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Research article

Weighted boundedness of multilinear integral operators for the endpoint cases

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Abstract: We prove the weighted boundedness for the multilinear operators associated to some integral operators for the endpoint cases. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

Keywords: multilinear operator; Littlewood-Paley operator; Marcinkiewicz operator; Bochner-Riesz operator; BMO space; Hardy space; Herz space
Mathematics Subject Classification: 42B20, 42B25

1. Introduction and preliminaries

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1, 3–6]. Let T be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$, a classical result of Coifman, Rochberg and Weiss stated that the commutator $[b, T]f = T(bf) - bTf$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [10], authors obtain the boundedness properties of the commutators for the extreme values of p. And note that $[b, T]$ is not bounded for the end point cases (that is $p = 1$ and $p = \infty$). In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see [8–11]). The purpose of this paper is to introduce some multilinear operators associated to certain non-convolution type integral operators and prove the weighted boundedness properties of the multilinear operators for the endpoint cases ($p = 1$ and $p = \infty$). In fact, we prove the endpoint boundedness of the multilinear operators only under the boundedness of the operators on Lebesgue spaces, that is the boundedness of the operators on $L^p(R^n)$ for $1 < p \leq \infty$ implies the boundedness of the multilinear operators from $L^\infty(R^n)$ to $BMO(R^n)$ for $p = \infty$ and from $H^1(R^n)$ to $L^1(R^n)$ for $p = 1$. Then, we apply the boundedness of the integral operator to some concrete including Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

First, let us introduce some preliminaries (see [7, 8, 14–16]). Throughout this paper, we denote by A_p the class of the Muckenhoupt weights for $1 \le p < \infty$ (see [9]). In this paper Q always stand for a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function b and a weight function w, let $w(Q) = \int_Q w(x)dx$, $b_Q = |Q|^{-1}\int_Q b(x)dx$ and $b_w^{\#}(x) = \sup\limits_{Q \ni x} w(Q)^{-1}\int_Q |b(y) - b_Q| w(y)dy$. b is said to belong to $BMO(w)$ if $b_w^{\#} \in L^{\infty}(w)$ and define $\|b\|_{BMO(w)} = \|b^{\#}\|_{L^{\infty}(w)}$, if $w = 1$, we denote $BMO(w) = BMO(R^n)$. We also define the weighted central BMO space by $CMO(w)$, which is the space of those functions $b \in L_{loc}(R^n)$ such that

$$\|b\|_{CMO(w)} = \sup_{r>1} w(Q(0,r))^{-1}\int_Q |b(x) - b_Q| w(x)dx < \infty.$$

It is well-known that (see [8, 9]

$$\|b\|_{CMO(w)} \approx \sup_{r>1}\inf_{c \in C} w(Q(0,r))^{-1}\int_Q |b(x) - c| w(x)dx.$$

Also, we give the concepts of the atom and weighted Hardy spaces $H^1(w)$. A function a is called an H^1 atom if there exists a cube Q such that a is supported on Q, $\|a\|_{L^2(w)} \le w(Q)^{-1/2}$ and $\int_{R^n} a(x)dx = 0$. It is well known that the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see [2, 9, 16, 17]).

For $k \in Z$, define $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_k$ the characteristic function of C_k for $k \ge 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 1. Let $1 < p < \infty$ and w_1, w_2 be two non-negative weight functions on R^n.

(1) The homogeneous weighted Herz space is defined by

$$\dot{K}_p(w_1, w_2; R^n) = \{f \in L_{loc}^p(R^n \setminus \{0\}) : \|f\|_{\dot{K}_p(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_p(w_1,w_2)} = \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p}\|f\chi_k\|_{L^p(w_2)};$$

(2) The nonhomogeneous weighted Herz space is defined by

$$K_p(w_1, w_2; R^n) = \{f \in L_{loc}^p(R^n) : \|f\|_{K_p(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{K_p(w_1,w_2)} = \sum_{k=0}^{\infty} [w_1(B_k)]^{1-1/p}\|f\tilde{\chi}_k\|_{L^p(w_2)};$$

(3) The homogeneous weighted Herz type Hardy space is defined by

$$H\dot{K}_p(w_1, w_2; R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_p(w_1, w_2; R^n)\},$$

where

$$\|f\|_{H\dot{K}_p(w_1,w_2)} = \|G(f)\|_{\dot{K}_p(w_1,w_2)};$$

(4) The nonhomogeneous weighted Herz type Hardy space is defined by

$$HK_p(w_1, w_2; R^n) = \{f \in S'(R^n) : G(f) \in K_p(w_1, w_2; R^n)\},$$

where

$$\|f\|_{HK_p(w_1,w_2)} = \|G(f)\|_{K_p(w_1,w_2)};$$

where $G(f)$ is the grand maximal function of f, that is

$$G(f)(x) = \sup_{\varphi\in K_m} \sup_{|x-y|<t} |f * \varphi_t(y)|,$$

where $K_m = \{\varphi \in S(R^n) : \sup_{x\in R^n,|\alpha|\le m}(1 + |u|)^{m+n}|D^\alpha\varphi(u)| \le 1\}$, $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$, m is a positive integer and $S(R^n)$ is the Schwartz class (see [18], p.88).

