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Research article

Qualitative analysis of nonlinear impulse langevin equation with helfer fractional order derivatives

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Abstract: In this manuscript, a class of impulsive Langevin equation with Hilfer fractional derivatives is considered. Using the techniques of nonlinear functional analysis, we establish appropriate conditions and results to discuss existence, uniqueness and different types of Ulam-Hyers stability results of our proposed model, with the help of Banach's fixed point theorem. An example is provided at the end to illustrate our results.

Keywords: Langevin equation; Hilfer fractional derivative; impulse; Ulam-Hyers stability
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1. Introduction

Hilfer [9] proposed a general operator for fractional derivative, called Hilfer fractional derivative, which combines Caputo and Riemann-Liouville fractional derivatives. Hilfer fractional derivative is performed, for example, in the theoretical simulation of dielectric relaxation in glass forming materials. Sandev et al. [28] derived the existence results of fractional diffusion equation with Hilfer fractional derivative which attained in terms of Mittag Leffler functions. Mahmudov and McKibben [16] studied the controllability of fractional dynamical equations with generalized Riemann Liouville fractional derivative by using Schauder fixed point theorem and fractional calculus. Recently, Gu and Trujillo [8] reported the existence results of fractional differential equations with Hilfer derivative based on noncompact measure method. The set of two parameters in Hilfer fractional derivative $D^{\alpha_1,\beta}(D^{\alpha_2,\beta}$ of order $0 \leq \alpha_1 \leq 1$ and $0 < \alpha_2 < 1$ permits one to connect between the Caputo and Riemann-Liouville derivatives [14, 34]. This set of parameters gives an extra degree of freedom on the initial conditions and produces more types of stationary states. Models with Hilfer fractional derivatives are discussed in [2, 23, 25, 30, 42, 43].

Langevin equation was introduced by Paul Langevin in 1908. These equations are used to described stochastic problems in physics, defence system, image processing, chemistry, astronomy, mechanical and electrical engineering. They are also used to describe Brownian motion when the random fluctuation force is assumed to be Gaussian noise. Fractional order differential equations remove the noise efficiently as compare to integer order differential equations. For more details, see [1, 6, 17, 20–22, 24].

At Wisconsin university, Ulam raised a question about the stability of functional equations in the year 1940. The question of Ulam was: Under what conditions does there exist an additive mapping near an approximately additive mapping [31]. In 1941, Hyers was the first mathematician who gave partial answer to Ulam's question [13], over Banach space. Afterwards, stability of such form is known as Ulam-Hyers stability. In 1978, Rassias [19], provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. For more information about the topic, we refer the reader to [23, 27, 29, 38, 39, 41].

Impulsive fractional differential equations describe physical sciences, social sciences and many dynamical systems. There are two types of impulsive fractional differential equations, instantaneous impulsive fractional differential equations and non-instantaneous impulsive fractional differential equations. From few decades, the theory of impulsive fractional differential equations is utilized in mechanical engineering, biology, ecology, astronomy and medicine etc. For details on impulsive fractional differential equations, see [4, 5, 10–12, 15, 26, 32, 35, 37, 40, 44].

Recently, the existence, uniqueness and different types of Ulam-Hyers stability of nonlinear implicit fractional differential equations with Helfer's fractional derivative have received a considerable attention, see [23, 29, 34, 39].

Wang et al. [33], studied generalized Ulam-Hyers-Rassias stability of the following fractional differential equation:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0,\upsilon}x(\upsilon) = f(\upsilon, x(\upsilon)), \quad \upsilon \in (\upsilon_i, s_i], \quad i = 0, 1, \ldots, m, \quad 0 < \alpha < 1, \\ x(\upsilon) = g_i(\upsilon, x(\upsilon)), \quad \upsilon \in (\varsigma_{i-1}, \upsilon_i], \quad i = 1, 2, \ldots, m. \end{cases}$$

