hardy-Hilbert's inequality involving partial sums, as follows:

If $\lambda_i \in (0,1]\cap(0,\lambda)$ $(\lambda \in (0,2];\ i=1,2)$, $\lambda_1+\lambda_2=\lambda$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<\lambda_1\lambda_2 B(\lambda_1,\lambda_2)(\sum_{m=1}^{\infty}m^{-p\lambda_1-1}A_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q)^{\frac{1}{q}}, \tag{4}$$

where the constant factor $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible, and the partial sums $A_m:=\sum_{i=1}^{m}a_i$ and $B_n:=\sum_{k=1}^{n}b_k\ (m,n\in \mathrm{N}=\{1,2,\Lambda\})$ satisfy

$$0<\sum_{m=1}^{\infty}m^{-p\lambda_1-1}A_m^p<\infty \ \ and\ \ 0<\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q<\infty. \tag{5}$$

Inequalities (1), (2) and the integral analogues play an important role in analysis and applications (cf. [5–18]).

In 2016, by means of the techniques of real analysis, Hong et al. [19] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters.

Motivated by the inequalities (2) and (4), in this paper, we establish a new Hardy-Hilbert-type inequality, which contains modified weight coefficients and partial sums. The main technical approaches are the constructing of weight coefficients and the use of Hermite-Hadamard's inequality, Euler-Maclaurin summation formula and Abel's partial summation formula. Moreover, the equivalent conditions of the best possible constant factor related to several parameters are discussed. As applications, we deal with some equivalent forms, the operator expressions and some special cases about the inequality obtained in the main result.

2. Some lemmas

In what follows, we suppose that $p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda \in (0,2]$,

$$\lambda_1 \in (0,\tfrac{3}{2}]\cap(0,\lambda+1),\quad \lambda_2 \in (0,\tfrac{1}{2}]\cap(0,\lambda),$$

$$\hat{\lambda}_1 := \tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q},\ \hat{\lambda}_2 := \tfrac{\lambda-\lambda_1}{q}+\tfrac{\lambda_2}{p},$$

$$\eta_i \in [0,\tfrac{1}{4}]\ (i=1,2),\ \eta := \eta_1+\eta_2.$$

We also assume that for $a_m, b_n \geq 0\ (m,n \in \mathrm{N} := \{1,2,\Lambda\})$, the partial sums $B_n := \sum_{k=1}^{n} b_k$ satisfy $B_n = o(e^{t(n-\eta_2)})\ (t>0; n\to\infty)$, and

$$0<\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p<\infty,\ 0<\sum_{n=1}^{\infty}(n-\eta_2)^{-q\hat{\lambda}_2-1}B_n^q<\infty.$$

Lemma 1. (cf. [5], (2.2.3)) (i) If $(-1)^i \frac{d^i}{dt^i}g(t)>0,\ t\in[m,\infty)\ (m\in\mathrm{N})$ with $g^{(i)}(\infty)=0\ \ (i=0,1,2,3)$, $P_i(t),\ B_i(i\in N)$ are the Bernoulli functions and the Bernoulli numbers of i-order, then

$$\int_m^{\infty}P_{2q-1}(t)g(t)dt=-\varepsilon_q\frac{B_{2q}}{2q}g(m)\ \ (0<\varepsilon_q<1; q=1,2,\Lambda). \tag{6}$$

In particular, for $q=1$, in view of $B_2=\frac{1}{6}$, we have

$$-\tfrac{1}{12}g(m)<\int_m^{\infty}P_1(t)g(t)dt<0; \tag{7}$$

for $q=2$, in view of $B_4=-\frac{1}{30}$, we have

$$0<\int_m^{\infty}P_3(t)g(t)dt<\tfrac{1}{120}g(m). \tag{8}$$

(ii) (cf. [5], (2.3.2)) If $f(t)(>0)\in C^3[m,\infty),\ f^{(i)}(\infty)=0\ (i=0,1,2,3)$, then we have the following Euler-Maclaurin summation formula:

$$\sum_{k=m}^{\infty}f(k)=\int_m^{\infty}f(t)dt+\tfrac{1}{2}f(m)+\int_m^{\infty}P_1(t)f'(t)dt, \tag{9}$$

$$\int_m^{\infty}P_1(t)f'(t)dt=-\tfrac{1}{12}f'(m)+\tfrac{1}{6}\int_m^{\infty}P_3(t)f'''(t)dt. \tag{10}$$

Lemma 2. Let $s\in(0,3],\ s_i\in(0,\frac{3}{2}]\cap(0,s),\ k_s(s_i):=B(s_i,s-s_i)\ (i=1,2)$, we define the following weight coefficient:

$$\varpi(s_2,m):=(m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty}\frac{(n-\eta_2)^{s_2-1}}{(m+n-\eta)^s}\ \ (m\in \mathrm{N})\,. \tag{11}$$

Then we have the following inequalities:

$$0<k_s(s_2)(1-O_1(\tfrac{1}{(m-\eta_1)^{s_2}}))<\varpi(s_2,m)<k_s(s_2)\ \ (m\in \mathrm{N}). \tag{12}$$

Where

$$O_1\,(\tfrac{1}{(m-\eta_1)^{s_2}}):=\tfrac{1}{k_s(s_2)}\int_0^{\frac{1-\eta_2}{m-\eta_1}}\tfrac{u^{s_2-1}}{(1+u)^s}\,du>0.$$

Proof. For estimating the series (11), we set the following real function: For fixed $m\in \mathrm{N}$,

$$g(m,t):=\tfrac{(t-\eta_2)^{s_2-1}}{(m-\eta+t)^s}\ \ (t>\eta_2).$$

In the following we divide two cases of $s_2\in(0,1)\cap(0,s)$ and $s_2\in[1,\frac{3}{2}]\cap(0,s)$ to prove (12).

