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Research article

Universal enveloping Hom-algebras of regular Hom-Poisson algebras

Xianguo Hu*

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

* **Correspondence:** Email: xghu@cnu.edu.cn.

Abstract: In this paper, we introduce universal enveloping Hom-algebras of Hom-Poisson algebras. Some properties of universal enveloping Hom-algebras of regular Hom-Poisson algebras are discussed. Furthermore, in the involutive case, it is proved that the category of involutive Hom-Poisson modules over an involutive Hom-Poisson algebra A is equivalent to the category of involutive Hom-associative modules over its universal enveloping Hom-algebra $U_{eh}(A)$.

Keywords: regular Hom-algebras; involutive Hom-algebras; Hom-Poisson algebras; Hom-Poisson modules; universal enveloping Hom-algebras
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1. Introduction

Poisson algebras originally arise from Hamiltonian mechanics and play an important role in Poisson geometry, algebraic geometry and deformation theory (see [3, 4, 9]). There are many interesting generalizations of Poisson structures developed by different authors from different perspectives (see, for instance, [5, 8, 12, 13, 22, 23]). One way to generalize Poisson algebras is to twist the structure by a homomorphism. Such a structure is called a Hom-Poisson algebra, which was first defined in [15] by Makhlouf and Silvestrov. As a natural generalization of Poisson algebras, Hom-Poisson algebras play the same role in the deformation of commutative Hom-associative algebras as Poisson algebras do in the deformation of commutative associative algebras. Generally, a Hom-type algebraic structure (e.g., algebra, Lie algebra, coalgebra, Hopf algebra, etc.) is a vector space, endowed with an endomorphism, such that the classical definition of this algebraic structure is "deformed" by this endomorphism. The origins of the study of Hom-type algebras can be found in [7], where the notion of Hom-Lie algebra was introduced as part of a study of deformations of the Witt and the Virasoro algebras. The theory of Hom-type algebras has been widely studied in the past two decades (see [1, 2, 6, 10, 11, 14, 16, 19, 20, 26] and the references therein).

In nowadays mathematics, much of the research on certain algebraic object is to study its representation theory. The representation theory of an algebraic object is very important since it reveals some of its profound structures hidden underneath, so is for Hom-Poisson algebra. Similar to the definition of Poisson modules over Poisson algebras, Hom-Poisson modules over Hom-Poisson algebras are defined in a natural way. In this paper, in order to study the representation theory of Hom-Poisson algebras, we introduce the notion of universal enveloping Hom-algebras of Hom-Poisson algebras.

The paper is organized as follows. In Section 2, we fix notation and recall some definitions and basic facts used throughout the paper. In particular, we recall the definitions of Hom-associative algebras, Hom-Lie algebras, Hom-Poisson algebras and Hom-Poisson modules. In section 3, we mainly study the universal enveloping Hom-algebra of a Hom-Poisson algebra. For any regular Hom-Poisson algebra $(A, \mu, [\cdot,\cdot], \alpha)$, basic properties of its universal enveloping Hom-algebra $U_{eh}(A)$ are discussed, including the relation to the usual universal enveloping algebra of A, whose Poisson structure is obtained by the action of α^{-1}. Moreover, in the involutive case, we show that the category of involutive Hom-Poisson modules over A is equivalent to the category of involutive Hom-associative modules over $U_{eh}(A)$.

Throughout this paper, all vector spaces and linear maps are over a fixed field k. In what follows, an unadorned $\otimes$ means $\otimes_k$. Given a k-module V, $\tau : V \otimes V \to V \otimes V$ interchanges the two variables, that is, $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$, for any $v_1, v_2 \in V$.

2. Preliminaries

In this section, we briefly recall some definitions and notation used in this paper.

By a Hom-module, we mean a pair (A, α) in which A is a vector space(i.e., k-module) and $\alpha : A \to A$ is a linear map, called the twisting map. Let (A, α) and (B, β) be two Hom-modules. A homomorphism $f : (A, \alpha) \to (B, \beta)$ of Hom-modules is a linear map $f : A \to B$ such that $\beta f = f\alpha$. A Hom-algebra is a triple (A, μ, α), where (A, α) is a Hom-module, and $\mu : A \otimes A \to A$ is a bilinear map, called the multiplication. For convenience, we shall write $\mu(a \otimes b)$ as ab, $\forall a, b \in A$, whenever this does not cause confusion.

Definition 2.1. Let (A, μ, α) be a Hom-algebra.

(1) The Hom-algebra A is called a Hom-associative algebra if there exists an element $1_A \in A$ such that

$$\alpha(1_A) = 1_A, 1_A a = \alpha(a) = a 1_A, \alpha(a)(bc) = (ab)\alpha(c)$$

for all $a, b, c \in A$. We usually denote a Hom-associative algebra by $(A, \mu, 1_A, \alpha)$, or simply by (A, μ, α) or A if no confusions arise.

(2) A Hom-associative algebra (A, μ, α) (resp. Hom-module (V, α_V)) is said to be involutive if $\alpha^2 = Id$ (resp. $\alpha_V^2 = Id$).

(3) A Hom-associative algebra (A, μ, α) (resp. Hom-module (V, α_V)) is said to be regular if α is bijective (resp. α_V is bijective).

(4) Let $(A, \cdot, 1_A, \alpha_A)$ and $(B, \bullet, 1_B, \alpha_B)$ be two Hom-associative algebras. A homomorphism $f : (A, \alpha_A) \to (B, \alpha_B)$ of Hom-modules (namely $\alpha_B f = f\alpha_A$) is a homomorphism of Hom-associative algebras if $f(1_A) = 1_B$ and $f(a \cdot b) = f(a) \bullet f(b)$ for all $a, b \in A$.

(5) Let $(A, \mu, 1_A, \alpha)$ be a Hom-associative algebra.

(5a) A submodule $S \subseteq A$ is called a Hom-associative subalgebra of A if $1_A \in S$, $ab \in S$ for all $a, b \in S$ and $\alpha(S) \subseteq S$;

(5b) A submodule $I \subseteq A$ is called a Hom-associative ideal of A if $ab \in I, ba \in I$ for all $a \in A, b \in I$ and $\alpha(I) \subseteq I$.

(6) Let $(A, \mu, 1_A, \alpha)$ be a Hom-associative algebra and (M, α_M) a Hom-module. We call M a (left) Hom-associative module over A provided that there is a bilinear map $\cdot : A \otimes M \to M$ such that

(i) $\alpha_M(a \cdot m) = \alpha(a) \cdot \alpha_M(m)$;

(ii) $(ab) \cdot \alpha_M(m) = \alpha(a) \cdot (b \cdot m)$;

(iii) $1_A \cdot m = \alpha_M(m)$,

for all elements $a, b \in A, m \in M$.

Remark 2.2. In the definition of a Hom-associative algebra $(A, \mu, 1_A, \alpha)$, the map α must satisfy the formula: $\alpha(ab) = \alpha(a)\alpha(b)$ for any $a, b \in A$, because

$$\alpha(ab) = 1_A(ab) = \alpha(1_A)(ab) = (1_A a)\alpha(b) = \alpha(a)\alpha(b).$$

Lemma 2.3. *(1) [24] Let $(V, \mu, 1_A)$ be an associative algebra and $\alpha : V \to V$ an algebra endomorphism. Then $(V, \mu_\alpha, 1_A, \alpha)$, where $\mu_\alpha := \alpha\mu$, is a Hom-associative algebra.*

(2) Let (A, μ, α) be a regular Hom-associative algebra. Then $(A, \alpha^{-1}\mu)$ is an associative algebra.