Zada et al. [36], studied existence, uniqueness of solutions by using Diaz Margolis's fixed point theorem and presented different types of Ulam-Hyers stability for a class of nonlinear implicit fractional differential equation with non-instantaneous integral impulses and nonlinear integral boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0,\upsilon}x(\upsilon) = f(\upsilon, x(\upsilon), {}^{c}\mathcal{D}^{\alpha}_{0,\upsilon}x(\upsilon)), \ \upsilon \in (\upsilon_i, s_i], \ i = 0, 1, \ldots, m, \ 0 < \alpha < 1, \ \upsilon \in (0, 1], \\ x(\upsilon) = I^{\alpha}_{s_{i-1},\upsilon_i}(\xi_i(\upsilon, x(\upsilon))), \ \upsilon \in (s_{i-1}, \upsilon_i], i = 1, 2, \ldots, m, \\ x(0) = \dfrac{1}{\Gamma\alpha}\displaystyle\int_0^T (T - \varsigma)^{\alpha-1}\eta(\varsigma, x(\varsigma))d\varsigma. \end{cases}$$

In this paper, we study a class of impulsive Langevin equation with Hilfer fractional derivatives of the form:

$$\begin{cases} D^{\alpha_1,\beta}(D^{\alpha_2,\beta} + \lambda)x(\upsilon) = f(\upsilon, x(\upsilon)), \ \ \upsilon \in J = [0, T], \ 0 < \alpha_1, \alpha_2 < 1, \ \ 0 \le \ \gamma \le 1, \\ \Delta\ x(\upsilon_i) = I_i(x(\upsilon_i)), \ \ i = 1, 2, \ldots, m, \\ I^{1-\gamma}x(0) = x_0, \quad \gamma = (\alpha_1 + \alpha_2)(1 - \beta) + \beta, \end{cases} \tag{1.1}$$

where $D^{\alpha_1,\beta}$ and $D^{\alpha_2,\beta}$ represents two Hilfer fractional derivatives [9], of order α_1 and α_2 respectively, β determines to the type of initial condition used in the problem. Further $f : J\times R \to R$ is continuous and $I_i : R \to R$ for all $i = 1, 2, \dots, m,$ represents impulsive nonlinear mapping and $\Delta x(\upsilon_i) = x(\upsilon_i^+) - x(\upsilon_i^-)$, where $x(\upsilon_i^+)$ and $x(\upsilon_i^-)$ represent the right and the lift limits, respectively, at $t = \upsilon_i$ for $i = 1, 2, \dots, m$.

2. Preliminaries

We recall some definitions of fractional calculus from [14, 18] as follows:

Definition 2.1. The fractional integral of order α from 0 to x for the function f is

$$I_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(\varsigma)(x-\varsigma)^{\alpha-1} d\varsigma, \quad x > 0,\ \alpha > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The Riemman–Liouville fractional derivative of fractional order α for f is

$${}^{L}\mathcal{D}_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\upsilon^n} \int_0^x \frac{f(\varsigma)}{(x-\varsigma)^{\alpha+1-n}} d\varsigma, \quad x > 0,\ n-1 < \alpha < n.$$

Definition 2.3. The Caputo derivative of fractional order α for f is

$${}^{c}\mathcal{D}_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\varsigma)^{n-\alpha-1} f^n(\varsigma) d\varsigma, \quad \text{where } n = [\alpha] + 1.$$

Definition 2.4. The classical Caputo derivative of order α of f is

$${}^{c}\mathcal{D}_{0,x}^{\alpha} = {}^{L}\mathcal{D}_{0,x}^{\alpha} \Big(f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0) \Big), \quad x > 0,\ n-1 < \alpha < n.$$

Definition 2.5. The Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \le \gamma \le 1$ of function $f(x)$ is

$$D^{\alpha,\gamma} f(x) = (I^{\gamma(1-\alpha)} D(I^{(1-\gamma)(1-\alpha)}(f))(x).$$

The Hilfer fractional derivative is used as an interpolator between the Riemman-Liouville and Caputo derivative.

Remark 2.1. **(a)** Operator $D^{\alpha,\gamma}$ also can be written as

$$D^{\alpha,\gamma} f(x) = (I^{\gamma(1-\alpha)} D(I^{(1-\gamma)(1-\alpha)} f)(x)) = I^{\gamma(1-\alpha)} D^{\eta} f(x), \eta = \alpha + \gamma - \alpha\gamma.$$

(b) If $\gamma = 0$, then $D^{\alpha,\gamma} = D^{\alpha,0}$ is called the Riemman-Liouville fractional derivative.

(c) If $\gamma = 1$, then $D^{\alpha,\gamma} = I^{1-\alpha} D$ is called the Caputo fractional derivative.