(i) For $s_2\in(0,1)\cap(0,s)$, since

$$(-1)^i g^{(i)}(m,t)>0\ \ (t>\eta_2;i=0,1,2),$$

by Hermite-Hadamard's inequality (cf. [20]), setting $u=\frac{t-\eta_2}{m-\eta_1}$, we have

$$\begin{aligned}\varpi(s_2,m)&=(m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty}g(m,n)<(m-\eta_1)^{s-s_2}\int_{\frac{1}{2}}^{\infty}g(m,t)dt\\&=(m-\eta_1)^{s-s_2}\int_{\frac{1}{2}}^{\infty}\frac{t^{s_2-1}}{(m-\eta_1+t-\eta_2)^s}dt=\int_{\frac{\frac{1}{2}-\eta_2}{m-\eta_1}}^{\infty}\frac{u^{s_2-1}}{(1+u)^s}du\\&\le\int_0^{\infty}\frac{u^{s_2-1}}{(1+u)^s}du=B(s_2,s-s_2)=k_\lambda(s_2).\end{aligned}$$

On the other hand, in view of the decreasingness property of series, setting $u=\frac{t-\eta_2}{m-\eta_1}$, we obtain

$$\begin{aligned}\varpi(s_2,m)&=(m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty}g(m,n)>(m-\eta_1)^{s-s_2}\int_1^{\infty}g(m,t)dt\\&=\int_{\frac{1-\eta_2}{m-\eta_1}}^{\infty}\frac{u^{s_2-1}}{(1+u)^s}du=B(s_2,s-s_2)-\int_0^{\frac{1-\eta_2}{m-\eta_1}}\frac{u^{s_2-1}}{(1+u)^s}du\\&=k_s(s_2)(1-O_1(\tfrac{1}{(m-\eta_1)^{s_2}}))>0,\end{aligned}$$

where

$$O_1(\tfrac{1}{(m-\eta_1)^{s_2}})=\tfrac{1}{k_s(s_2)}\int_0^{\frac{1-\eta_2}{m-\eta_1}}\tfrac{u^{s_2-1}}{(1+u)^s}du>0,$$

satisfying

$$0<\int_0^{\frac{1-\eta_2}{m-\eta_1}}\tfrac{u^{s_2-1}}{(1+u)^s}du<\int_0^{\frac{1-\eta_2}{m-\eta_1}}u^{s_2-1}du=\tfrac{1}{s_2}(\tfrac{1-\eta_2}{m-\eta_1})^{s_2}\ \ (m\in \mathrm{N}).$$

Hence, we obtain (12).

(ii) For $s_2 \in [1, \frac{3}{2}] \cap (0, s)$, by (9), we have

$$\sum_{n=1}^{\infty} g(m,n) = \int_1^{\infty} g(m,t)dt + \tfrac{1}{2} g(m,1) + \int_1^{\infty} P_1(t) g'(m,t)dt$$
$$= \int_{\eta_2}^{\infty} g(m,t)dt - h(m),$$

where, $h(m)$ is indicated as

$$h(m) := \int_{\eta_2}^{1} g(m,t)dt - \tfrac{1}{2} g(m,1) - \int_1^{\infty} P_1(t) g'(m,t)dt.$$

We obtain $-\frac{1}{2} g(m,1) = \frac{-(1-\eta_2)^{s_2-1}}{2(m-\eta+1)^s}$, and integrating by parts, it follows that

$$\int_{\eta_2}^{1} g(m,t)dt = \int_{\eta_2}^{1} \frac{(t-\eta_2)^{s_2-1}}{(m-\eta+t)^s} dt = \frac{1}{s_2} \int_{\eta_2}^{1} \frac{d(t-\eta_2)^{s_2}}{(m-\eta+t)^s}$$
$$= \frac{1}{s_2} \frac{(t-\eta_2)^{s_2}}{(m-\eta+t)^s} \Big|_{\eta_2}^{1} + \frac{s}{s_2} \int_{\eta_2}^{1} \frac{(t-\eta_2)^{s_2} dt}{(m-\eta+t)^{s+1}} = \frac{1}{s_2} \frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)} \int_{\eta_2}^{1} \frac{d(t-\eta_2)^{s_2+1}}{(m-\eta+t)^{s+1}}$$
$$> \frac{1}{s_2} \frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)} \left[\frac{(t-\eta_2)^{s_2+1}}{(m-\eta+t)^{s+1}}\right]_{\eta_2}^{1}$$
$$+ \frac{s(s+1)}{s_2(s_2+1)(m-\eta+1)^{s+2}} \int_{\eta_2}^{1} (t-\eta_2)^{s_2+1} dt$$
$$= \frac{1}{s_2} \frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)} \frac{(1-\eta_2)^{s_2+1}}{(m-\eta+1)^{s+1}} + \frac{s(s+1)(1-\eta_2)^{s_2+2}}{s_2(s_2+1)(s_2+2)(m-\eta+1)^{s+2}}.$$

We find

$$-g'(m,t) = -\frac{(s_2-1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-1}}{(m-\eta+t)^{s+1}}$$
$$= \frac{(1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-2}[(m-\eta+t)-(m-\eta_1)]}{(m-\eta+t)^{s+1}}$$
$$= \frac{(1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} - \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}$$
$$= \frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} - \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}},$$

and for $s_2 \in [1, \frac{3}{2}] \cap (0, s)$, it follows that

$$(-1)^i \frac{d^i}{dt^i}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}\right] > 0, (-1)^i \frac{d^i}{dt^i}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\right] > 0 \ \ (t > \eta_2; i = 0,1,2,3).$$