(3) Let (A, μ, α) be a Hom-associative algebra. If I is a Hom–associative ideal of A, then $(A/I, \overline{\mu}, \overline{\alpha})$, where $\overline{\mu}(a + I) := \mu(a) + I, \overline{\alpha}(a + I) := \alpha(a) + I$ for all $a \in A$, is a Hom-associative algebra.

Let (M, α_M) be a regular Hom-module. Set $End_k(M) := \{f : M \to M \mid f \text{ is a linear map}\}$, μ_E is the composition of the endomorphism algebra. Then $(End_k(M), \mu_E)$ is an associative algebra. Define a linear map $\alpha_E : End_k(M) \to End_k(M)$ sending $f \in End_k(M)$ to $\alpha_M f \alpha_M^{-1}$. Clearly, α_E is an algebra endomorphism. By Lemma 2.3(1), $(End_k(M), \alpha_E\mu_E, \alpha_E)$ is a Hom-associative algebra. In the following, we always set $End_k(M)_\alpha := (End_k(M), \alpha_E\mu_E, \alpha_E)$ if no confusions arise. Immediately, we have the following basic observations.

Lemma 2.4. *Let (A, μ, α) be a Hom-associative algebra and (M, α_M) a regular Hom-module. If there exists a Hom-associative algebra morphism $f : A \to End_k(M)_\alpha$, define $\cdot : A \otimes M \to M$ by $a \cdot m = f(a)(\alpha_M(m))$ for any $a \in A, m \in M$. Then $(M, \cdot, \alpha_M)$ is a Hom-associative module over A. Conversely, if $\alpha_M^2 = Id_M$ and $(M, \bullet, \alpha_M)$ is a Hom-associative module over A, then the linear map $g : A \to End_k(M)_\alpha$, given by $g(a)(m) = a \bullet \alpha_M(m)$ for any $a \in A, m \in M$, is a Hom-associative algebra morphism.*

Proof. First, we show that $(M, \cdot, \alpha_M)$ is a Hom-associative module over A. Since $f : A \to End_k(M)_\alpha$ is a Hom-associative algebra morphism, for any $a, b \in A$, $m \in M$, we have

$$f(1_A) = Id_{End_k(M)}, f(\alpha(a)) = \alpha_E(f(a)) = \alpha_M f(a)\alpha_M^{-1},$$
$$f(ab) = \alpha_E(f(a)f(b)) = \alpha_M(f(a)f(b))\alpha_M^{-1},$$

and then

$$1_A \cdot m = f(1_A)(\alpha_M(m)) = Id_{End_k(M)}(\alpha_M(m)) = \alpha_M(m),$$

$$\alpha(a)\cdot\alpha_M(m)=f(\alpha(a))(\alpha_M(\alpha_M(m)))=\alpha_M f(a)\alpha_M^{-1}(\alpha_M^2(m))=\alpha_M f(a)\alpha_M(m)=\alpha_M(a\cdot m),$$
$$\begin{aligned}\alpha(a)\cdot(b\cdot m)&=\alpha(a)\cdot(f(b)\alpha_M(m))=f(\alpha(a))(\alpha_M f(b)\alpha_M(m))=\alpha_M f(a)\alpha_M^{-1}\alpha_M f(b)\alpha_M(m)\\&=\alpha_M f(a)f(b)\alpha_M(m)=\alpha_M(f(a)f(b))\alpha_M^{-1}\alpha_M^2(m)\\&=f(ab)(\alpha_M^2(m))=(ab)\cdot\alpha_M(m).\end{aligned}$$

Thus, $(M,\cdot,\alpha_M)$ is a Hom-associative module over A.

Next, we prove that g is a Hom-associative algebra morphism. Note that $(M,\bullet,\alpha_M)$ is a Hom-associative module over A, for any $a,b\in A$, $m\in M$, we have

$$\begin{aligned}g(1_A)(m)&=1_A\bullet\alpha_M(m)=\alpha_M^2(m)=m(\because\ \alpha_M^2=Id_M),\\ \alpha_E(g(a)g(b))(m)&=\alpha_M(g(a)g(b))\alpha_M^{-1}(m)=\alpha_M g(a)(b\bullet\alpha_M\alpha_M^{-1}(m))=\alpha_M(a\bullet\alpha_M(b\bullet m))\\&=\alpha(a)\bullet\alpha_M\alpha_M(b\bullet m)=\alpha(a)\bullet(b\bullet m)(\because\ \alpha_M^2=Id_M)\\&=(ab)\bullet\alpha_M(m)=g(ab)(m),\\ \alpha_E(g(a))(m)&=\alpha_M g(a)\alpha_M^{-1}(m)=\alpha_M(a\bullet\alpha_M\alpha_M^{-1}(m))=\alpha_M(a\bullet m)\\&=\alpha(a)\bullet\alpha_M(m)=g(\alpha(a))(m).\end{aligned}$$

Hence, g is a Hom-associative algebra morphism. □

Other examples and properties of Hom-associative algebras can be found in [16] and the references therein.

Definition 2.5. (1) A Hom-Lie algebra is a triple $(L,[\cdot,\cdot],\alpha)$, which consists of a k-module L, a bilinear map $[\cdot,\cdot]:L\otimes L\to L$ and a linear map $\alpha:L\to L$, satisfying

$$\begin{aligned}&\alpha([a,b])=[\alpha(a),\alpha(b)],\\&[a,b]=-[b,a],\\&[\alpha(a),[b,c]]+[\alpha(b),[c,a]]+[\alpha(c),[a,b]]=0,\end{aligned}$$

for all elements $a,b,c\in L$.

(2) A Hom-Lie algebra $(L,[\cdot,\cdot],\alpha)$ is said to be involutive (resp. regular) if $\alpha^2=Id$ (resp. α is bijective).

(3) Let $(L,[\cdot,\cdot],\alpha)$ and $(L',[\cdot,\cdot]',\alpha')$ be two Hom-Lie algebras. A linear map $f:L\to L'$ is called a homomorphism of Hom-Lie algebras if $\alpha'(f(a))=f(\alpha(a))$, and $f([a,b])=[f(a),f(b)]'$ for all $a,b\in L$.

(4) Let $(A,[\cdot,\cdot],\alpha)$ be a Hom-Lie algebra and (M,α_M) a Hom-module. We call M a (left) Hom-Lie module over A if the following holds.

(i) There is a bilinear map $[\cdot,\cdot]_M:A\otimes M\to M$ such that $\alpha_M([a,m]_M)=[\alpha(a),\alpha_M(m)]_M$;
(ii) $[\cdot,\cdot]_M$ satisfies the formula: $[[a,b],\alpha_M(m)]_M=[\alpha(a),[b,m]_M]_M-[\alpha(b),[a,m]_M]_M$,

for all elements $a,b\in A,m\in M$.