Remark 2.2. **(i)** If $f(\cdot) \in C^m([0,\infty), R)$, then

$${}^{L}\mathcal{D}_{0,x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^m(\varsigma)}{(x-\varsigma)^{\alpha+1-m}} d\varsigma = I_{0,x}^{m-\alpha} f^{(m)}(x), \quad x > 0,\ m-1 < \alpha < m.$$

(ii) In Definition 2.4, the integrable function f can be discontinuous. This fact can support us to consider impulsive fractional problems in the sequel.

Lemma 2.1. *[18] Let $\alpha > 0$ and $\beta > 0$, $f \in L^1([a,b])$.*

$$\text{Then } I^{\alpha}I^{\beta}f(x) = I^{\alpha+\beta}f(x), {}^{c}\mathcal{D}^{\alpha}_{0,x}({}^{c}\mathcal{D}^{\beta}_{0,x}f(x)) = {}^{c}\mathcal{D}^{\alpha+\beta}_{0,x}f(x) \text{ and } I^{\alpha}\mathcal{D}^{\alpha}_{0,x}f(x) = f(x), x \in [a,b].$$

Theorem 2.1. *[[3](Banach's fixed point theorem)]. Let B be a Banach space. Then any contraction mapping $N : B \to B$ has a unique fixed point.*

3. Existence and uniqueness

In this section, we investigate the existence, uniqueness of solutions to the proposed Langevin equation using two Hilfer fractional derivatives.

Lemma 3.1. *A function $f : (0,T] \times R \to R$ is equivalent to the integral equation*

$$x(\upsilon) = \begin{cases} \dfrac{x_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \dfrac{\lambda}{\Gamma(\alpha_1)}\displaystyle\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \quad \upsilon \in J_0 \\ \dfrac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + \dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma + \dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_0^{\upsilon_1}(\upsilon_1-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma \\ -\dfrac{\lambda}{\Gamma(\alpha_1)}\displaystyle\int_0^{\upsilon_1}(\upsilon_1-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma - \dfrac{\lambda}{\Gamma(\alpha_1)}\displaystyle\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + I_1(x(\upsilon_1)), \quad \upsilon \in J_1, \\ \dfrac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \displaystyle\sum_{i=1}^{m}\dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \displaystyle\sum_{i=1}^{m}\dfrac{\lambda}{\Gamma(\alpha_1)}\displaystyle\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \\ +\displaystyle\sum_{i=1}^{m}I_i(x(\upsilon_i)), \quad \upsilon \in J_i \quad i = 1,2,\ldots,m. \end{cases} \tag{3.1}$$

is the only solution of the problem (1.1).

Proof. Let x satisfies (1.1), then for any $\upsilon \in J_0$, there exists a constant $c \in R$, such that

$$x(\upsilon) = c + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma. \tag{3.2}$$

Using the condition $I^{1-\gamma}x(0) = x_0$, Eq (3.2) yields that

$$x(\upsilon) = \frac{x_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma, \quad \upsilon \in J_0.$$

Similarly for $\upsilon \in J_1$, there exists a constant $d_1 \in R$, such that

$$x(\upsilon) = d_1 + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma.$$

Hence, we have

$$x(\upsilon_1^-) = \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma,$$

$$x(\upsilon_1^+) = d_1.$$

In view of

$$\Delta\, x(\upsilon_1) = x(\upsilon_1^+) - x(\upsilon_1^-) = I_1(x(\upsilon_1)),$$

we get

$$x(\upsilon_1^+) - x(\upsilon_1^-) = d_1 - \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma + \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma,$$

$$I_1(x(\upsilon_1)) = d_1 - \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma + \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma,$$

$$d_1 = \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\beta-1}x(\varsigma)d\varsigma + I_1(x(\upsilon_1)).$$

For this value of d_1, we have

$$\begin{aligned} x(\upsilon) &= \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma \\ &\quad -\frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + I_1(x(\upsilon_1)). \end{aligned}$$

Similarly for $\upsilon \in J_i$, we get

$$\begin{aligned} x(\upsilon) &= \frac{x_0}{\Gamma(\gamma)}\upsilon_i^{\gamma-1} + \sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma - \sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \\ &\quad + \sum_{i=1}^{m} I_i(x(\upsilon_i)). \end{aligned}$$

Conversely, let that x satisfies (3.1), then it can be easily proved that the solution $x(\upsilon)$ given by (3.1) satisfies (1.1) along with its impulsive and integral boundary conditions. □

Consider some assumptions as follows:

(H_1) $f \in C(J \times R, R)$ is continuous.