By (8)–(10), for $a := 1 - \eta_2 (\in [\frac{3}{4}, 1])$, we obtain

$$(s+1-s_2) \int_1^{\infty} P_1(t) \frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} dt > -\frac{s+1-s_2}{12(m-\eta+1)^s} a^{s_2-2},$$

$$-(m-\eta_1)s\int_1^\infty P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}dt > \frac{(m-\eta_1)sa^{s_2-2}}{12(m-\eta+1)^{s+1}} - \frac{(m-\eta_1)s}{720}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\right]''_{t=1}$$

$$> \frac{(m-\eta+1)s-as}{12(m-\eta+1)^{s+1}}a^{s_2-2}$$

$$-\frac{(m-\eta+1)s}{720}\left[\frac{(s+1)(s+2)}{(m-\eta+1)^{s+3}}a^{s_2-2}+\frac{2(s+1)(2-s_2)}{(m-\eta+1)^{s+2}}a^{s_2-3}+\frac{(2-s_2)(3-s_2)}{(m-\eta+1)^{s+1}}a^{s_2-4}\right]$$

$$= \frac{sa^{s_2-2}}{12(m-\eta+1)^{s}} - \frac{sa^{s_2-1}}{12(m-\eta+1)^{s+1}}$$

$$-\frac{s}{720}\left[\frac{(s+1)(s+2)}{(m-\eta+1)^{s+2}}a^{s_2-2}+\frac{2(s+1)(2-s_2)}{(m-\eta+1)^{s+1}}a^{s_2-3}+\frac{(2-s_2)(3-s_2)}{(m-\eta+1)^{s}}a^{s_2-4}\right],$$

and then we have

$$h(m) > \frac{a^{s_2-4}}{(m-\eta+1)^s}h_1 + \frac{sa^{s_2-3}}{(m-\eta+1)^{s+1}}h_2 + \frac{s(s+1)a^{s_2-2}}{(m-\eta+1)^{s+2}}h_3,$$

where, h_i $(i=1,2,3)$ are indicated as

$$h_1 := \frac{a^4}{s_2} - \frac{a^3}{2} - \frac{(1-s_2)a^2}{12} - \frac{s(2-s_2)(3-s_2)}{720},$$
$$h_2 := \frac{a^4}{s_2(s_2+1)} - \frac{a^2}{12} - \frac{(s+1)(2-s_2)}{360}, \ \ and$$

$$h_3 := \frac{a^4}{s_2(s_2+1)(s_2+2)} - \frac{s+2}{720}.$$

For $s\in(0,3], s_2\in[1,\frac{3}{2}]\cap(0,s)$, $a\in[\frac{3}{4},1]$, we find

$$h_1 > \frac{a^2}{12s_2}[s_2^2-(6a+1)s_2+12a^2]-\frac{1}{120}.$$

In view of

$$\frac{d}{da}[s_2^2-(6a+1)s_2+12a^2] = 6(4a-s_2) \ge 6(4\cdot\tfrac{3}{4}-\tfrac{3}{2}) > 0,$$

and

$$\frac{d}{ds_2}[s_2^2-(6a+1)s_2+12a^2] = 2s_2-(6a+1) \le 2\cdot\tfrac{3}{2}-(6\cdot\tfrac{3}{4}+1) = 3-\tfrac{11}{2} < 0,$$

we obtain

$$h_1 \ge \frac{(3/4)^2}{12(3/2)}[(\tfrac{3}{2})^2-(6\cdot\tfrac{3}{4}+1)\tfrac{3}{2}+12(\tfrac{3}{4})^2]-\tfrac{1}{120} = \tfrac{3}{128}-\tfrac{1}{120} > 0,$$

$$h_2 > a^2(\tfrac{4a^2}{15}-\tfrac{1}{12})-\tfrac{1}{90} \ge (\tfrac{3}{4})^2[\tfrac{4}{15}(\tfrac{3}{4})^2-\tfrac{1}{12}]-\tfrac{1}{90} = \tfrac{3}{80}-\tfrac{1}{90} > 0,$$

$$h_3 \ge \tfrac{8a^4}{105}-\tfrac{5}{720} \ge \tfrac{8}{105}(\tfrac{3}{4})^4-\tfrac{1}{144} = \tfrac{27}{1120}-\tfrac{1}{144} > 0,$$

and then $h(m) > 0$.

On the other hand, we also have

$$\sum_{n=1}^\infty g(m,n) = \int_1^\infty g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)dt$$
$$= \int_1^\infty g(m,t)dt + H(m),$$

where, $H(m)$ is indicated as

$$H(m) := \frac{1}{2} g(m,1) + \int_1^\infty P_1(t) g'(m,t) dt.$$

We have obtained that

$$\frac{1}{2} g(m,1) = \frac{a^{s_2-1}}{2(m-\eta+1)^s}$$

and

$$g'(m,t) = -\frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}.$$

For $s_2 \in \left[1, \frac{3}{2}\right] \cap (0,s), 0 < s \leq 3$, by (7), we obtain

$$-(s+1-s_2)\int_1^\infty P_1(t) \frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} dt > 0,$$

$$(m-\eta_1)s\int_1^\infty P_1(t) \frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}} dt > \frac{-(m-\eta_1)s}{12(m-\eta+1)^{s+1}} a^{s_2-2} = \frac{-(m-\eta+1)s+as}{12(m-\eta+1)^{s+1}} a^{s_2-2}$$
$$= \frac{-s}{12(m-\eta+1)^s} a^{s_2-2} + \frac{s}{12(m-\eta+1)^{s+1}} a^{s_2-1} > \frac{-s}{12(m-\eta+1)^s} a^{s_2-2}.$$