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 2. Let $1 < p < \infty$ and $w_1, w_2 \in A_1$. A function $a(x)$ on R^n is called a central $(n(1 - 1/p), p; w_1, w_2)$-atom (or a central $(n(1 - 1/p), p; w_1, w_2)$-atom of restrict type), if

1) supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \ge 1$);
2) $\|a\|_{L^p(w_2)} \le [w_1(B(0, r))]^{1/p-1}$;
3) $\int_{R^n} a(x)dx = 0$.

Lemma 1. (see [7,16]). Let $w_1, w_2 \in A_1$ and $1 < p < \infty$. A tempered distribution f belongs to $H\dot{K}_p(w_1, w_2; R^n)$(or $HK_p(w_1, w_2; R^n)$) if and only if there exist central $(n(1 - 1/p), p; w_1, w_2)$-atoms (or central $(n(1 - 1/p), p; w_1, w_2)$-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and

$$\|f\|_{H\dot{K}_p(w_1,w_2)}(\text{ or } \|f\|_{HK_p(w_1,w_2)}) \approx \sum_j |\lambda_j|.$$

Definition 3. Let $1 < p < \infty$ and w be a non-negative weight functions on R^n. We shall call $B_p(w)$ the space of those functions f on R^n such that

$$\|f\|_{B_p(w)} = \sup_{r>1}[w(Q(0, r))]^{-1/p}\|f\chi_{Q(0,r)}\|_{L^p(w)} < \infty.$$

2. Theorems

In this paper, we will consider a class of multilinear operators associated to some non-convolution type integral operators as following.

Let m_j be the positive integers($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and b_j be the locally integrable functions on R^n ($j = 1, \cdots, l$). Set

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha,$$

and

$$Q_{m_j+1}(b_j; x, y) = R_{m_j}(b_j; x, y) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} D^\alpha b_j(x)(x - y)^\alpha.$$

Let $F_t(x,y)$ define on $R^n \times R^n \times [0,+\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x,y)f(y)dy,$$

and

$$F_t^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m} F_t(x,y)f(y)dy,$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : \|h\| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^b(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear operator associated to F_t is defined by

$$T^b(f)(x) = \|F_t^b(f)(x)\|,$$

where F_t satisfies: for fixed $0 < \varepsilon \le 1$,

$$\|F_t(x,y)\| \le C|x-y|^{-n},$$

and

$$\|F_t(y,x) - F_t(z,x)\| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon},$$

if $2|y-z| \le |x-z|$. Let $T(f)(x) = \|F_t(f)(x)\|$. We also consider the variant of T^b, which is defined by

$$\tilde{T}^b(f)(x) = \|\tilde{F}_t^b(f)(x)\|,$$

where

$$\tilde{F}_t^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(b_j;x,y)}{|x-y|^m} F_t(x,y)f(y)dy.$$

Note that when $m = 0$, T^b is just the higher order commutators of T and b (see [1,12,13,20]). The operator $\tilde{T}^b$ is a variant of T^b, and it has not any form in the commutator, thus, it is a non-trivial variant of the commutator. It is well-known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3–6]). In [3], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the weighted boundedness properties of the multilinear operators T^A and $\tilde{T}^A$ for the extreme cases. In Section 4, some examples of theorems in this paper are given.

We shall prove the following theorems in Section 3.

Theorem 1. Let $w \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$. Suppose that T is bounded on $L^q(u)$ for any $1 < q \le \infty$ and $u \in A_1$. Then T^b is bounded from $L^\infty(w)$ to $BMO(w)$.

Theorem 2. Let $w \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$. Suppose that $\tilde{T}^b$ is bounded on $L^q(u)$ for any $1 < q \le \infty$ and $u \in A_1$. Then $\tilde{T}^b$ is bounded from $H^1(w)$ to $L^1(w)$.

Theorem 3. Let $1 < p < \infty$, $w \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$. Suppose that T is bounded on $L^q(u)$ for any $1 < q \le \infty$ and $u \in A_1$. Then T^b is bounded from $B_p(w)$ to $CMO(w)$.

Theorem 4. Let $1 < p < \infty$, $w_1, w_2 \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\cdots,l$. Suppose that $\tilde{T}^b$ is bounded on $L^q(u)$ for any $1 < q \le \infty$ and $u \in A_1$. Then $\tilde{T}^b$ is bounded from $\dot{H}K_p(w_1,w_2;R^n)$ (or $HK_p(w_1,w_2;R^n)$) to $\dot{K}_p(w_1,w_2;R^n)$ (or $HK_p(w_1,w_2;R^n)$).

3. Proof of theorems

To prove the theorem, we need the following lemma.