Lemma 2.6. *(1) [24] Let $(L,[\cdot,\cdot])$ be a Lie algebra and $\alpha:L\to L$ a Lie algebra morphism. Then $(L,[\cdot,\cdot]_\alpha:=\alpha[\cdot,\cdot],\alpha)$ is a Hom-Lie algebra.*

(2) Let $(L,[\cdot,\cdot],\alpha)$ be a regular Hom-Lie algebra. Then $(L,\alpha^{-1}[\cdot,\cdot])$ is a Lie algebra.

As is noted in [17], given a Hom-associative algebra (A, μ, α), the triple $(A, [\cdot,\cdot]_L, \alpha)$ is a Hom-Lie algebra, where

$$[a, b]_L := ab - ba,$$

for all $a, b \in A$. We denote this Hom-Lie algebra by $A_L := (A_L, [\cdot,\cdot]_L, \alpha)$.

Let us recall the definition of a Hom-Poisson algebra.

Definition 2.7. A Hom-Poisson algebra is a quadruple $(A, \mu, [\cdot,\cdot], \alpha)$ consisting of a Hom-module (A, α), bilinear maps $\mu : A \otimes A \to A$ and $[\cdot,\cdot] : A \otimes A \to A$, called the Hom-Poisson bracket, satisfying

(i) (A, μ, α) is a commutative Hom-associative algebra;

(ii) $(A, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra;

(iii) For all $a, b, c \in A$,

$$[\alpha(a), bc] = \alpha(b)[a, c] + [a, b]\alpha(c). \tag{2.1}$$

Moreover, we call A an involutive (resp. regular) Hom-Poisson algebra provided that $\alpha^2 = Id$ (resp. α is bijective).

By the anti-symmetry of the Hom-Poisson bracket $[\cdot,\cdot]$, The formula (2.1) can be reformulated equivalently as

$$[ab, \alpha(c)] = \alpha(a)[b, c] + [a, c]\alpha(b).$$

Let $(A, \cdot, [\cdot,\cdot]_A, \alpha_A)$ and $(B, \bullet, [\cdot,\cdot]_B, \alpha_B)$ be two Hom-Poisson algebras. A linear map $f : A \to B$ is called a homomorphism of Hom-Poisson algebras if $\alpha_B(f(a)) = f(\alpha_A(a))$, $f(a \cdot b) = f(a) \bullet f(b)$ and $f([a, b]_A) = [f(a), f(b)]_B$ for all $a, b \in A$.

Lemma 2.8. *(1) [25] Let $(A, \mu, [\cdot,\cdot])$ be a Poisson algebra and $\alpha : A \to A$ a Poisson algebra morphism. Then $(A, \mu_\alpha := \alpha\mu, [\cdot,\cdot]_\alpha := \alpha[\cdot,\cdot], \alpha)$ is a Hom-Poisson algebra.*

(2) Let $(A, \mu, [\cdot,\cdot], \alpha)$ be a regular Hom-Poisson algebra. Then $(A, \alpha^{-1}\mu, \alpha^{-1}[\cdot,\cdot])$ is a Poisson algebra.

Lemma 2.9. *Let $(A, \mu, [\cdot,\cdot], \alpha)$ be a Hom-Poisson algebra. Then $(A^{op} := A, \mu_{op}, [\cdot,\cdot]_{op}, \alpha_{op})$ is also a Hom-Poisson algebra, where*

$$\begin{aligned} \mu_{op}(a \otimes b) &:= \mu\tau(a \otimes b) = \mu(a \otimes b), \\ [\cdot,\cdot]_{op}(a \otimes b) &:= [\cdot,\cdot]\tau(a \otimes b) = -[\cdot,\cdot](a \otimes b), \\ \alpha_{op} &:= \alpha, \end{aligned}$$

for any elements $a, b \in A$.

Lemma 2.10. *[25] Let $(A, \mu_A, [\cdot,\cdot]_A, \alpha_A)$ and $(B, \mu_B, [\cdot,\cdot]_B, \alpha_B)$ be Hom-Poisson algebras. Define the linear maps $\alpha : A \otimes B \to B \otimes A$ and $\mu, [\cdot,\cdot] : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ by the following ways:*

$$\begin{aligned} \alpha &:= \alpha_A \otimes \alpha_B, \\ \mu(a_1 \otimes b_1, a_2 \otimes b_2) &:= \mu_A(a_1, a_2) \otimes \mu_B(b_1, b_2), \\ [a_1 \otimes b_1, a_2 \otimes b_2] &:= [a_1, a_2]_A \otimes \mu_B(b_1, b_2) + \mu_A(a_1, a_2) \otimes [b_1, b_2]_B \end{aligned}$$

for all $a_i \in A, b_i \in B, i = 1, 2$. Then $(A \otimes B, \mu, [\cdot,\cdot], \alpha)$ is a Hom-Poisson algebra.

Example 2.11. *Let $(k[x,y],\mu)$ be the commutative polynomial algebra in two variables. Define a Poisson structure on $A := k[x,y]$ by setting*

$$[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

for all $f,g \in A$, then A is a Poisson algebra. Define an algebra endomorphism $\alpha : A \to A$ on the affine plane A by setting

$$\alpha(x) = y \quad \text{and} \quad \alpha(y) = -x.$$

It is easy to check that α is a Poisson algebra morphism with $\alpha^4 = Id$. By Lemma 2.8(1), $(A, \alpha\mu, \alpha[\cdot,\cdot], \alpha)$ is a Hom-Poisson algebra.

In the following, we will consider Hom-Poisson modules over Hom-Poisson algebras.

Definition 2.12. Let $(A,\mu,[\cdot,\cdot],\alpha)$ be a Hom-Poisson algebra and (M,α_M) a Hom-module. We call M a (left) Hom-Poisson module over A provided that

(i) There exists a bilinear map $\cdot : A \otimes M \to M$ such that $(M,\cdot,\alpha_M)$ is a Hom-associative module over the Hom-associative algebra (A,μ,α);
(ii) There is a bilinear map $[\cdot,\cdot]_M : A \otimes M \to M$ such that $(M,[\cdot,\cdot]_M,\alpha_M)$ is a Hom-Lie module over the Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$;
(iii) The bilinear map $\cdot$ is compatible with the bracket $[\cdot,\cdot]_M$. That is, we have

$$[ab, \alpha_M(m)]_M = \alpha(b)\cdot[a,m]_M + \alpha(a)\cdot[b,m]_M;$$
$$[a,b]\cdot\alpha_M(m) = [\alpha(a), b\cdot m]_M - \alpha(b)\cdot[a,m]_M,$$

for all $a,b \in A, m \in M$.

In addition, if A is an involutive (resp. regular) Hom-Poisson algebra, then the Hom-Poisson module (M,α_M) is called involutive (resp. regular) if $\alpha_M^2 = Id$ (resp. α_M is bijective).

Similar to the Lemmas 2.4 and 2.8, we have the following remarks, the proofs of which are left as easy exercises to the reader.