Hence, we have

$$H(m) > \frac{a^{s_2-1}}{2(m-\eta+1)^s} - \frac{sa^{s_2-2}}{12(m-\eta+1)^s} = \left(\frac{a}{2} - \frac{s}{12}\right)\frac{a^{s_2-2}}{(m-\eta+1)^s}$$
$$\geq \left(\frac{1}{2}\cdot\frac{3}{4} - \frac{3}{12}\right)\frac{a^{s_2-2}}{(m-\eta+1)^s} = \left(\frac{3}{8} - \frac{3}{12}\right)\frac{a^{s_2-2}}{(m-\eta+1)^s} > 0.$$

Therefore, we obtain the following inequalities:

$$\int_1^\infty g(m,t)dt < \sum_{n=1}^\infty g(m,n) < \int_{\eta_2}^\infty g(m,t)dt.$$

In view of the the results in case (i), we have

$$\varpi(s_2,m) > (m-\eta_1)^{s-s_2}\int_1^\infty g(m,t)dt = k_s(s_2)\left(1 - O_1\left(\frac{1}{(m-\eta_1)^{s_2}}\right)\right) > 0,$$

$$\varpi(s_2,m) < (m-\eta_1)^{s-s_2}\int_{\eta_2}^\infty g(m,t)dt \leq B(s_2, s-s_2) = k_\lambda(s_2).$$

Hence, we obtain (12). The Lemma 2 is proved.

Lemma 3. Let $s \in (0,3]$, $s_i \in \left(0, \frac{3}{2}\right] \cap (0,s) (i = 1,2)$. Then we have the following Hardy-Hilbert-type inequality:

$$I = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n-\eta)^s} \leq (k_s(s_2))^{\frac{1}{p}} (k_s(s_1))^{\frac{1}{q}}$$
$$\times\left\{\sum_{m=1}^\infty (m-\eta_1)^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty (n-\eta_2)^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}. \quad (13)$$

Proof. In the same way as the proof of inequality (12) along with the symmetry of the parameters

s_1, n, η_2 and $s_2,\ m,\ \eta_1$, we obtain the following inequalities for the next weight coefficient:

$$0 < k_s(s_1)(1 - O_2(\tfrac{1}{(n-\eta_2)^{s_1}}))$$

$$< \omega(s_1, n) := (n-\eta_2)^{s-s_1} \sum_{m=1}^{\infty} \tfrac{(m-\eta_1)^{s_1-1}}{(m+n-\eta)^s} < k_s(s_1) \ \ (n \in \mathrm{N}), \tag{14}$$

where

$$O_2(\tfrac{1}{(n-\eta_2)^{s_1}}) := \tfrac{1}{k_s(s_1)} \int_0^{\frac{1-\eta_1}{n-\eta_2}} \tfrac{u^{s_1-1}}{(1+u)^s} du > 0.$$

By Hölder's inequality (cf. [20]), we obtain

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tfrac{1}{(m+n-\eta)^s} [\tfrac{(m-\eta_1)^{(1-s_1)/q}}{(n-\eta_2)^{(1-s_2)/p}} a_m][\tfrac{(n-\eta_2)^{(1-s_2)/p}}{(m-\eta_1)^{(1-s_1)/q}} b_n]$$

$$\le [\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tfrac{1}{(m+n-\eta)^s} \tfrac{(m-\eta_1)^{(1-s_1)(p-1)} a_m^p}{(n-\eta_2)^{1-s_2}}]^{\frac{1}{p}} [\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tfrac{1}{(m+n-\eta)^s} \tfrac{(n-\eta_2)^{(1-s_2)(q-1)} b_n^q}{(m-\eta_1)^{1-s_1}}]^{\frac{1}{q}}$$

$$= \{\sum_{m=1}^{\infty} \varpi(s_2, m)(m-\eta_1)^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})]-1} a_m^p\}^{\frac{1}{p}}$$

$$\times \{\sum_{n=1}^{\infty} \omega(s_1, n)(n-\eta_2)^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})]-1} b_n^q\}^{\frac{1}{q}}.$$

Then by (12) and (14), we obtain (13). The proof Lemma 3 is complete.

Remark 1. In particular, for

$$s = \lambda + 1 \in (1,3], s_1 = \lambda_1 \in \left(0, \frac{3}{2}\right] \cap (0, \lambda + 1), s_2 = \lambda_2 + 1 \in \left(1, \frac{3}{2}\right] \cap (1, \lambda + 1),$$

in (13), then

$$\lambda \in (0,2], \lambda_1 \in (0, \tfrac{3}{2}] \cap (0, \lambda + 1), \lambda_2 \in (0, \tfrac{1}{2}] \cap (0, \lambda),$$

replacing b_n by B_n, in view of the assumptions of a_m *and* B_n, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tfrac{a_m B_n}{(m+n-\eta)^{\lambda+1}} < (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$

$$\times [\sum_{m=1}^{\infty} (m-\eta_1)^{p(1-\hat{\lambda}_1)-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} (n-\eta_2)^{-q\hat{\lambda}_2-1} B_n^q]^{\frac{1}{q}}. \tag{15}$$

Lemma 4. If $t > 0$, then the following inequality holds

$$\sum_{n=1}^{\infty} e^{-t(n-\eta_2)} b_n \le t \sum_{n=1}^{\infty} e^{-t(n-\eta_2)} B_n. \tag{16}$$

Proof. In view of $B_n e^{-t(n-\eta_2)} = o(1)\ (n \to \infty)$, by Abel's summation by parts formula, we find

$$\sum_{n=1}^{\infty} e^{-t(n-\eta_2)} b_n = \lim_{n\to\infty} B_n e^{-t(n-\eta_2)} + \sum_{n=1}^{\infty} B_n [e^{-t(n-\eta_2)} - e^{-t(n-\eta_2+1)}]$$

$$= \sum_{n=1}^{\infty} B_n [e^{-t(n-\eta_2)} - e^{-t(n-\eta_2+1)}] = (1-e^{-t}) \sum_{n=1}^{\infty} e^{-t(n-\eta_2)} B_n. \quad (17)$$

Since $1-e^{-t} < t \ (t>0)$, by (17), we have inequality

$$\sum_{n=1}^{\infty} e^{-t(n-\eta_2)} b_n \le t \sum_{n=1}^{\infty} e^{-t(n-\eta_2)} B_n,$$

namely, (16) follows. The Lemma 4 is proved.