(H_2) There exists $0 < Ł_f < 1$ such that

$$\left|f(\upsilon,u) - f(\upsilon,v)\right| \le Ł_f|u-v|, \text{ for each } \upsilon \in J_i,\ i = 1,2,\ldots,m, \text{ and all } u,v \in R.$$

(H_3) There exists $0 < Ł_k < 1$, such that

$$\left|I_i(u) - I_i(v)\right| \le Ł_k|u-v|, \text{ for each } \upsilon \in J_i,\ i = 1,2,\ldots,m, \text{ and for all } u,v \in R.$$

Theorem 3.1. *Let assumptions* $(H_1)-(H_3)$ *be satisfied and if*

$$\Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2}+\frac{m\lambda}{\Gamma\alpha_1+1}T^{\alpha_1-1}+mŁ_k\Big)<1, \tag{3.3}$$

then (1.1) *has a unique solution* x *in* $C_{1-\gamma}[0,T]$.

Proof. We define a mapping $N: C_{1-\gamma}[0,T]\to C_{1-\gamma}[0,T]$

$$\begin{cases}(Nx)(\upsilon)=\dfrac{x_0}{\Gamma(\gamma)}\upsilon^{\gamma-1}+\dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma-\dfrac{\lambda}{\Gamma(\alpha_1)}\displaystyle\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \quad \upsilon\in J_0,\\ (Nx)(\upsilon)=\dfrac{x_0}{\Gamma(\gamma)}\upsilon_i^{\gamma-1}+\displaystyle\sum_{i=1}^{m}\dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma,x(\varsigma))d\varsigma-\displaystyle\sum_{i=1}^{m}\dfrac{\lambda}{\Gamma(\alpha_1)}\displaystyle\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma\\ \qquad+\displaystyle\sum_{i=1}^{m}I_i(x(\upsilon_i)) \quad \upsilon\in J_i \quad i=1,2,\dots,m.\end{cases}$$

For any $x,y\in C_{1-\gamma}[0,T]$ and $\upsilon\in J_i$, consider the following

$$\begin{aligned}|(Nx)(\upsilon)-(Ny)(\upsilon)| &\le \sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}|f(\varsigma,x(\varsigma))-f(\varsigma,y(\varsigma))|d\varsigma\\ &\quad-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma+\sum_{i=1}^{m}|I_i(x(\upsilon_i))-I_i(y(\upsilon_i))|\\ &\le \sum_{i=1}^{m}\frac{Ł_f}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}|x(\varsigma)-y(\varsigma)|d\varsigma\\ &\quad-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma+Ł_k\sum_{i=1}^{m}|x(\upsilon)-y(\upsilon)|\\ &\le \Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(\upsilon_i-\upsilon_{i-1})^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(\upsilon_i-\upsilon_{i-1})^{\alpha_1-1}+mŁ_k\Big)|x(\upsilon)-y(\upsilon)|\\ &\le \Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2}+\frac{m\lambda}{\Gamma\alpha_1+1}T^{\alpha_1-1}+mŁ_k\Big)|x(\upsilon)-y(\upsilon)|.\end{aligned}$$

Hence N is a contraction according to Banach's contraction theorem and so it has only one fixed point, which is the only one solution of (1.1). □

4. Ulam-Hyers stability analysis

Let $\varepsilon>0$ and $\varphi: J\to R^+$ be a continuous function. Consider

$$\begin{cases}|D^{\alpha_1,\beta}(D^{\alpha_2,\beta}+\lambda)z(\upsilon)-f(\upsilon,z(\upsilon))|\le\varepsilon, \quad \upsilon\in J_i, \quad i=1,2,\dots,q,\\ |\Delta\, z(\upsilon_i)-I_i(z(\upsilon_i))|\le\varepsilon, \quad i=1,2,\dots,m,\end{cases} \tag{4.1}$$