Lemma 2. (see [5]). Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(b; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |T^b(f)(x) - C_Q| w(x) dx \leq C||f||_{L^\infty(w)}$$

holds for any cube Q. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_m(b_j; x, y) = R_m(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^b(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} F_t(x,y) f(y) dy = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} F_t(x,y) f_2(y) dy \\ &+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} F_t(x,y) f_1(y) dy \\ &- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) F_t(x,y) f_1(y) dy \\ &- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) F_t(x,y) f_1(y) dy \\ &+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} F_t(x,y) f_1(y) dy, \end{aligned} \tag{3.1}$$

then

$$\begin{aligned} & \left| T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0) \right| = \left| ||F_t^b(f)(x)|| - ||F_t^{\tilde{b}}(f_2)(x_0)|| \right| \\ \leq\ & ||F_t^b(f)(x) - F_t^{\tilde{b}}(f_2)(x_0)|| \leq \left\| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} F_t(x,y) f_1(y) dy \right\| \\ &+ \left\| \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) F_t(x,y) f_1(y) dy \right\| \\ &+ \left\| \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) F_t(x,y) f_1(y) dy \right\| \\ &+ \left\| \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} F_t(x,y) f_1(y) dy \right\| \\ &+ \left| T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \right| \end{aligned} \tag{3.2}$$

$$:= \quad I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),$$

thus,

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q \left|T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)\right| w(x)dx \le \frac{1}{w(Q)}\int_Q I_1(x)w(x)dx \\ &+\frac{1}{w(Q)}\int_Q I_2(x)w(x)dx + \frac{1}{w(Q)}\int_Q I_3(x)w(x)dx + \frac{1}{w(Q)}\int_Q I_4(x)w(x)dx + \frac{1}{w(Q)}\int_Q I_5(x)w(x)dx \\ := \;& I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{3.3}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 2, we get

$$R_{m_j}(\tilde{b}_j; x, y) \le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO},$$

thus, by the $L^\infty(w)$-boundedness of T, we get

$$\begin{aligned} I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\frac{1}{w(Q)}\int_Q |T(f_1)(x)|w(x)dx \\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|T(f_1)\|_{L^\infty(w)} \\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}. \end{aligned} \tag{3.4}$$

For I_2, since $w \in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^q dx\right)^{1/q} \le \frac{C}{|Q|}\int_Q w(x)dx$$

for all cube Q and some $1 < q < \infty$ (see [9]). Thus, by the $L^p(w)$-boundedness of T for $p > 1$ and Hölder'inequality, we get

$$\begin{aligned} I_2 &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{w(Q)}\int_Q |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|w(x)dx \\ &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^p w(x)dx\right)^{1/p} \\ &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x) f_1(x)|^p w(x)dx\right)^{1/p} \\ &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}b_1(x) - (D^{\alpha_1}b_1)_{\tilde{Q}}|^{pq'} dx\right)^{1/pq'} \\ &\quad \times w(Q)^{-1/p}|Q|^{1/p}\left(\frac{1}{|Q|}\int_{\tilde{Q}} w(x)^q dx\right)^{1/pq}\|f\|_{L^\infty(w)} \end{aligned} \tag{3.5}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}.$$

For I_3, similar to the proof of I_2, we get

$$I_3\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}.$$

Similarly, for I_4, choose $1<r_1,r_2<\infty$ such that $1/r_1+1/r_2+1/q=1$, we obtain, by Hölder'inequality,

$$\begin{aligned} I_4 &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x)|^pw(x)dx\right)^{1/p} \qquad (3.6)\\ &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}w(Q)^{-1/p}\left(\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x)|^pw(x)dx\right)^{1/p}\\ &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(x)|^{pr_1}dx\right)^{1/pr_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(x)|^{pr_2}dx\right)^{1/pr_2}\\ &\quad\times w(Q)^{-1/p}|Q|^{1/p}\left(\frac{1}{|Q|}\int_{\tilde{Q}}w(x)^qdx\right)^{1/pq}\|f\|_{L^\infty(w)}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}. \end{aligned}$$

For I_5, we write

$$\begin{aligned} &F_t^{\tilde{b}}(f_2)(x)-F_t^{\tilde{b}}(f_2)(x_0)=\int_{R^n}\left(\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,y)f_2(y)dy \qquad (3.7)\\ &+\int_{R^n}\left(R_{m_1}(\tilde{b}_1;x,y)-R_{m_1}(\tilde{b}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{b}_2;x,y)}{|x_0-y|^m}F_t(x_0,y)f_2(y)dy\\ &+\int_{R^n}\left(R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{b}_1;x_0,y)}{|x_0-y|^m}F_t(x_0,y)f_2(y)dy\\ &-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}F_t(x,y)-\frac{R_{m_2}(\tilde{b}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}F_t(x_0,y)\right]\\ &\times D^{\alpha_1}\tilde{b}_1(y)f_2(y)dy\\ &-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{R_{m_1}(\tilde{b}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]\\ &\times D^{\alpha_2}\tilde{b}_2(y)f_2(y)dy\\ &+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]\\ &\times D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)f_2(y)dy \end{aligned}$$

$$= \quad I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 2 and the following inequality (see [18])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|)\|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned} |R_{m_j}(\tilde{b}_j; x, y)| &\le C|x-y|^{m_j} \sum_{|\alpha|=m_j} (\|D^\alpha b_j\|_{BMO} + |(D^\alpha b_j)_{\tilde{Q}(x,y)} - (D^\alpha b_j)_{\tilde{Q}}|) \\ &\le Ck|x-y|^{m_j} \sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}. \end{aligned} \tag{3.8}$$