Remark 2.13. Let $(A,\mu,[\cdot,\cdot],\alpha)$ be a regular Hom-Poisson algebra and $(M,\mu_M,[\cdot,\cdot]_M,\alpha_M)$ a Hom-Poisson module over A. Suppose that $\alpha_M : M \to M$ is a linear isomorphism. Then $(M,(\alpha_M)^{-1}\mu_M,(\alpha_M)^{-1}[\cdot,\cdot]_M)$ is a Poisson module over $(A,\alpha^{-1}\mu,\alpha^{-1}[\cdot,\cdot])$.

Remark 2.14. Let $(A,\mu,[\cdot,\cdot],\alpha)$ be a Hom-Poisson algebra and (M,α_M) a regular Hom-module. If there are a Hom-associative algebra morphism γ and a Hom-Lie algebra morphism δ from A into $(End_k(M)_\alpha, * := \alpha_E\mu_E, \alpha_E)$, such that

$$\gamma([a,b]) = \delta(a)*\gamma(b) - \gamma(b)*\delta(a),$$
$$\delta(ab) = \gamma(a)*\delta(b) + \gamma(b)*\delta(a),$$

for all $a,b \in A$. Define $\cdot : A\otimes M \to M$ by $a\cdot m = \gamma(a)(\alpha_M(m))$, and $[\cdot,\cdot]_M : A\otimes M \to M$ by $[a,m]_M = \delta(a)(\alpha_M(m))$ for any $a \in A, m \in M$. Then $(M,\cdot,[\cdot,\cdot]_M,\alpha_M)$ is a Hom-Poisson module.

Remark 2.15. Let $(A, \mu, [\cdot,\cdot], \alpha)$ be a Hom-Poisson algebra and (M, α_M) an involutive Hom-module. If $(M, \cdot, [\cdot,\cdot]_M, \alpha_M)$ is a Hom-Poisson module, define $\gamma : A \to End_k(M)_\alpha := (End_k(M), * := \alpha_E\mu_E, \alpha_E)$ by $\gamma(a)(m) = a \cdot \alpha_M(m)$, and $\delta : A \to End_k(M)_\alpha$ by $\delta(a)(m) = [a, \alpha_M(m)]_M$ for all $a \in A, m \in M$. Then γ is a Hom-associative algebra morphism, and δ is a Hom-Lie algebra morphism, such that

$$\gamma([a,b]) = \delta(a) * \gamma(b) - \gamma(b) * \delta(a),$$
$$\delta(ab) = \gamma(a) * \delta(b) + \gamma(b) * \delta(a),$$

for all $a, b \in A$.

3. Universal enveloping Hom-algebras of Hom-Poisson algebras

In this section, we study universal enveloping Hom-algebras of Hom-Poisson algebras.

3.1. Definition and examples

The universal enveloping algebra of an ordinary Poisson algebra is given in [18]. Our aim is to generalize the definition to the Hom-setting.

Definition 3.1. Given a Hom-Poisson algebra $(A, \mu, [\cdot,\cdot], \alpha)$, let $(U_{eh}(A), \cdot, \alpha_U, \eta, \theta)$ be a quintuple, which has property **P** described as

(P1) $(U_{eh}(A), \cdot, \alpha_U)$ is a Hom-associative algebra and $\eta : A \to U_{eh}(A)$ is a Hom-associative algebra morphism;
(P2) $\theta : A \to U_{eh}(A)_L$ is a Hom-Lie algebra morphism;
(P3) $\eta([a,b]) = \theta(a) \cdot \eta(b) - \eta(b) \cdot \theta(a)$, and
(P4) $\theta(ab) = \eta(a) \cdot \theta(b) + \eta(b) \cdot \theta(a)$ for all $a, b \in A$.

Then $(U_{eh}(A), \eta, \theta)$ is called the universal enveloping Hom-algebra of A if for any other quintuple $(D, \bullet, \alpha_D, \gamma, \delta)$ that satisfies property **P**, there exists a unique Hom-associative algebra morphism $\varphi : U_{eh}(A) \to D$, making the diagram

$$\begin{array}{ccc} (A, \mu, [\cdot,\cdot], \alpha) & \xrightarrow{\eta,\ \theta} & (U_{eh}(A), \cdot, \alpha_U) \\ & {\scriptstyle \gamma,\ \delta} \searrow \quad \swarrow {\scriptstyle \exists ! \varphi} & \\ & (D, \bullet, \alpha_D) & \end{array}$$

"bi-commute", i.e., $\varphi\eta = \gamma$ and $\varphi\theta = \delta$.

The relation about the universal enveloping Hom-algebra of a Hom-Poisson algebra and the universal enveloping algebra of a Poisson algebra is discussed in the following result, which shows that the universal enveloping algebra deforms into the universal enveloping Hom-algebra via an algebra homomorphism.

Proposition 3.2. *Let $(A, \mu, [\cdot,\cdot])$ be a Poisson algebra and $(U(A), \mu')$ the universal enveloping algebra of A. Assume that $\alpha : A \to A$ is a Poisson algebra isomorphism. Then there exists an algebra homomorphism $\alpha_U : U(A) \to U(A)$, such that $(U(A), \alpha_U\mu', \alpha_U)$ is the universal enveloping Hom-algebra of $(A, \alpha\mu, \alpha[\cdot,\cdot], \alpha)$.*

Proof. Let $(A, \mu, [\cdot, \cdot])$ be a Poisson algebra. The universal enveloping algebra $(U(A), \mu')$ of A can be constructed explicitly. Let $m_A = \{m_a : a \in A\}$ and $h_A = \{h_a : a \in A\}$ be two copies of the vector space A endowed with two linear isomorphisms $m : A \to m_A$ sending a to m_a and $h : A \to h_A$ sending a to h_a. Then $(U(A), \mu')$ is defined to be the quotient algebra of the free algebra generated by m_A and h_A, subject to the following relations:

(i) $m_{1_A} = 1_{U(A)}, \quad m_{ab} = m_a m_b$,
(ii) $h_{[a,b]} = h_a h_b - h_b h_a$,
(iii) $m_{[a,b]} = h_a m_b - m_b h_a$,
(iv) $h_{ab} = m_a h_b + m_b h_a$

for all elements $a, b \in A$. Define (η, θ) as follows:

$$\eta : A \to U(A), \qquad \eta(a) = m_a;$$
$$\theta : A \to U(A)_L, \qquad \theta(a) = h_a.$$

By the section 2 of [21], $(U(A), \mu', \eta, \theta)$ is the universal enveloping algebra of A.

Suppose that $\alpha : A \to A$ is a Poisson algebra isomorphism. By Lemma 2.8, $(A_\alpha := A, \cdot := \alpha\mu, [\cdot, \cdot]_\alpha := \alpha[\cdot, \cdot], \alpha)$ is a Hom-Poisson algebra. Let $\alpha_U : U(A) \to U(A)$ be an algebra homomorphism determined by

$$\alpha_U(m_a) = m_{\alpha(a)}, \alpha_U(h_a) = h_{\alpha(a)}$$

for any element $a \in A$. Then $(U(A)_{\alpha_U} := U(A), \bullet := \alpha_U \mu', \alpha_U)$ is the universal enveloping Hom-algebra of A_α by the following steps.