$$\begin{cases}|D^{\alpha_1,\beta}(D^{\alpha_2,\beta}+\lambda)z(\upsilon)-f(\upsilon,z(\upsilon))|\le\varphi(\upsilon), \quad \upsilon\in J_i, \quad i=1,2,\dots,q,\\ |\Delta\, z(\upsilon_i)-I_i(z(\upsilon_i))|\le\psi, \quad i=1,2,\dots,m,\end{cases} \tag{4.2}$$

and

$$\begin{cases} |D^{\alpha_1,\beta}(D^{\alpha_2,\beta}+\lambda)z(\upsilon)-f(\upsilon,z(\upsilon))| \le \varepsilon\varphi(\upsilon), & \upsilon\in J_i,\ i=1,2,\dots,q, \\ |\Delta\, z(\upsilon_i)-I_i(z(\upsilon_i))| \le \varepsilon\psi, & i=1,2,\dots,m. \end{cases} \tag{4.3}$$

Definition 4.1. The problem (1.1) is Ulam-Hyers stable if there exist a real number $C_{f,i,q,\sigma}$ such that for each solution $\varepsilon > 0$ and for each solution $z \in C_{1-\gamma}[0,T]$ of the inequality (4.1), there exist a solution $x \in C_{1-\gamma}[0,T]$ of the problem (1.1) such that

$$|z(\upsilon)-x(\upsilon)| \le C_{f,i,q,\sigma}\ \varepsilon\ \ \upsilon\in J. \tag{4.4}$$

Definition 4.2. The problem (1.1) is generalized Ulam-Hyers stable if there exist $\phi_{f,i,q,\sigma} \in C_{1-\gamma}[0,T]$, $\phi_{f,i,q,\sigma}(0) = 0$ and $\varepsilon > 0$ such that for each solution $z \in C_{1-\gamma[0,T]}$ of the inequality (4.1), there exist a solution $x \in C_{1-\gamma}[0,T]$ of the problem (1.1) such that

$$|z(\upsilon)-x(\upsilon)| \le \phi_{f,i,q,\sigma}\ \varepsilon\ \ \upsilon\in J. \tag{4.5}$$

Remark 4.1. Keep in mind that Definition 4.1 $\Rightarrow$ Definition 4.2.

Definition 4.3. The problem (1.1) is Ulam-Hyers-Rassias stable with respect to (φ,ψ) if there exist $C_{f,i,q,\sigma,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C_{1-\gamma}[0,T]$ of inequality (4.3) there is a solution $x \in C_{1-\gamma}[0,T]$ of the problem (1.1) with

$$|z(\upsilon)-x(\upsilon)| \le C_{f,i,q,\sigma,\varphi}\varepsilon(\varphi(\upsilon)+\psi)\ \varepsilon\ \ \upsilon\in J. \tag{4.6}$$

Definition 4.4. The problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (φ,ψ) if there exist $C_{f,i,q,\sigma,\varphi} > 0$ such that for each solution $z \in C_{1-\gamma}[0,T]$ of inequality (4.2) there is a solution $x \in C_{1-\gamma}[0,T]$ of the problem (1.1) with

$$|z(\upsilon)-x(\upsilon)| \le C_{f,i,q,\sigma,\varphi}(\varphi(\upsilon)+\psi)\ \varepsilon\ \ \upsilon\in J. \tag{4.7}$$

Remark 4.2. It should be noted that Definition 4.3 implies Definition 4.4.

Remark 4.3. A function $z \in C_{1-\gamma}[0,T]$ is a solution of the inequality (4.1) if and only if there exists a function $g \in C_{1-\gamma}[0,T]$ and a sequence $g_i, i = 1,2,\dots,m$, depending on g, such that

(a) $|g(\upsilon)| \le \varepsilon,\ |g_i| \le \varepsilon\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(b) $D^{\alpha_1,\beta}(D^{\alpha_2,\beta}+\lambda)z(\upsilon) = f(\upsilon,z(\upsilon))+g(\upsilon),\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(c) $\Delta\, x(\upsilon_i) = I_i(x(\upsilon_i))+g_i,\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

Remark 4.4. A function $z \in C_{1-\gamma}[0,T]$ satisfies (4.2) if and only if there exists $g \in C_{1-\gamma}[0,T]$ and a sequence $g_i, i = 1,2,\dots,m$, depending on g, such that