3. Main results

Theorem 1. Let $p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda \in (0,2],\ \lambda_1 \in (0,\frac{3}{2}] \cap (0,\lambda+1),\ \lambda_2 \in (0,\frac{1}{2}] \cap (0,\lambda),\ \hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q},$ $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p},\ \eta_i \in [0,\frac{1}{4}]\ (i=1,2),\ \eta = \eta_1 + \eta_2,\ B_n = \sum_{k=1}^{n} b_k,\ B_n = o(e^{t(n-\eta_2)})\ (t>0; n\to\infty),$ and let

$$0 < \sum_{m=1}^{\infty} (m-\eta_1)^{p(1-\hat{\lambda}_1)-1} a_m^p < \infty,\ 0 < \sum_{n=1}^{\infty} (n-\eta_2)^{-q\hat{\lambda}_2-1} B_n^q < \infty.$$

Then, we have the following Hardy-Hilbert-type inequality:

$$I := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}} < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$

$$\times [\sum_{m=1}^{\infty} (m-\eta_1)^{p(1-\hat{\lambda}_1)-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} (n-\eta_2)^{-q\hat{\lambda}_2-1} B_n^q]^{\frac{1}{q}}. \quad (18)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$ ($\lambda_1 \in (0,\frac{3}{2}] \cap (0,\lambda), \lambda_2 \in (0,\frac{1}{2}] \cap (0,\lambda)$), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}} < \lambda_2 B(\lambda_1,\lambda_2) [\sum_{m=1}^{\infty} (m-\eta_1)^{p(1-\lambda_1)-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} (n-\eta_2)^{-q\lambda_2-1} B_n^q]^{\frac{1}{q}}. \quad (19)$$

Proof. In view of the formula that

$$\frac{1}{(m+n-\eta)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-(m+n-\eta)t} dt,$$

by (16), it follows that

$$I = \frac{1}{\Gamma(\lambda)} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n \int_0^{\infty} t^{\lambda-1} e^{-(m+n-\eta)t} dt$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} \sum_{m=1}^{\infty} e^{-(m-\eta_1)t} a_m \sum_{n=1}^{\infty} e^{-(n-\eta_2)t} b_n dt$$

$$\le \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{1} \sum_{m=1}^{\infty} e^{-(m-\eta_1)t} a_m \sum_{n=1}^{\infty} e^{-(n-\eta_2)t} B_n dt$$

$$= \frac{1}{\Gamma(\lambda)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m B_n \int_0^{\infty} t^{(\lambda+1)-1} e^{-(m+n-\eta)t} dt$$
$$= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m B_n}{(m+n-\eta)^{\lambda+1}}.$$

Then by (15), we have (18).

For $\lambda_1 + \lambda_2 = \lambda \ (\in (0,2]) \ (\lambda_1 \in (0, \frac{3}{2}] \cap (0,\lambda), \lambda_2 \in (0, \frac{1}{2}] \cap (0,\lambda))$, we have

$$k_{\lambda+1}(\lambda_2 + 1) = k_{\lambda+1}(\lambda_1) = B(\lambda_1, \lambda_2 + 1)$$
$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2+1)}{\Gamma(\lambda+1)} = \frac{\lambda_2 \Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda+1)} = \frac{\lambda_2 \Gamma(\lambda)}{\Gamma(\lambda+1)} B(\lambda_1, \lambda_2),$$

inequality (18) reduces to (19). The Theorem 1 is proved.

Theorem 2. Suppose that $\lambda \in (0,2], \lambda_1 \in (0, \frac{3}{2}] \cap (0,\lambda), \lambda_2 \in (0, \frac{1}{2}] \cap (0,\lambda)$. If $\lambda_1 + \lambda_2 = \lambda$, then the constant factor

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$

in (18) is the best possible. On the other hand, if the same constant factor in (18) is the best possible, then for $\lambda - \lambda_1 \le \frac{1}{2}, \lambda - \lambda_2 \le \frac{3}{2}$, we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. We now prove that the constant factor $\lambda_2 B(\lambda_1, \lambda_2)$ in (19) is the best possible. For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set

$$\tilde{a}_m := m^{\lambda_1 - \frac{\varepsilon}{p} - 1}, \tilde{b}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \ (m, n \in \mathrm{N}).$$

Since for $\lambda_2 \le \frac{1}{2}$, $g(t) := t^{\lambda_2 - \frac{\varepsilon}{q} - 1}$ is strictly decreasing with respect to $t > 0$, by the decreasingness property of series, we have

$$\tilde{B}_n := \sum_{k=1}^{n} \tilde{b}_k = \sum_{k=1}^{n} k^{\lambda_2 - \frac{\varepsilon}{q} - 1} < \int_0^n t^{\lambda_2 - \frac{\varepsilon}{q} - 1} dt = \frac{1}{\lambda_2 - \frac{\varepsilon}{q}} n^{\lambda_2 - \frac{\varepsilon}{q}}.$$

If there exists a positive constant $M \le \lambda_2 B(\lambda_1, \lambda_2)$, such that (19) is valid when we replace $\lambda_2 B(\lambda_1, \lambda_2)$ by M, then in particular, for $\eta_i = \eta = 0 \ (i = 1,2)$, substitution of $a_m = \tilde{a}_m$, $b_n = \tilde{b}_n$ and $B_n = \tilde{B}_n$ in (19), we have

$$\tilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\lambda}} \tilde{a}_m \tilde{b}_n < M [\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} \tilde{a}_m^p]^{\frac{1}{p}} (\sum_{n=1}^{\infty} n^{-q\lambda_2 - 1} \tilde{B}_n^q)^{\frac{1}{q}}. \quad (20)$$

In the following, we obtain that $M \ge \lambda_2 B(\lambda_1, \lambda_2)$, which follows that $M = \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor in (19).