Step 1: α_U is a well-defined algebra homomorphism. It suffices to prove the following equations for $a, b \in A$:

$$\begin{aligned}
\alpha_U(m_{1_A} - 1_{U(A)}) &= 0, \\
\alpha_U(m_{ab} - m_a m_b) &= 0, \\
\alpha_U(h_{[a,b]} - (h_a h_b - h_b h_a)) &= 0, \\
\alpha_U(m_{[a,b]} - (h_a m_b - m_b h_a)) &= 0, \\
\alpha_U(h_{ab} - (m_a h_b + m_b h_a)) &= 0,
\end{aligned}$$

which follows from

$$\alpha_U(m_{1_A} - 1_{U(A)}) = m_{\alpha(1_A)} - 1_{U(A)} = m_{1_A} - 1_{U(A)} = 0,$$

$$\alpha_U(m_{ab} - m_a m_b) = m_{\alpha(ab)} - \alpha_U(m_a)\alpha_U(m_b) = m_{\alpha(a)\alpha(b)} - m_{\alpha(a)} m_{\alpha(b)} = 0,$$

$$\begin{aligned}
\alpha_U(h_{[a,b]} - (h_a h_b - h_b h_a)) =& h_{\alpha([a,b])} - (\alpha_U(h_a)\alpha_U(h_b) - \alpha_U(h_b)\alpha_U(h_a)) \\
=& h_{[\alpha(a),\alpha(b)]} - (h_{\alpha(a)} h_{\alpha(b)} - h_{\alpha(b)} h_{\alpha(a)}) = 0,
\end{aligned}$$

$$\alpha_U(m_{[a,b]} - (h_a m_b - m_b h_a)) = m_{\alpha([a,b])} - (\alpha_U(h_a)\alpha_U(m_b) - \alpha_U(m_b)\alpha_U(h_a))$$

$$=m_{[\alpha(a),\alpha(b)]}-(h_{\alpha(a)}m_{\alpha(b)}-m_{\alpha(b)}h_{\alpha(a)})=0$$

and

$$\begin{aligned}\alpha_U(h_{ab}-(m_ah_b+m_bh_a)) =&h_{\alpha(ab)}-(\alpha_U(m_a)\alpha_U(h_b)+\alpha_U(m_b)\alpha_U(h_a))\\ =&h_{\alpha(a)\alpha(b)}-(m_{\alpha(a)}h_{\alpha(b)}+m_{\alpha(b)}h_{\alpha(a)})=0.\end{aligned}$$

Step 2: $(U(A)_{\alpha_U},\bullet,\alpha_U,\eta,\theta)$ satisfies property **P**.

By step 1, α_U is an algebra homomorphism. Then by Lemma 2.3, $(U(A)_{\alpha_U},\bullet,\alpha_U)$ is a Hom-associative algebra. Moreover, for any $a,b\in A$, we have

$$\begin{aligned}&\eta(1_A)=m_{1_A}=1_{U(A)},\\ &\alpha_U\eta(a)=\alpha_U(m_a)=m_{\alpha(a)}=\eta\alpha(a),\\ &\alpha_U\theta(a)=\alpha_U(h_a)=h_{\alpha(a)}=\theta\alpha(a),\end{aligned}$$

$$\begin{aligned}\eta(a\cdot b)=&\eta(\alpha(ab))=\eta(\alpha(a)\alpha(b))=\eta(\alpha(a))\eta(\alpha(b))=m_{\alpha(a)}m_{\alpha(b)}\\ =&\alpha_U(m_a)\alpha_U(m_b)=\alpha_U(m_am_b)=m_a\bullet m_b=\eta(a)\bullet\eta(b),\end{aligned}$$

$$\begin{aligned}\theta([a,b]_\alpha)=&\theta(\alpha([a,b]))=\theta([\alpha(a),\alpha(b)])=\theta(\alpha(a))\theta(\alpha(b))-\theta(\alpha(b))\theta(\alpha(a))\\ =&h_{\alpha(a)}h_{\alpha(b)}-h_{\alpha(b)}h_{\alpha(a)}=\alpha_U(h_a)\alpha_U(h_b)-\alpha_U(h_b)\alpha_U(h_a)\\ =&h_a\bullet h_b-h_b\bullet h_a=[h_a,h_b]_L=[\theta(a),\theta(b)]_L,\end{aligned}$$

$$\begin{aligned}\eta([a,b]_\alpha)=&\eta(\alpha([a,b]))=\eta([\alpha(a),\alpha(b)])=\theta(\alpha(a))\eta(\alpha(b))-\eta(\alpha(b))\theta(\alpha(a))\\ =&h_{\alpha(a)}m_{\alpha(b)}-m_{\alpha(b)}h_{\alpha(a)}=\alpha_U(h_a)\alpha_U(m_b)-\alpha_U(m_b)\alpha_U(h_a)\\ =&h_a\bullet m_b-m_b\bullet h_a=\theta(a)\bullet\eta(b)-\eta(b)\bullet\theta(a)\end{aligned}$$

and

$$\begin{aligned}\theta(a\cdot b)=&\theta(\alpha(ab))=\theta(\alpha(a)\alpha(b))=\eta(\alpha(a))\theta(\alpha(b))+\eta(\alpha(b))\theta(\alpha(a))\\ =&m_{\alpha(a)}h_{\alpha(b)}+m_{\alpha(b)}h_{\alpha(a)}=\alpha_U(m_a)\alpha_U(h_b)+\alpha_U(m_b)\alpha_U(h_a)\\ =&m_a\bullet h_b+m_b\bullet h_a=\eta(a)\bullet\theta(b)+\eta(b)\bullet\theta(a).\end{aligned}$$

Thus, $(U(A)_{\alpha_U},\bullet,\alpha_U,\eta,\theta)$ satisfies property **P**.

Step 3: The universal property is true. For any Hom-associative algebra $(D,*,\alpha_D,\gamma,\delta)$ satisfying property **P**, define an algebra homomorphism $\varphi: U(A)_{\alpha_U}\to D$ by the rules: $\varphi(m_a):=\gamma(a),\varphi(h_a)=\delta(a)$. We show that φ is well-defined. Note that the Poisson algebra homomorphism α is bijective, then the relations of $U(A)_{\alpha_U}$ become the following relations:

(i) $m_{1_{A_\alpha}}=1_{U(A)_{\alpha_U}},\quad m_{a\cdot b}=m_a\bullet m_b$,
(ii) $h_{[a,b]_\alpha}=h_a\bullet h_b-h_b\bullet h_a$,
(iii) $m_{[a,b]_\alpha}=h_a\bullet m_b-m_b\bullet h_a$,
(iv) $h_{a\cdot b}=m_a\bullet h_b+m_b\bullet h_a$