(a) $|g(\upsilon)| \le \varphi(\upsilon),\ |g_i| \le \psi\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(b) $D^{\alpha_1,\beta}(D^{\alpha_2,\beta}+\lambda)z(\upsilon) = f(\upsilon,z(\upsilon))+g(\upsilon),\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(c) $\Delta\, x(\upsilon_i) = I_i(x(\upsilon_i))+g_i,\ \ \upsilon\in J_i,\ i=1,2,\dots,m.$

Remark 4.5. A function $z \in C_{1-\gamma}[0,T]$ satisfies (4.2) if and only if there exists $g \in C_{1-\gamma}[0,T]$ and a sequence $g_i, i = 1,2,\ldots,m$, depending on g, such that

(a) $|g(\upsilon)| \le \varepsilon\varphi(\upsilon),\ |g_i| \le \varepsilon\psi\ \ \upsilon \in J_i,\ i = 1,2,\ldots,m,$

(b) $D^{\alpha_1,\beta}(D^{\alpha_2,\beta} + \lambda)z(\upsilon) = f(\upsilon, z(\upsilon)) + g(\upsilon),\ \ \upsilon \in J_i,\ i = 1,2,\ldots,m,$

(c) $\Delta\, x(\upsilon_i) = I_i(x(\upsilon_i)) + g_i,\ \ \upsilon \in J_i,\ i = 1,2,\ldots,m.$

Theorem 4.1. *If the assumptions* $(H1)$–$(H3)$ *and the inequality* (3.3) *hold, then Eq* (1.1) *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $y \in C_{1-\gamma}[0,T]$ satisfies (4.1) and let x be the only one solution of

$$\begin{cases} D^{\alpha_1,\beta}(D^{\alpha_2,\beta} + \lambda)x(\upsilon) = f(\upsilon, x(\upsilon)),\ \ \upsilon \in J = [0,T],\ 0 < \alpha_1, \alpha_2 < 1,\ \ 0 \le\ \gamma \le 1, \\ \Delta\, x(\upsilon_i) = I_i(x(\upsilon_i)),\ \ i = 1,2,\ldots,m, \\ I^{1-\gamma}x(0) = x_0, \quad \gamma = (\alpha_1 + \alpha_2)(1-\beta) + \beta. \end{cases}$$

By Lemma 3.1, we have for each $\upsilon \in J_i$

$$\begin{aligned} x(\upsilon) &= \frac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} f(\varsigma, x(\varsigma))d\varsigma - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} x(\varsigma)d\varsigma \\ &\quad + \sum_{i=1}^{m} I_i(x(\upsilon_i)) \quad \upsilon \in J_i \quad i = 1,2,\ldots,m. \end{aligned}$$

Since y satisfies inequality (4.1), so by Remark 4.3, we get

$$\begin{cases} D^{\alpha_1,\beta}(D^{\alpha_2,\beta} + \lambda)y(\upsilon) = f(\upsilon, y(\upsilon)) + g_i,\ \ \upsilon \in J = [0,T],\ 0 < \alpha_1, \alpha_2 < 1,\ \ 0 \le\ \gamma \le 1, \\ \Delta\, x(\upsilon_i) = I_i(y(\upsilon_i)) + g_i,,\ \ i = 1,2,\ldots,m, \\ I^{1-\gamma}y(0) = y_0, \quad \gamma = (\alpha_1 + \alpha_2)(1-\beta) + \beta. \end{cases} \tag{4.8}$$

Obviously the solution of (4.8) will be

$$y(\upsilon) = \begin{cases} \frac{y_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^{\upsilon} (\upsilon - \varsigma)^{\alpha_1+\alpha_2-1} f(\varsigma, y(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)} \int_0^{\upsilon} (\upsilon - \varsigma)^{\alpha_1-1} y(\varsigma)d\varsigma \\ + \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^{\upsilon} (\upsilon - \varsigma)^{\alpha_1+\alpha_2-1} g_i(\varsigma)d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)} \int_0^{\upsilon} (\upsilon - \varsigma)^{\alpha_1-1} g_i(\varsigma)d\varsigma \quad \upsilon \in J_0 \\ \frac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} f(\varsigma, y(\varsigma))d\varsigma - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} y(\varsigma)d\varsigma \\ + \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} g_i(\varsigma)d\varsigma - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} g_i(\varsigma)d\varsigma \\ + \sum_{i=1}^{m} I_i(x(\upsilon_i)) + \sum_{i=1}^{m} g_i \quad \upsilon \in J_i \quad i = 1,2,\ldots,m. \end{cases}$$