Note that $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition on F_t,

$$\begin{aligned} \|I_5^{(1)}\| &\le C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}} \right) \prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)||f_2(y)|dy \\ &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right) |f(y)|dy \\ &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \|f\|_{L^\infty(w)} \\ &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}. \end{aligned} \tag{3.9}$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{b}_j; x, x_0)(x-y)^\beta,$$

and Lemma 2, we have

$$|R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y)| \le C \sum_{|\beta|<m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|} |x-y|^{|\beta|} \|D^\alpha b_j\|_{BMO},$$

thus

$$\begin{aligned} \|I_5^{(2)}\| &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)|dy \\ &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}. \end{aligned} \tag{3.10}$$

Similarly,

$$\|I_5^{(3)}\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.$$

For $I_5^{(4)}$, similar to the proof of $I_5^{(1)}$ and $I_5^{(2)}$, we get

$$\begin{aligned} \|I_5^{(4)}\| &\leq C\sum_{|\alpha_1|=m_1}\int_{R^n}\left\|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0,y)}{|x_0-y|^m}\right\||R_{m_2}(\tilde{b}_2;x,y)||D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|dy \quad (3.11)\\ &\quad +C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)|\frac{\|(x_0-y)^{\alpha_1}F_t(x_0,y)\|}{|x_0-y|^m}|D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|dy\\ &\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|dy\right)\|f\|_{L^\infty(w)}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}. \end{aligned}$$

Similarly,

$$\|I_5^{(5)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}.$$

For $I_5^{(6)}$, taking $1<r_1,r_2<\infty$ such that $1/r_1+1/r_2=1$, then

$$\begin{aligned} \|I_5^{(6)}\| &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\left\|\frac{(x-y)^{\alpha_1+\alpha_2}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}F_t(x_0,y)}{|x_0-y|^m}\right\| \quad (3.12)\\ &\quad\times|D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f_2(y)|dy\\ &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\|f\|_{L^\infty(w)}\\ &\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{r_1}dy\right)^{1/r_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{r_2}dy\right)^{1/r_2}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}. \end{aligned}$$

Thus

$$I_5\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{L^\infty(w)}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. First, we prove that there exists a constant $C>0$ such that for every $H^1(w)$-atom a (that is that a satisfy: supp $a\subset Q=Q(x_0,r)$, $\|a\|_{L^2(w)}\leq w(Q)^{-1/2}$ and $\int_{R^n}a(y)dy=0$ (see [7])), the following inequality holds:

$$\|\tilde{T}^b(a)\|_{L^1(w)}\leq C.$$

Without loss of generality, we may assume $l=2$. Write

$$\int_{R^n}\tilde{T}^b(a)(x)w(x)dx=\left[\int_{2Q}+\int_{(2Q)^c}\right]\tilde{T}^b(a)(x)w(x)dx:=J_1+J_2.$$

For J_1, by the $L^2(w)$-boundedness of $\tilde{T}^b$, we get

$$J_1\leq C\|\tilde{T}^b(a)\|_{L^2(w)}w(2Q)^{1/2}\leq C\|a\|_{L^2(w)}w(Q)\leq C.$$

To obtain the estimate of J_2, we denote that $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha_j|=m_j} \frac{1}{\alpha!}(D^{\alpha} b_j)_{2Q} x^{\alpha}$. Then $Q_{m_j}(b_j; x, y) = Q_{m_j}(\tilde{b}_j; x, y)$. We write, by the vanishing moment of a,

$$\begin{aligned}
\tilde{F}_t^b(a)(x) \;=\; & \int_{R^n} \left[\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x,x_0)}{|x-x_0|^m} \right] R_{m_1}(\tilde{b}_1; x, y) R_{m_2}(\tilde{b}_2; x, y) a(y) dy \qquad (3.13)\\
& + \int_{R^n} \frac{F_t(x,x_0)}{|x-x_0|^m} [R_{m_1}(\tilde{b}_1; x, y) R_{m_2}(\tilde{b}_2; x, y) - R_{m_1}(\tilde{b}_1; x, x_0) R_{m_2}(\tilde{b}_2; x, x_0)] a(y) dy\\
& - \sum_{|\alpha_2|=m_2} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\alpha_2}}{|x-y|^m} - \frac{F_t(x,x_0)(x-x_0)^{\alpha_2}}{|x-x_0|^m} \right] R_{m_1}(\tilde{b}_1; x, y) D^{\alpha_2}\tilde{b}_2(x) a(y) dy\\
& - \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{F_t(x,x_0)(x-x_0)^{\alpha_2}}{|x-x_0|^m} [R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x, x_0)] D^{\alpha_2}\tilde{b}_2(x) a(y) dy\\
& - \sum_{|\alpha_1|=m_1} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\alpha_1}}{|x-y|^m} - \frac{F_t(x,x_0)(x-x_0)^{\alpha_1}}{|x-x_0|^m} \right] R_{m_2}(\tilde{b}_2; x, y) D^{\alpha_1}\tilde{b}_1(x) a(y) dy\\
& - \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{F_t(x,x_0)(x-x_0)^{\alpha_1}}{|x-x_0|^m} [R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x, x_0)] D^{\alpha_1}\tilde{b}_1(x) a(y) dy\\
& + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} - \frac{F_t(x,x_0)(x-x_0)^{\alpha_1+\alpha_2}}{|x-x_0|^m} \right]\\
& \times D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x) a(y) dy,
\end{aligned}$$