Then for any $a, b \in A$, we have

$$\begin{aligned}
\varphi(m_{1_{A_\alpha}}) &= \gamma(1_{A_\alpha}) = 1_D = \varphi(1_{U(A)_{\alpha_U}}),\\
\varphi(m_{a\cdot b}) &= \gamma(a \cdot b) = \gamma(a) * \gamma(b) = \varphi(m_a) * \varphi(m_b) = \varphi(m_a \bullet m_b),\\
\varphi(h_{[a,b]_\alpha}) &= \delta([a, b]_\alpha) = \delta(a) * \delta(b) - \delta(b) * \delta(a)\\
&= \varphi(h_a) * \varphi(h_b) - \varphi(h_b) * \varphi(h_a) = \varphi(h_a \bullet h_b - h_b \bullet h_a),\\
\varphi(m_{[a,b]_\alpha}) &= \gamma([a, b]_\alpha) = \delta(a) * \gamma(b) - \gamma(b) * \delta(a)\\
&= \varphi(h_a) * \varphi(m_b) - \varphi(m_b) * \varphi(h_a) = \varphi(h_a \bullet m_b - m_b \bullet h_a),\\
\varphi(h_{a\cdot b}) &= \delta(a \cdot b) = \gamma(a) * \delta(b) + \gamma(b) * \delta(a)\\
&= \varphi(m_a) * \varphi(h_b) + \varphi(m_b) * \varphi(h_a) = \varphi(m_a \bullet h_b + m_b \bullet h_a).
\end{aligned}$$

Hence $\varphi : U(A)_{\alpha_U} \to D$ is a well-defined algebra morphism. Further,

$$\varphi\alpha_U(m_a) = \varphi(m_{\alpha(a)}) = \gamma(\alpha(a)) = \gamma\alpha(a) = \alpha_D\gamma(a) = \alpha_D\varphi(m_a),$$

$$\varphi\alpha_U(h_a) = \varphi(h_{\alpha(a)}) = \delta(\alpha(a)) = \delta\alpha(a) = \alpha_D\delta(a) = \alpha_D\varphi(h_a).$$

Therefore, $\varphi\alpha_U = \alpha_D\varphi$, which means φ is a Hom-associative algebra morphism. By the construction of φ, we have $\varphi\eta = \gamma$ and $\varphi\theta = \delta$. Note that $U(A)_{\alpha_U}$ is generated by $m(A)$ and $h(A)$. Since two Hom-associative algebra homomorphisms that coincide on generators are necessarily identical, the uniqueness of φ is true, as claim.

□

Corollary 3.3. *Given a regular Hom-Poisson algebra $(A, \mu, [\cdot, \cdot], \alpha)$, its universal enveloping Hom-algebra exists and is unique up to isomorphisms.*

Proof. The uniqueness of the universal enveloping Hom-algebra of A, up to isomorphisms, follows immediately from the universal mapping property. Hence, it suffices to prove that the universal enveloping Hom-algebra of A exists.

Let $(A, \mu, [\cdot, \cdot], \alpha)$ be a regular Hom-Poisson algebra. Note that α is bijective, by Lemma 2.8(2), $(A_{\alpha^{-1}} := A, \alpha^{-1}\mu, \alpha^{-1}[\cdot, \cdot])$ is a Poisson algebra. Suppose that $(U(A), \mu')$ is the universal enveloping algebra of $A_{\alpha^{-1}}$. Note that $\alpha : A \to A$ is a Poisson algebra isomorphism. By Lemma 2.8(1), $(A, \alpha(\alpha^{-1}\mu), \alpha(\alpha^{-1}[\cdot, \cdot]), \alpha)$ is a Hom-Poisson algebra. By Proposition 3.2, there exists an algebra homomorphism $\alpha_U : U(A) \to U(A)$, such that $(U(A), \alpha_U\mu', \alpha_U)$ is the universal enveloping Hom-algebra of $(A, \alpha(\alpha^{-1}\mu), \alpha(\alpha^{-1}[\cdot, \cdot]), \alpha)$, which exactly is $(A, \mu, [\cdot, \cdot], \alpha)$. □

Example 3.4. *Let $(A := k[x, y], \mu, [\cdot, \cdot])$ be the Poisson polynomial algebra in two variables. Here, $[\cdot, \cdot] : A \otimes A \to A$ is defined by*

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

for all $f, g \in A$. By the Example 4 of [18], the Weyl algebra $(A_2, \mu', \eta, \theta)$ is the universal enveloping algebra of $(A, \mu, [\cdot, \cdot])$, where A_2 is the associative algebra given by generators x_1, x_2, y_1, y_2 and defined relations

$$x_1x_2 = x_2x_1, \qquad y_1y_2 = y_2y_1, \qquad y_ix_j - x_jy_i = \delta_{ij}, i, j = 1, 2.$$

The linear maps η and θ are given by

$$\eta(x) = y_2, \qquad \eta(y) = x_1, \qquad \theta(x) = y_1, \qquad \theta(y) = x_2.$$

Define an algebra endomorphism $\alpha : A \to A$ on the affine plane A by setting

$$\alpha(x) = y \quad \text{and} \quad \alpha(y) = -x.$$

By Example 2.11, α is a Poisson algebra automorphism and $(A_\alpha := A, \alpha\mu, \alpha[\cdot,\cdot], \alpha)$ is a Hom-Poisson algebra. Define an associative algebra homomorphism $\alpha_{A_2} : A_2 \to A_2$ such that

$$\alpha_{A_2}(x_1) = -y_2, \qquad \alpha_{A_2}(x_2) = -y_1, \qquad \alpha_{A_2}(y_1) = x_2, \qquad \alpha_{A_2}(y_2) = x_1.$$

Then by the method in Proposition 3.2, $(A_2, \alpha_{A_2}\mu', \alpha_{A_2})$ is the universal enveloping Hom-algebra of $(A_\alpha, \alpha\mu, \alpha[\cdot,\cdot], \alpha)$.

3.2. Some basis properties

Note that the universal enveloping Hom-algebra exists for any regular Hom-Poisson algebra, from now on, we always consider regular Hom-Poisson algebras. In particular, involutive Hom-Poisson algebras are also considered.

Proposition 3.5. *Let $(A, \mu_A, [\cdot,\cdot]_A, \alpha_A)$ and $(B, \mu_B, [\cdot,\cdot]_B, \alpha_B)$ be regular Hom-Poisson algebras. Then we have*

(i) $U_{eh}(A^{op}) \cong (U_{eh}(A))^{op}$,
(ii) $U_{eh}(A \otimes B) \cong U_{eh}(A) \otimes U_{eh}(B)$,
(iii) $U_{eh}(A \otimes A^{op}) \cong U_{eh}(A) \otimes U_{eh}(A^{op})$.

Proof. We only prove (ii) here. We can get (i) from the same fashion, and (iii) is a corollary of (i) and (ii). Note that $(A, \mu_A, [\cdot,\cdot]_A, \alpha_A)$ is a regular Hom-Poisson algebra, we know α_A is invertible. By Lemma 2.8(2), $(A, {\alpha_A}^{-1}\mu_A, {\alpha_A}^{-1}[\cdot,\cdot]_A)$ is a Poisson algebra. Suppose that $(U(A), \mu_{U(A)})$ is the universal enveloping algebra of $(A, {\alpha_A}^{-1}\mu_A, {\alpha_A}^{-1}[\cdot,\cdot]_A)$, where $U(A)$ is generated by m_A and h_A, subject to some relations. Let $\alpha_{U(A)} : U(A) \to U(A)$ be an algebra homomorphism determined by

$$\alpha_{U(A)}(m_a) = m_{\alpha_A(a)}, \alpha_{U(A)}(h_a) = h_{\alpha_A(a)}$$

for any element $a \in A$. then $(U(A), \alpha_{U(A)}\mu_{U(A)}, \alpha_{U_A})$ is the universal enveloping Hom-algebra of $(A, \mu_A, [\cdot,\cdot]_A, \alpha_A)$. Similarly, we get $(U(B), \alpha_{U(B)}\mu_{U(B)}, \alpha_{U_B})$ is the universal enveloping Hom-algebra of $(B, \mu_B, [\cdot,\cdot]_B, \alpha_B)$. Here, $(U(B), \mu_{U(B)})$ is the universal enveloping algebra of $(B, {\alpha_B}^{-1}\mu_B, {\alpha_B}^{-1}[\cdot,\cdot]_B)$.