Therefore, for each $\upsilon \in J_i$, we have the following

$$|x(\upsilon) - y(\upsilon)| \ \le\ \sum_{i=1}^{m} \int_{\upsilon_{i-1}}^{\upsilon_i} \frac{(\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} |f(\varsigma, x(\varsigma)) - f(\varsigma, y(\varsigma))|d\varsigma - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} |x(\varsigma) - y(\varsigma)|d\varsigma$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}g_i(\varsigma)d\varsigma-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}g_i(\varsigma)d\varsigma\\
&+\sum_{i=1}^{m}|I_i(x(\upsilon_i))-I_i(y(\upsilon_i))|+\sum_{i=1}^{m}g_i\\
\leq\ &\sum_{i=1}^{m}\frac{Ł_f}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}|x(\varsigma)-y(\varsigma)|d\varsigma-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma\\
&+\sum_{i=1}^{m}\frac{\varepsilon}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}d\varsigma-\sum_{i=1}^{m}\frac{\varepsilon\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}d\varsigma\\
&+Ł_k\sum_{i=1}^{m}|x(\upsilon)-y(\upsilon)|+\sum_{i=1}^{m}\varepsilon\\
\leq\ &\Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+mŁ_k\Big)|x(\upsilon)-y(\upsilon)|\\
&+\frac{m\varepsilon}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\varepsilon\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m\varepsilon,
\end{aligned}
$$

which implies that

$$
|x(\upsilon)-y(\upsilon)|\ \leq\ \varepsilon\left(\frac{\frac{m}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+mŁ_k)}\right).
$$

Thus

$$
|x(\upsilon)-y(\upsilon)|\leq\varepsilon C_{f,g,\alpha_1,\alpha_2},
$$

where

$$
C_{f,g,\alpha_1,\alpha_2}=\frac{\frac{m}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+mŁ_k)}.
$$

So Eq (1.1) is Ulam-Hyers stable and if we set $\phi(\varepsilon)=\varepsilon C_{f,g,\alpha_1,\alpha_2}$, $\phi(0)=0$, then Eq (1.1) is generalized Ulam-Hyers stable. □

Theorem 4.2. *If the assumptions* (H_1)–(H_3) *and the inequality* (3.3) *are satisfied, then the problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* (φ,ψ)*, consequently generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y\in C_{1-\gamma}[0,T]$ be a solution of the inequality (4.3) and let x be the only one solution of the following problem

$$
\begin{cases}
D^{\alpha_1,\beta}(D^{\alpha_2,\beta}+\lambda)x(\upsilon)=f(\upsilon,x(\upsilon)),\ \ \upsilon\in J=[0,T],\ 0<\alpha_1,\alpha_2<1,\ \ 0\leq\ \gamma\leq1,\\
\Delta\ x(\upsilon_i)=I_i(x(\upsilon_i)),\ \ i=1,2,\ldots,m,\\
I^{1-\gamma}x(0)=x_0,\quad \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta.
\end{cases}
$$

From Theorem 4.1, for all $\upsilon \in J_i$, we get

$$\begin{aligned}
|x(\upsilon)-y(\upsilon)| &\leq \sum_{i=1}^{m}\int_{\upsilon_{i-1}}^{\upsilon_i}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}|f(\varsigma,x(\varsigma))-f(\varsigma,y(\varsigma))|d\varsigma-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma \\
&+\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}g_i(\varsigma)d\varsigma-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}g_i(\varsigma)d\varsigma \\
&+\sum_{i=1}^{m}|I_i(x(\upsilon_i))-I_i(y(\upsilon_i))|+\sum_{i=1}^{m}g_i \\
&\leq \sum_{i=1}^{m}\frac{Ł_f}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}|x(\varsigma)-y(\varsigma)|d\varsigma-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma \\
&+\sum_{i=1}^{m}\frac{\varepsilon}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}\varphi(\varsigma)d\varsigma-\sum_{i=1}^{m}\frac{\varepsilon\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}\varphi(\varsigma)d\varsigma \\
&+Ł_k\sum_{i=1}^{m}|x(\upsilon)-y(\upsilon)|+\sum_{i=1}^{m}\psi \\
&\leq \Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(\upsilon_i-\upsilon_{i-1})^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(\upsilon_i-\upsilon_{i-1})^{\alpha_1}+mŁ_k\Big)|x(\upsilon)-y(\upsilon)| \\
&+\frac{m\varepsilon\lambda_\varphi\varphi(\upsilon)}{\Gamma(\alpha_1+\alpha_2+1)}(\upsilon_i-\upsilon_{i-1})^{\alpha_1+\alpha_2}-\frac{m\varepsilon\lambda_\varphi\varphi(\upsilon)\lambda}{\Gamma\alpha_1+1}(\upsilon_i-\upsilon_{i-1})^{\alpha_1},+m\varepsilon\psi,
\end{aligned}$$