notice that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)} \le C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$. Thus, by Hölder's inequality and the reverse of Hölder's inequality for $w \in A_1$ and $1 < q < \infty$, we obtain, similar to the proof of Theorem 1,

$$\begin{aligned}
J_2 \;\le\; & C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \left(\frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right) \qquad (3.14)\\
& + C \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} b_1\|_{BMO} \sum_{k=1}^{\infty} k (2^{-k} + 2^{-\varepsilon k}) \sum_{|\alpha_2|=m_2} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^{\alpha_2}\tilde{b}_2(x)|^{q'} dx \right)^{1/q'}\\
& \times \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^q dx \right)^{1/q}\\
& + C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{k=1}^{\infty} k (2^{-k} + 2^{-\varepsilon k}) \sum_{|\alpha_1|=m_1} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^{\alpha_1}\tilde{b}_1(x)|^{q'} dx \right)^{1/q'}\\
& \times \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^q dx \right)^{1/q}\\
& + C \sum_{k=1}^{\infty} (2^{-k} + 2^{-\varepsilon k}) \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^q dx \right)^{1/q}\\
& \times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^{\alpha_1}\tilde{b}_1(x)|^{r_1} dx \right)^{1/r_1} \sum_{|\alpha_2|=m_2} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^{\alpha_2}\tilde{b}_2(x)|^{r_2} dx \right)^{1/r_2}\\
\le\; & C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right) \le C.
\end{aligned}$$

Now, for $f \in H^1(w)$ with $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where $a_j' s$ are the $H^1(w)$-atom and $\sum_j |\lambda_j| \le C\|f\|_{H^1(w)}$. From above, we get

$$\sum_{j=1}^{\infty} |\lambda_j| \|\tilde{T}^b(a_j)\|_{L^1(w)} \le C \sum_{j=1}^{\infty} |\lambda_j| \le C\|f\|_{H^1(w)},$$

that is $\sum_{j=1}^{\infty} |\lambda_j| |\tilde{T}^b(a_j)| \in L^1(w)$, and

$$\|\tilde{T}^b(f)\|_{L^1(w)} \le \sum_{j=1}^{\infty} |\lambda_j| \|\tilde{T}^A(a_j)\|_{L^1(w)} \le C \sum_{j=1}^{\infty} |\lambda_j| \le C\|f\|_{H^1(w)}.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |T^b(f)(x) - C_Q| w(x)dx \le C\|f\|_{B_p(w)}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(b_j; x, y) = R_{m_j}(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. Similar to the proof of Theorem 1, we write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
& \frac{1}{w(Q)} \int_Q \left| T^b(f)(x) - T^{\tilde{b}}(f_2)(0) \right| w(x)dx \qquad (3.15)\\
\le\ & \frac{1}{w(Q)} \int_Q \left\| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} F_t(x, y) f_1(y)dy \right\| w(x)dx \\
& + \frac{1}{w(Q)} \int_Q \left\| \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{b}_1(y) F_t(x, y) f_1(y)dy \right\| w(x)dx \\
& + \frac{1}{w(Q)} \int_Q \left\| \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{b}_2(y) F_t(x, y) f_1(y)dy \right\| w(x)dx \\
& + \frac{1}{w(Q)} \int_Q \left\| \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} F_t(x, y) f_1(y)dy \right\| w(x)dx \\
& + \frac{1}{w(Q)} \int_Q \left| T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(0) \right| w(x)dx \\
:=\ & L_1 + L_2 + L_3 + L_4 + L_5.
\end{aligned}$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}
L_1 &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{w(Q)} \int_Q |T(f_1)(x)|^p w(x)dx \right)^{1/p} \qquad (3.16)\\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(\tilde{Q})^{-1/p} \|f\chi_{\tilde{Q}}\|_{L^p(w)}
\end{aligned}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.$$

For L_2, taking $r,s>1$ such that $rs<p$ and $q=(ps-rs)/(p-rs)$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}
L_2 &\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|^r w(x)dx\right)^{1/r} \qquad (3.17)\\
&\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}w(Q)^{-1/r}\|D^{\alpha_1}\tilde{b}_1f_1\|_{L^r(w)}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|Q|^{1/rs'}w(Q)^{-1/r}\left(\int_{\tilde{Q}}|f(x)|^p w(x)dx\right)^{1/p}\left(\int_{\tilde{Q}}w(x)^q dx\right)^{(p-r)/pqr}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)w(\tilde{Q})^{-1/p}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.\\
L_3 &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.
\end{aligned}$$