By (20) and the decreasingness property of series, we obtain

$$\widetilde{I} < M(\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} m^{p\lambda_1 - p-\varepsilon})^{\frac{1}{p}} \frac{1}{\lambda_2 - \frac{\varepsilon}{q}} (\sum_{n=1}^{\infty} n^{-q\lambda_2 -1} n^{q\lambda_2 - \varepsilon})^{\frac{1}{q}}$$

$$= \frac{M}{\lambda_2 - \frac{\varepsilon}{q}} (1 + \sum_{m=2}^{\infty} m^{-\varepsilon-1})$$

$$< \frac{M}{\lambda_2 - \frac{\varepsilon}{q}} (1 + \int_1^{\infty} x^{-\varepsilon-1} dx) = \frac{M}{\varepsilon} (\frac{1}{\lambda_2 - \frac{\varepsilon}{q}})(\varepsilon + 1).$$

By (14), for $\eta_i = \eta = 0$ $(i = 1,2)$, $s = \lambda$, $s_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0, \frac{3}{2}) \cap (0, \lambda))$, $s_2 = \lambda_2 + \frac{\varepsilon}{p}$ $(\in (0, \lambda))$, we have

$$\widetilde{I} = \sum_{n=1}^{\infty} [n^{(\lambda_2 + \frac{\varepsilon}{p})} \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\lambda}} m^{(\lambda_1 - \frac{\varepsilon}{p})-1}] n^{-\varepsilon-1}$$

$$= \sum_{n=1}^{\infty} \omega(\lambda_1 - \frac{\varepsilon}{p}, n) n^{-\varepsilon-1} > k_{\lambda}(\lambda_1 - \frac{\varepsilon}{p}) \sum_{n=1}^{\infty} [1 - O_2(\frac{1}{n^{\lambda_1 - \frac{\varepsilon}{p}}})] n^{-\varepsilon-1}$$

$$> k_{\lambda}(\lambda_1 - \frac{\varepsilon}{p}) [\int_1^{\infty} y^{-\varepsilon-1} dy - \sum_{n=1}^{\infty} O_2(\frac{1}{n^{\lambda_1 + \frac{\varepsilon}{q} + 1}})]$$

$$= \frac{1}{\varepsilon} B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p})(1 - \varepsilon O(1)).$$

By (20) and the above results, we have

$$B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p})(1 - \varepsilon O(1)) < \varepsilon \widetilde{I} < M(\frac{1}{\lambda_2 - \frac{\varepsilon}{q}})(\varepsilon + 1).$$

Setting $\varepsilon \to 0^+$, in view of the continuity of the beta function, we find $\lambda_2 B(\lambda_1, \lambda_2) \le M$. Hence, $M = \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor in (19).

On the other hand, for $\hat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, and $\lambda - \lambda_2 \le \frac{3}{2}, \lambda - \lambda_1 \le \frac{1}{2}$, we find

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p} = \lambda,$$

$$0 < \hat{\lambda}_1, \hat{\lambda}_2 < \lambda, \hat{\lambda}_1 \le \frac{3}{2p} + \frac{3}{2q} = \frac{3}{2}, \hat{\lambda}_2 \le \frac{1}{2},$$

and $\hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) \in \mathrm{R}_+ = (0, \infty)$. By (19), we still have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}} < \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) [\sum_{m=1}^{\infty} (m - \eta_1)^{p(1-\hat{\lambda}_1)-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} (n - \eta_2)^{-q\hat{\lambda}_2 - 1} B_n^q]^{\frac{1}{q}}. \tag{21}$$

If the constant factor $\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$ in (18) is the best possible, then for any M, when we replace $\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$ by M, we have

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \le M,$$

and then by (21), we have the following inequality:

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \le \hat{\lambda}_2 B(\hat{\lambda}_1, \hat{\lambda}_2) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} k_{\lambda+1}(\hat{\lambda}_1).$$

It follows that

$$k_{\lambda+1}(\hat{\lambda}_1) \geq (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}.$$

By using Hölder's inequality (cf. [20]), we obtain

$$\begin{aligned} k_{\lambda+1}(\hat{\lambda}_1) &= k_{\lambda+1}(\tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q}) \\ &= \int_0^\infty \tfrac{1}{(1+u)^{\lambda+1}} u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1} du = \int_0^\infty \tfrac{1}{(1+u)^{\lambda+1}} (u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}}) du \\ &\leq [\int_0^\infty \tfrac{1}{(1+u)^{\lambda+1}} u^{\lambda-\lambda_2-1} du]^{\frac{1}{p}} [\int_0^\infty \tfrac{1}{(1+u)^{\lambda+1}} u^{\lambda_1-1} du]^{\frac{1}{q}} \\ &= [\int_0^\infty \tfrac{1}{(1+v)^{\lambda+1}} v^{(\lambda_2+1)-1} dv]^{\frac{1}{p}} [\int_0^\infty \tfrac{1}{(1+u)^{\lambda+1}} u^{\lambda_1-1} du]^{\frac{1}{q}} \\ &= (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}. \end{aligned} \tag{22}$$

Then we have

$$k_{\lambda+1}(\hat{\lambda}_1) = (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}},$$

namely, (22) keeps the form of equality.