which implies that

$$\begin{aligned}
|x(\upsilon)-y(\upsilon)| &\leq \varepsilon\left(\frac{\frac{m\lambda_\varphi\varphi(\upsilon)}{\Gamma(\alpha_1+\alpha_2+1)}(\upsilon_i-\upsilon_{i-1})^{\alpha_1+\alpha_2}-\frac{m\lambda_\varphi\varphi(\upsilon)\lambda}{\Gamma\alpha_1+1}(\upsilon_i-\upsilon_{i-1})^{\alpha_1}+m\psi}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(\upsilon_i-\upsilon_{i-1})^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(\upsilon_i-\upsilon_{i-1})^{\alpha_1}+mŁ_k)}\right) \\
&\leq \left(\frac{\frac{m\lambda_\varphi}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda_\varphi\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+mŁ_k)}\right)\varepsilon\big(\varphi(\upsilon)+\psi\big).
\end{aligned}$$

Thus

$$|x(\upsilon)-y(\upsilon)|\leq C_{f,g,\alpha_1,\alpha_2,\varphi,\psi}\varepsilon\big(\varphi(\upsilon)+\psi\big),$$

where

$$C_{f,g,\alpha_1,\alpha_2,\varphi,\psi}=\left(\frac{\frac{m\lambda_\varphi}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda_\varphi\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+mŁ_k)}\right).$$

Hence (1.1) is Ulam-Hyers-Rassias stable and is obviously generalized Ulam-Hyers-Rassias stable. □

Finally we give an example to illustrate our main result.

Example 4.1.

$$\begin{cases} D^{(\frac{1}{2},\frac{1}{2})}(D^{(\frac{1}{3},\frac{1}{2})}+\dfrac{1}{2})x(\upsilon)=\dfrac{|x(\upsilon)|}{8+e^{\upsilon}+\upsilon^2}, & \upsilon\in J=[0,1],\\ I_i x(\dfrac{1}{2})=\dfrac{x|(\frac{1}{2})|}{70+|x(\frac{1}{2})|}, & i=1,2,\ldots,m,\\ I^{1-\gamma}x(0)=0, \quad \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta. \end{cases} \tag{4.9}$$

Let $J_0=[0,\frac{1}{2}]$, $J_1=[\frac{1}{2},1]$ $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{1}{3}$, $\lambda=\lambda_\varphi=\frac{1}{2}$, $Ł_f=Ł_k=\frac{1}{90e^2}$ and $m=T=1$. Obviously

$$\left(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2}+\frac{m\lambda}{\Gamma\alpha_1+1}T^{\alpha_1-1}+mŁ_k\right)<1.$$

Thus, thanks to Theorem 3.1, the given problem (4.9) has a unique solution. Further the conditions of Theorem 4.1 are satisfied so the solution of the given problem (4.9) is Ulam-Hyers stable and generalized Ulam-Hyers stable. Further it is also easy to check the conditions of Theorem 4.2 hold and thus the problem (4.9) is Ulam-Hyers-Rassias stable and consequently generalized Ulam-Hyers-Rassias stable.

5. Conclusions

In this article, we consider a class of impulsive Langevin equation with Hilfer fractional derivative. Some conditions are made to beat the hurdles to investigate the existence, uniqueness and to discuss different types of Ulam-Hyers stability of our considered model, using Banach's fixed point theorem.

Conflict of interest

The authors declare that they have no competing interest regarding this research work.

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