For L_4, taking $r,t_1,t_2,t_3>1$ such that $1/t_1+1/t_2+1/t_3=1$, $rt_3<p$ and $q=(pt_3-rt_3)/(p-rt_3)$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}
L_4 &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{w(Q)}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x)|^r w(x)dx\right)^{1/r} \qquad (3.18)\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}w(Q)^{-1/r}\left(\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x)|^r w(x)dx\right)^{1/r}\\
&\leq C\sum_{|\alpha_1|=m_1}\left(\int_{\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(x)|^{rt_1}dx\right)^{1/rt_1}\sum_{|\alpha_2|=m_2}\left(\int_{\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(x)|^{rt_2}dx\right)^{1/rt_2}\\
&\quad\times w(Q)^{-1/r}\left(\int_{\tilde{Q}}|f(x)|^{rt_3}w(x)^{t_3}dx\right)^{1/rt_3}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|Q|^{1/rt_1+1/rt_2}w(Q)^{-1/r}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\left(\int_{\tilde{Q}}w(x)^q dx\right)^{(p-rt_3)/prt_3}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)w(\tilde{Q})^{-1/p}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.
\end{aligned}$$

For L_5, we write, for $x \in Q$,

$$F_t^{\tilde{b}}(f_2)(x) - F_t^{\tilde{b}}(f_2)(0) = \int_{R^n} \left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(0,y)}{|y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) f_2(y) dy \tag{3.19}$$
$$+ \int_{R^n} \left(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; 0, y) \right) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|y|^m} F_t(0,y) f_2(y) dy$$
$$+ \int_{R^n} \left(R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; 0, y) \right) \frac{R_{m_1}(\tilde{b}_1; 0, y)}{|y|^m} F_t(0,y) f_2(y) dy$$
$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x,y) - \frac{R_{m_2}(\tilde{b}_2; 0, y)(-y)^{\alpha_1}}{|y|^m} F_t(0,y) \right] D^{\alpha_1} \tilde{b}_1(y) f_2(y) dy$$
$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x,y) - \frac{R_{m_1}(\tilde{b}_1; 0, y)(-y)^{\alpha_2}}{|y|^m} F_t(0,y) \right] D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy$$
$$+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} F_t(x,y) - \frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m} F_t(0,y) \right]$$
$$\times D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy$$
$$= L_5^{(1)} + L_5^{(2)} + L_5^{(3)} + L_5^{(4)} + L_5^{(5)} + L_5^{(6)}.$$

Similar to the proof of Theorem 1 and notice that $w \in A_1 \subset A_p$, we get

$$\|L_5^{(1)}\| \le C \int_{R^n} \left(\frac{|x|}{|y|^{m+n+1}} + \frac{|x|^\varepsilon}{|y|^{m+n+\varepsilon}} \right) \prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)| |f_2(y)| dy \tag{3.20}$$
$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k^2 \left(\frac{|x|}{|y|^{n+1}} + \frac{|x|^\varepsilon}{|y|^{n+\varepsilon}} \right) |f(y)| dy$$
$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) w(2^k \tilde{Q})^{-1/p} \left(\int_{2^k \tilde{Q}} |f(y)|^p w(y) dy \right)^{1/p}$$
$$\times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/p} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(p-1)} dy \right)^{(p-1)/p}$$
$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \|f\|_{B_p(w)}$$
$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.$$

$$\|L_5^{(2)}\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k \frac{|x|}{|y|^{n+1}} |f(y)| dy$$
$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.$$

$$\|L_5^{(3)}\| \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.$$

For $L_5^{(4)}$, choose $1<r<p$, notice that $w\in A_1\subset A_{p/r}$, we get

$$\begin{aligned}
\|L_5^{(4)}\| &\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\left(\frac{|x|}{|y|^{n+1}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon}}\right)|D^{\alpha_1}\tilde{b}_1(y)||f(y)|dy \qquad (3.21)\\
&\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{k=0}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)\left(\int_{2^{k+1}\tilde{Q}}|f(y)|^r dy\right)^{1/r}dy\\
&\quad\times\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{r'}dy\right)^{1/r'}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\left(\int_{2^k\tilde{Q}}|f(y)|^p w(y)dy\right)^{1/p}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-r/(p-r)}dy\right)^{(p-r)/pr}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.
\end{aligned}$$

$$\|L_5^{(5)}\| \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.$$

For $L_5^{(6)}$, choose $1<r_1,r_2,r_3<\infty$ such that $r_3<p$ and $1/r_1+1/r_2+1/r_3=1$, notice that $w\in A_1\subset A_{p/r_3}$, we get

$$\begin{aligned}
\|L_5^{(6)}\| &\leq C\sum_{k=0}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)\left(\int_{2^{k+1}\tilde{Q}}|f(y)|^{r_3}dy\right)^{1/r_3}dy \qquad (3.22)\\
&\quad\times\sum_{|\alpha_1|=m_1}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{r_1}dy\right)^{1/r_1}\sum_{|\alpha_2|=m_2}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{r_2}dy\right)^{1/r_2}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\left(\int_{2^k\tilde{Q}}|f(y)|^p w(y)dy\right)^{1/p}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-r_3/(p-r_3)}dy\right)^{(p-r_3)/pr_3}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.
\end{aligned}$$

Thus

$$L_5\leq C\prod_{j=1}^{2}\left(\sum_{\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.$$

This finishes the proof of Theorem 3.