We observe that (22) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero satisfying (cf. [20])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \ a.e. \quad \text{in} \quad \mathrm{R}_+.$$

Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ $a.e.$ in R_+, and then $\lambda-\lambda_2-\lambda_1=0$, namely, $\lambda_1+\lambda_2=\lambda$. This completes the proof of Theorem 2.

4. Equivalent forms and operator expressions

Theorem 3. We have the following inequality equivalent to (18):

$$\begin{aligned} J &:= \left\{ \sum_{m=1}^\infty (m-\eta_1)^{q\hat{\lambda}_1-1} \left[\sum_{n=1}^\infty \tfrac{b_n}{(m+n-\eta)^\lambda} \right]^q \right\}^{\frac{1}{q}} \\ &< \tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} [\sum_{n=1}^\infty (n-\eta_2)^{-q\hat{\lambda}_2-1} B_n^q]^{\frac{1}{q}}. \end{aligned} \tag{23}$$

In particular, for $\lambda_1+\lambda_2=\lambda$ $(\lambda_1 \in (0,\frac{3}{2}] \cap (0,\lambda),\ \lambda_2 \in (0,\frac{1}{2}] \cap (0,\lambda))$, we have the following inequality equivalent to (19):

$$\left\{ \sum_{m=1}^\infty (m-\eta_1)^{q\lambda_1-1} \left[\sum_{n=1}^\infty \tfrac{b_n}{(m+n-\eta)^\lambda} \right]^q \right\}^{\frac{1}{q}} < \lambda_2 B(\lambda_1,\lambda_2) [\sum_{n=1}^\infty (n-\eta_2)^{-q\lambda_2-1} B_n^q]^{\frac{1}{q}}. \tag{24}$$

Proof. Suppose that (23) is valid. By using Hölder's inequality (cf. [20]), we have

$$I=\sum_{m=1}^{\infty}\left[(m-\eta_1)^{\frac{1}{q}-\hat{\lambda}_1}a_m\right]\left[(m-\eta_1)^{\frac{-1}{q}+\hat{\lambda}_1}\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\eta)^{\lambda}}\right]$$

$$\leq\left[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}J. \tag{25}$$

Then by (23), we have (18). On the other hand, assuming that (18) is valid, we set

$$a_m:=(m-\eta_1)^{q\hat{\lambda}_1-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m+n-\eta)^{\lambda}}\right]^{q-1},m\in\mathrm{N}.$$

Then, it follows that $J=\left[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{q}}$.

If $J=0$, then (23) is naturally valid; if $J=\infty$, then it is impossible that makes (23) valid, namely, $J<\infty$. Suppose that $0<J<\infty$. By (18), we have

$$\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p=J^q=I$$

$$<\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}J^{q-1}[\sum_{n=1}^{\infty}(n-\eta_2)^{-q\hat{\lambda}_2-1}B_n^q]^{\frac{1}{q}},$$

$$J<\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}[\sum_{n=1}^{\infty}(n-\eta_2)^{-q\hat{\lambda}_2-1]-1}B_n^q]^{\frac{1}{q}},$$

namely, (23) follows, which is equivalent to (18). The Theorem 3 is proved.

Theorem 4. Suppose that $\lambda\in(0,2],\lambda_1\in(0,\frac{3}{2}]\cap(0,\lambda),\lambda_2\in(0,\frac{1}{2}]\cap(0,\lambda)$. If $\lambda_1+\lambda_2=\lambda$, then the constant factor $\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$ in (23) is the best possible; On the other hand, if the same constant factor in (23) is the best possible, then for $\lambda-\lambda_2\leq\frac{3}{2},\lambda-\lambda_1\leq\frac{1}{2}$, we have $\lambda_1+\lambda_2=\lambda$.

Proof. If $\lambda_1+\lambda_2=\lambda$, then by Theorem 2, the constant factor

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}(k_{\lambda+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$$

in (18) is the best possible. Then by (25), the constant factor in (23) is the best possible.

On the other hand, if the same constant factor in (23) is the best possible, then by the equivalency of (23) and (18), in view of $J^q=I$ (in the proof of Theorem 3), it follows that the same constant factor in (18) is the best possible. By Theorem 2, in view of $\lambda-\lambda_2\leq\frac{3}{2}$, $\lambda-\lambda_1\leq\frac{1}{2}$, we have $\lambda_1+\lambda_2=\lambda$. The theorem is proved.

Setting

$$\varphi(m):=(m-\eta_1)^{p(1-\hat{\lambda}_1)-1},\psi(n):=(n-\eta_2)^{q(1-\hat{\lambda}_2)-1},\Psi(n):=(n-\eta_2)^{-q\hat{\lambda}_2-1},$$

where from,

$$\varphi^{1-q}(m)=(m-\eta_1)^{q\hat{\lambda}_1-1}\ (\mathrm{m},n\in\mathrm{N}),$$

we define the following normed linear spaces:

$$l_{p,\varphi} := \{a = \{a_m\}_{m=1}^{\infty}; \| a \|_{p,\varphi} := (\sum_{m=1}^{\infty} \varphi(m) | a_m |^p)^{\frac{1}{p}} < \infty\},$$

$$l_{q,\psi} := \{b = \{b_n\}_{n=1}^{\infty}; \| b \|_{q,\psi} := (\sum_{n=1}^{\infty} \psi(n) | b_n |^q)^{\frac{1}{q}} < \infty\},$$

$$l_{q,\Psi} := \{B = \{B_n\}_{n=1}^{\infty}; \| B \|_{q,\Psi} := (\sum_{n=1}^{\infty} \Psi(n) | B_n |^q)^{\frac{1}{q}} < \infty\},$$

$$l_{q,\varphi^{1-q}} := \{c = \{c_m\}_{m=1}^{\infty}; \| c \|_{q,\varphi^{1-q}} := (\sum_{m=1}^{\infty} \varphi^{1-q}(m) | c_m |^q)^{\frac{1}{q}} < \infty\}.$$

For $b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi}$, setting $c = \{c_m\}_{m=1}^{\infty} : c_m := \sum_{n=1}^{\infty} \frac{b_n}{(m+n-\eta)^{\lambda}}$, we can rewrite (23) as follows:

$$\| c \|_{q,\varphi^{1-q}} < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \| B \|_{q,\Psi} < \infty,$$

namely, $c \in l_{q,\varphi^{1-q}}$. The proof of Theorem 4 is complete.