On the one hand, $(U(A)\otimes U(B), (\mu_{U(A)}\otimes\mu_{U(B)})(Id\otimes\tau_{U(A),U(B)}\otimes Id))$ is the universal enveloping algebra of $(A\otimes B, ({\alpha_A}^{-1}\mu_A\otimes{\alpha_B}^{-1}\mu_B)(Id\otimes\tau_{A,B}\otimes Id), ({\alpha_A}^{-1}[\cdot,\cdot]_A\otimes{\alpha_B}^{-1}\mu_B+{\alpha_A}^{-1}\mu_A\otimes{\alpha_B}^{-1}[\cdot,\cdot]_B)(Id\otimes\tau_{A,B}\otimes Id))$. Set $\alpha_{U_A\otimes U_B} := \alpha_{U_A}\otimes\alpha_{U_B}$, then $(U(A)\otimes U(B), \alpha_{U_A\otimes U_B}(\mu_{U(A)}\otimes\mu_{U(B)})(Id\otimes\tau_{U(A),U(B)}\otimes Id), \alpha_{U_A\otimes U_B})$ is the universal enveloping Hom-algebra of $(A\otimes B, (\mu_A\otimes\mu_B)(Id\otimes\tau_{A,B}\otimes Id), ([\cdot,\cdot]_A\otimes\mu_B+\mu_A\otimes[\cdot,\cdot]_B)(Id\otimes\tau_{A,B}\otimes Id), \alpha_A\otimes\alpha_B)$.

On the other hand,

$$\begin{aligned}&(U(A), \alpha_{U(A)}\mu_{U(A)}, \alpha_{U_A}) \otimes (U(B), \alpha_{U(B)}\mu_{U(B)}, \alpha_{U_B})\\ \cong&(U(A) \otimes U(B), (\alpha_{U(A)}\mu_{U(A)} \otimes \alpha_{U(B)}\mu_{U(B)})(Id \otimes \tau_{U(A),U(B)} \otimes Id), \alpha_{U_A} \otimes \alpha_{U_B}),\end{aligned}$$

which is equal to $(U(A)\otimes U(B), \alpha_{U_A\otimes U_B}(\mu_{U(A)}\otimes\mu_{U(B)})(Id\otimes\tau_{U(A),U(B)}\otimes Id), \alpha_{U_A\otimes U_B})$. Hence $U_{eh}(A\otimes B) \cong U_{eh}(A) \otimes U_{eh}(B)$. □

Recall that a Hom-Poisson algebra $(A, \mu, [\cdot, \cdot], \alpha)$ is involutive if $\alpha^2 = Id$. Generally, if there exists $t > 0$ such that $\alpha^t = Id$, then we call A is t-involutive. Particularly, when $t = 1$, A is a Poisson algebra. When $t = 2$, A is an involutive Hom-Poisson algebra.

Proposition 3.6. *Let $A := (A, \mu, [\cdot, \cdot], \alpha)$ be a t-involutive Hom-Poisson algebra with $t > 0$.*

(a) Let $(B, \cdot, \alpha_B)$ be a Hom-associative algebra, $f : (A, \mu, \alpha) \to (B, \cdot, \alpha_B)$ a homomorphism of Hom-associative algebras and $g : (A, [\cdot, \cdot], \alpha) \to (B_L, [\cdot, \cdot]_L, \alpha_B)$ a homomorphism of Hom-Lie algebras. Suppose that E is the Hom-associative subalgebra of B generated by $f(A)$ and $g(A)$. Then E is t-involutive.

(b) The universal enveloping Hom-algebra $(U_{eh}(A), \eta, \theta)$ of A is t-involutive.

(c) In order to verify the universal property of $U_{eh}(A)$ in Definition 3.1, we only need to consider t-involutive Hom-associative algebras $(D, \bullet, \alpha_D, \gamma, \delta)$.

Proof. (a) Let

$$C := \{b \in B \mid \alpha_B^t(b) = b\}.$$

Note that A is t-involutive, f is a homomorphism of Hom-associative algebras and g is a homomorphism of Hom-Lie algebras. Then for any element $a \in A$, we have

$$\alpha_B^t(f(a)) = f(\alpha^t(a)) = f(a) \qquad \textit{and} \qquad \alpha_B^t(g(a)) = g(\alpha^t(a)) = g(a).$$

Thus $f(A)$ and $g(A)$ are contained in C. In order to prove (a), it remains to show C is a Hom-associative subalgebra of B, which means C contains E. Here E is the Hom-associative subalgebra of B generated by $f(A)$ and $g(A)$. Indeed, C is a submodule of B. For $b, c \in C$, by the formula $\alpha_B(bc) = \alpha_B(b)\alpha_B(c)$, we have

$$\alpha_B^t(bc) = \alpha_B^t(b)\alpha_B^t(c) = bc \qquad \text{and} \qquad \alpha_B^t(\alpha_B(b)) = \alpha_B(\alpha_B^t(b)) = \alpha_B(b),$$

and hence C is a Hom-associative subalgebra of B. Therefore E is t-involutive.

(b) By the universal property of $(U_{eh}(A), \eta, \theta)$, $U_{eh}(A)$ is the Hom-associative algebra generated by $\eta(A)$ and $\theta(A)$, and so (b) is a special case of (a).

(c) For any quintuple $(D, \bullet, \alpha_D, \gamma, \delta)$ satisfies property **P**, where D is a Hom-associative algebra. Let $C' := \{d \in D \mid \alpha_D^t(d) = d\}$ be a t-involutive Hom-associative subalgebra of D defined in the proof of (a). By (a), $\gamma(A)$ and $\delta(A)$ are contained in C' and thus γ(resp. δ) is the composition of a homomorphism $\gamma_{C'} : A \to C'$ (resp. $\delta_{C'} : A \to C'$) of Hom-associative algebras (resp. Hom-Lie algebras) with the inclusion $C' \hookrightarrow D$. Note that $(C', \gamma_{C'}, \delta_{C'})$ also satisfies property **P**. By the assumption, there is a homomorphism $\varphi_{C'} : U_{eh}(A) \to C'$ of Hom-associative algebras such that $\varphi_{C'}\eta = \gamma_{C'}$ and $\varphi_{C'}\theta = \delta_{C'}$. Then composing with the inclusion $C' \hookrightarrow D$, we obtain a homomorphism $\varphi : U_{eh}(A) \to D$ of Hom-associative algebras such that $\varphi\eta = \gamma$ and $\varphi\theta = \delta$. Note that $U_{eh}(A)$ is a t-involutive Hom-associative algebra by (b), similar to the previous proof, it is obvious to see the uniqueness of φ such that $\varphi\eta = \gamma$ and $\varphi\theta = \delta$, which completes the proof. □