Proof of Theorem 4. We only give the proof of homogeneous Herz type Hardy spaces. Without loss of generality, we may assume $l = 2$. Let $f \in H\dot{K}_p(w_1, w_2; R^n)$, by Lemma 1, $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $a_j's$ are the central $(n(1-1/p), p; w_1, w_2)$-atom with supp$a_j \subset B_j = B(0, 2^j)$ and $\|f\|_{H\dot{K}_p(w_1,w_2)} \sim \sum_j |\lambda_j|$. Write

$$\begin{aligned}\|\tilde{T}^b(f)\|_{\dot{K}_p(w_1,w_2)} &= \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \|\chi_k \tilde{T}^b(f)\|_{L^p(w_2)} \qquad (3.23)\\ \leq\ & \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=-\infty}^{k-1} |\lambda_j| \|\chi_k \tilde{T}^b(a_j)\|_{L^p(w_2)} + \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| \|\chi_k \tilde{T}^b(a_j)\|_{L^p(w_2)}\\ =\ & M_1 + M_2.\end{aligned}$$

For M_2, by the $L^p(w)$-boundedness of $\tilde{T}^b$ for $1 < p < \infty$ and $w \in A_1$, we get

$$\begin{aligned}M_2 &\leq C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| \|a_j\|_{L^p(w_2)} \leq C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| [w_1(B_j)]^{-(1-1/p)} \qquad (3.24)\\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=-\infty}^{j} \left[\frac{w_1(B_k)}{w_1(B_j)}\right]^{1-1/p} \leq C \sum_{j=-\infty}^{\infty} |\lambda_j| \leq C\|f\|_{H\dot{K}_p(w_1,w_2)}.\end{aligned}$$

To estimate M_1, we denote that $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha_j|=m_j} \frac{1}{\alpha!}(D^\alpha b_j)_{2Q} x^\alpha$. Then $Q_{m_j}(b_j; x, y) = Q_{m_j}(\tilde{b}_j; x, y)$. We write, by the moment condition of a_j,

$$\begin{aligned}\tilde{F}_t^b(a_j)(x) =\ & \int_{R^n} \left[\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x,0)}{|x|^m}\right] R_{m_1}(\tilde{b}_1; x, y) R_{m_2}(\tilde{b}_2; x, y) a_j(y) dy \qquad (3.25)\\ &+ \int_{R^n} \frac{F_t(x,0)}{|x|^m} [R_{m_1}(\tilde{b}_1; x, y) R_{m_2}(\tilde{b}_2; x, y) - R_{m_1}(\tilde{b}_1; x, 0) R_{m_2}(\tilde{b}_2; x, 0)] a_j(y) dy\\ &- \sum_{|\alpha_2|=m_2} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\alpha_2}}{|x-y|^m} - \frac{F_t(x,0)x^{\alpha_2}}{|x|^m}\right] R_{m_1}(\tilde{b}_1; x, y) D^{\alpha_2}\tilde{b}_2(x) a_j(y) dy\\ &- \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{F_t(x,0)x^{\alpha_2}}{|x|^m} [R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x, 0)] D^{\alpha_2}\tilde{b}_2(x) a_j(y) dy\\ &- \sum_{|\alpha_1|=m_1} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\alpha_1}}{|x-y|^m} - \frac{F_t(x,0)x^{\alpha_1}}{|x|^m}\right] R_{m_2}(\tilde{b}_2; x, y) D^{\alpha_1}\tilde{b}_1(x) a_j(y) dy\\ &- \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{F_t(x,0)x^{\alpha_1}}{|x|^m} [R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x, 0)] D^{\alpha_1}\tilde{b}_1(x) a_j(y) dy\\ &+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} - \frac{F_t(x,0)x^{\alpha_1+\alpha_2}}{|x|^m}\right] D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x) a_j(y) dy.\end{aligned}$$

Going through a similar argument to Theorem 2, we obtain

$$|\tilde{T}^b(a_j)(x)| \qquad (3.26)$$

$$
\begin{aligned}
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\|a_j\|_{L^p(w_2)}\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&+C\sum_{|\alpha_1|=m_1}\|D^{\alpha_1}b_1\|_{BMO}\sum_{|\alpha_2|=m_2}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^{\alpha_2}\tilde{b}_2(x)|\|a_j\|_{L^p(w_2)}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&+C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^{\alpha_1}\tilde{b}_1(x)|\|a_j\|_{L^p(w_2)}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_1}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^{\alpha_1}\tilde{b}_1(x)||D^{\alpha_2}\tilde{b}_2(x)|\|a_j\|_{L^p(w_2)}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right][w_1(B_j)]^{1/p-1}\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&+C\sum_{|\alpha_1|=m_1}\|D^{\alpha_1}b_1\|_{BMO}\sum_{|\alpha_2|=m_2}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^{\alpha_2}\tilde{b}_2(x)|[w_1(B_j)]^{1/p-1}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&+C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^{\alpha_1}\tilde{b}_1(x)|[w_1(B_j)]^{1/p-1}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_1}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^{\alpha_1}\tilde{b}_1(x)||D^{\alpha_2}\tilde{b}_2(x)|[w_1(B_j)]^{1/p-1}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p},
\end{aligned}
$$