Definition 1. Define a more accurate Hardy-Hilbert's operator $T : l_{q,\psi} \to l_{q,\varphi^{1-q}}$ as follows: For any $b \in l_{q,\psi}$, there exists a unique representation $c = Tb \in l_{q,\varphi^{1-q}}$, such that for any $m \in \mathrm{N}$, $Tb(m) = c_m$. Define the formal inner product of Tb and $a \in l_{p,\varphi}$ and the norm of T as follows:

$$(Tb, a) := \sum_{m=1}^{\infty} a_m \sum_{n=1}^{\infty} \frac{b_n}{(m+n-\eta)^{\lambda}},$$

$$\| T \| := \sup_{b(\neq 0) \in l_{q,\psi}} \frac{\|Tb\|_{q,\varphi^{1-q}}}{\|B\|_{q,\Psi}}.$$

By Theorems 2–4, we have

Theorem 5. Suppose that

$$\lambda \in (0,2], \lambda_1 \in (0, \tfrac{3}{2}] \cap (0, \lambda), \lambda_2 \in (0, \tfrac{1}{2}] \cap (0, \lambda).$$

If

$$b \ (\geq 0) \in l_{q,\psi}, B \in l_{q,\Psi}, a \ (\geq 0) \in l_{p,\varphi}, \| b \|_{q,\psi} > 0, \| B \|_{q,\Psi} > 0 \| a \|_{p,\varphi} > 0,$$

then we have the following equivalent inequalities:

$$(Tb, a) < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \| a \|_{p,\varphi} \| B \|_{q,\Psi}, \tag{26}$$

$$\| Tb \|_{q,\varphi^{1-q}} < \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}} \| B \|_{q,\Psi}. \tag{27}$$

Moreover, for $\lambda_1 + \lambda_2 = \lambda$, the constant factor $\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$ in (26) and (27) is the best possible, namely

$$\| T \| = \lambda_2 B(\lambda_1, \lambda_2).$$

On the other hand, if the same constant factor in (26) (or (27)) is the best possible, then for $\lambda - \lambda_2 \le \frac{3}{2}$, $\lambda - \lambda_1 \le \frac{1}{2}$, we have $\lambda_1 + \lambda_2 = \lambda$.

Remark 3. Taking $\lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$ in (19) and (24), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{2}$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-\eta} < \frac{\pi}{4}\,[\sum_{m=1}^{\infty}(m-\eta_1)^{\frac{p}{2}-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(n-\eta_2)^{-\frac{q}{2}-1}B_n^q]^{\frac{1}{q}}, \tag{28}$$

$$\left\{\sum_{m=1}^{\infty}(m-\eta_1)^{\frac{q}{2}-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{m+n-\eta}\right]^q\right\}^{\frac{1}{q}} < \frac{\pi}{2}\,[\sum_{n=1}^{\infty}(n-\eta_2)^{-\frac{q}{2}-1}B_n^q]^{\frac{1}{q}}. \tag{29}$$

In particular, putting $\eta_1 = \eta_2 = \eta = 0$ in (28) and (29), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{4}\,(\sum_{m=1}^{\infty}m^{\frac{p}{2}-1}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}n^{-\frac{q}{2}-1}B_n^q)^{\frac{1}{q}}, \tag{30}$$

$$\left[\sum_{m=1}^{\infty}m^{\frac{q}{2}-1}\left(\sum_{n=1}^{\infty}\frac{b_n}{m+n}\right)^q\right]^{\frac{1}{q}} < \frac{\pi}{2}\,(\sum_{n=1}^{\infty}n^{-\frac{q}{2}-1}B_n^q)^{\frac{1}{q}}. \tag{31}$$

Putting $\eta_1 = \eta_2 = \frac{1}{4}, \eta = \frac{1}{2}$ in (28) and (29), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-\frac{1}{2}} < \frac{\pi}{4}\,[\sum_{m=1}^{\infty}(m-\tfrac{1}{4})^{\frac{p}{2}-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(n-\tfrac{1}{4})^{-\frac{q}{2}-1}B_n^q]^{\frac{1}{q}}, \tag{32}$$

$$\left\{\sum_{m=1}^{\infty}(m-\tfrac{1}{4})^{\frac{q}{2}-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{m+n-\frac{1}{2}}\right]^q\right\}^{\frac{1}{q}} < \frac{\pi}{2}\,[\sum_{n=1}^{\infty}(n-\tfrac{1}{4})^{-\frac{q}{2}-1}B_n^q]^{\frac{1}{q}}. \tag{33}$$

5. Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters and the techniques of real analysis, using Hermite-Hadamard's inequality, the Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate Hardy-Hilbert-type inequality involving one partial sums is given in Theorem 1. The equivalent conditions of the best possible constant factor related to several parameters are provided in Theorem 2. We also consider the equivalent forms, the operator expressions and some particular inequalities in Theorem 3, Theorem 4, Theorem 5 and Remark 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Conflict of interest

The authors declare that they have no competing interests.

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