Proposition 3.7. *Let $A := (A, \mu, [\cdot, \cdot], \alpha)$ be an involutive Hom-Poisson algebra and $U_{eh}(A) := (U_{eh}(A), \eta, \theta)$ the universal enveloping Hom-algebra of A. Then η is injective.*

Proof. Note that $(A, \mu, [\cdot, \cdot], \alpha)$ is an involutive Hom-Poisson module, define $\gamma : A \to End_k(A)_\alpha := (End_k(A), * := \alpha_E\mu_E, \alpha_E)$ by $\gamma(a)(m) = a \cdot \alpha(m)$, and $\delta : A \to End_k(A)_\alpha$ by $\delta(a)(m) = [a, \alpha(m)]_M$ for all

$a \in A, m \in M$. By Remark 2.15, γ is a Hom-associative algebra morphism, and δ is a Hom-Lie algebra morphism, such that

$$\gamma([a,b]) = \delta(a) * \gamma(b) - \gamma(b) * \delta(a),$$
$$\delta(ab) = \gamma(a) * \delta(b) + \gamma(b) * \delta(a),$$

for all $a, b \in A$. By Definition 3.1, there is a Hom-associative algebra morphism φ from $U_{eh}(A)$ into $End_k(A)_\alpha$ such that $\varphi\eta = \gamma$ and $\varphi\theta = \delta$. If $a \in ker(\eta)$, then $0 = \varphi\eta(a) = \gamma(a)$. Thus, $0 = \gamma(a)(1_A) = a \cdot \alpha(1_A) = a \cdot 1_A = \alpha(a)$. But $\alpha^2 = Id$, we have $a = 0$, as required. □

Now our main result is stated as follows.

Theorem 3.8. *Let $A := (A, \mu, [\cdot,\cdot], \alpha)$ be a regular Hom-Poisson algebra and $U_{eh}(A) := (U_{eh}(A), \alpha_U, \eta, \theta)$ the universal enveloping Hom-algebra of A.*

(1) If M is a regular Hom-associative module over $U_{eh}(A)$, then M is a regular Hom-Poisson module over A.

(2) Assume that A is involutive. If M is an involutive Hom-Poisson module over A, then M is an involutive Hom-associative module over $U_{eh}(A)$.

Proof. (1) If $(M, \circ, \alpha_M)$ is a regular Hom-associative module over $(U_{eh}(A), \alpha_U)$, define $\bullet : A \otimes M \to M$ by $a \bullet m = \eta(a) \circ m$, and $[\cdot,\cdot]_M : A \otimes M \to M$ by $[a,m]_M = \theta(a) \circ m$ for any $a \in A, m \in M$. In fact, for any $a, b \in A$, $m \in M$, we have

$$\begin{aligned}
1_A \bullet m &= \eta(1_A) \circ m = 1_{U_{eh}(A)} \circ m = \alpha_M(m),\\
\alpha_M(a \bullet m) &= \alpha_M(\eta(a) \circ m) = \alpha_U(\eta(a)) \circ \alpha_M(m) = \eta\alpha(a) \circ \alpha_M(m) = \alpha(a) \bullet \alpha_M(m),\\
(ab) \bullet \alpha_M(m) &= \eta(ab) \circ \alpha_M(m) = (\eta(a)\eta(b)) \circ \alpha_M(m) = \alpha_U(\eta(a)) \circ (\eta(b) \circ m)\\
&= \eta\alpha(a) \circ (b \bullet m) = \alpha(a) \bullet (b \bullet m),
\end{aligned}$$

$$\begin{aligned}
\alpha_M([a,m]_M) &= \alpha_M(\theta(a) \circ m) = \alpha_U(\theta(a)) \circ \alpha_M(m) = \theta\alpha(a) \circ \alpha_M(m) = [\alpha(a), \alpha_M(m)]_M,\\
[[a,b], \alpha_M(m)]_M &= \theta([a,b]) \circ \alpha_M(m) = (\theta(a)\theta(b) - \theta(b)\theta(a)) \circ \alpha_M(m)\\
&= \alpha_U(\theta(a)) \circ (\theta(b) \circ m) - \alpha_U(\theta(b)) \circ (\theta(a) \circ m)\\
&= \theta\alpha(a) \circ ([b,m]_M) - \theta\alpha(b) \circ ([a,m]_M) = [\alpha(a), [b,m]_M]_M - [\alpha(b), [a,m]_M]_M,
\end{aligned}$$

$$\begin{aligned}
[ab, \alpha_M(m)]_M &= \theta(ab) \circ \alpha_M(m) = (\eta(a)\theta(b) + \eta(b)\theta(a)) \circ \alpha_M(m)\\
&= \alpha_U(\eta(a)) \circ (\theta(b) \circ m) + \alpha_U(\eta(b)) \circ (\theta(a) \circ m)\\
&= \eta\alpha(a) \circ ([b,m]_M) + \eta\alpha(b) \circ ([a,m]_M) = \alpha(a) \bullet [b,m]_M + \alpha(b) \bullet [a,m]_M,\\
[a,b] \bullet \alpha_M(m) &= \eta([a,b]) \circ \alpha_M(m) = (\theta(a)\eta(b) - \eta(b)\theta(a)) \circ \alpha_M(m)\\
&= \alpha_U(\theta(a)) \circ (\eta(b) \circ m) - \alpha_U(\eta(b)) \circ (\theta(a) \circ m)\\
&= \theta\alpha(a) \circ (b \bullet m) - \eta\alpha(b) \circ ([a,m]_M) = [\alpha(a), b \bullet m]_M - \alpha(b) \bullet [a,m]_M.
\end{aligned}$$

Therefore, $(M, \bullet, [\cdot,\cdot]_M, \alpha_M)$ is a regular Hom-Poisson module over A.

(2) Assume that A is involutive, if M is an involutive Hom-Poisson module over A, by Remark 2.15, there exist a Hom-associative algebra morphism γ, and a Hom-Lie algebra morphism δ from A into $End_k(M)_\alpha := (End_k(M), * := \alpha_E\mu_E, \alpha_E)$, such that

$$\gamma([a,b]) = \delta(a) * \gamma(b) - \gamma(b) * \delta(a),$$
$$\delta(ab) = \gamma(a) * \delta(b) + \gamma(b) * \delta(a),$$

for all $a, b \in A$. By the definition of the universal enveloping Hom-algebra of A, there is a unique Hom-associative algebra morphism $\varphi : U_{eh}(A) \to End_k(M)_\alpha$, such that $\varphi\eta = \gamma$ and $\varphi\theta = \delta$. By Lemma 2.4, we have M is an involutive Hom-associative module over $U_{eh}(A)$. □

4. Conclusions

We first introduced universal enveloping Hom-algebras of Hom-Poisson algebras, and discussed their properties. Moreover, we proved that the category of involutive Hom-Poisson modules over an involutive Hom-Poisson algebra A is equivalent to the category of involutive Hom-associative modules over its universal enveloping Hom-algebra $U_{eh}(A)$.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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