thus

$$
\begin{aligned}
M_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-1/p}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\\
&\quad\times[w_1(B_j)]^{1/p-1}\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}[w_2(B_k)]^{1/p}
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&+C\sum_{|\alpha_1|=m_1}\|D^{\alpha_1}b_1\|_{BMO}\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-1/p}\sum_{j=-\infty}^{k-1}|\lambda_j|\sum_{|\alpha_2|=m_2}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\\
&\times[w_1(B_j)]^{1/p-1}\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\left(\int_{B_k}|D^{\alpha_2}\tilde{b}_2(x)|^p w_2(x)dx\right)^{1/p}\\
&+C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-1/p}\sum_{j=-\infty}^{k-1}|\lambda_j|\sum_{|\alpha_1|=m_1}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\\
&\times[w_1(B_j)]^{1/p-1}\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\left(\int_{B_k}|D^{\alpha_1}\tilde{b}_1(x)|^p w_2(x)dx\right)^{1/p}\\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_1}\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-1/p}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right][w_1(B_j)]^{1/p-1}\\
&\times\left(\int_{B_j}w_2(y)^{-1/(p-1)}dy\right)^{(p-1)/p}\left(\int_{B_k}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)|^p w_2(x)dx\right)^{1/p}\\
\le\;& C\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=j+1}^{\infty}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\left[\frac{w_1(B_k)}{w_1(B_j)}\frac{|B_j|}{|B_k|}\right]^{1-1/p}\left[\frac{w_2(B_k)}{w_2(B_j)}\frac{|B_j|}{|B_k|}\right]^{1/p}|B_k|\\
\le\;& C\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=j+1}^{\infty}[2^{j-k}+2^{(j-k)\varepsilon}]\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\le C\|f\|_{H\dot{K}_p(w_1,w_2)}.
\end{aligned}
$$

This completes the proof of Theorem 4.

4. Applications

Now we apply the theorems of this paper to some concrete including Littlewood-Paley operators, Marcinkiewicz operators and the Bochner-Riesz operator.

Example 1. Littlewood-Paley operators.

Fixed $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n}\psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y,t)\in R^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by (see [13]),

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} \psi_t(x-y) f(y) dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, z)}{|x-z|^m} f(z) \psi_t(y-z) dz,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. The variants of g_ψ^A, S_ψ^A and g_μ^A are defined by

$$\tilde{g}_\psi^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$\tilde{S}_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

and

$$\tilde{g}_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j; x, y)}{|x-y|^m} \psi_t(x-y) f(y) dy,$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j; x, z)}{|x-z|^m} \psi_t(y-z) f(z) dz.$$

Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which are the Littlewood-Paley operators (see [19]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\},$$

or

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left\|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right\|, \ \ S_\psi(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|,$$

and

$$g_\mu^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right\|, \ g_\mu(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right\|.$$

It is easy to see that g_ψ, S_ψ and g_μ satisfy the conditions of Theorems 1–4, thus Theorems 1–4 hold for g_ψ^A and $\tilde{g}_\psi^A$, S_ψ^A and $\tilde{S}_\psi^A$, g_μ^A and $\tilde{g}_\mu^A$.

Example 2. Marcinkiewicz operators.

Fixed $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy,$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz;$$

The variants of μ_Ω^A, μ_S^A and μ_λ^A are defined by

$$\tilde{\mu}_\Omega^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\tilde{\mu}_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

and

$$\tilde{\mu}_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy,$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators (see [20]). Let H be the space

$$H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\},$$

or

$$H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = \|F_t^A(f)(x)\|, \quad \mu_\Omega(f)(x) = \|F_t(f)(x)\|,$$

$$\mu_S^A(f)(x) = \left\|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right\|, \ \mu_S(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|,$$

and

$$\mu_\lambda^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right\|, \ \mu_\lambda(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right\|.$$

It is easy to see that μ_Ω, μ_S and μ_λ satisfy the conditions of Theorems 1–4, thus Theorems 1–4 hold for μ_Ω^A and $\tilde{\mu}_\Omega^A$, μ_S^A and $\tilde{\mu}_S^A$, μ_λ^A and $\tilde{\mu}_\lambda^A$.

Example 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(\hat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y)dy,$$

and

$$\tilde{F}_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y)dy.$$

The maximal Bochner-Riesz multilinear operator and its the variants are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)| \ \text{ and } \ \tilde{B}_{\delta,*}^A(f)(x) = \sup_{t>0} |\tilde{B}_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the maximal Bochner-Riesz operator (see [14]). Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B^A_{\delta,*}(f)(x) = \|B^A_{\delta,t}(f)(x)\|, \quad B^{\delta}_{*}(f)(x) = \|B^{\delta}_{t}(f)(x)\|.$$

It is easy to see that $B_{\delta,*}$ satisfies the conditions of Theorems 1–4, thus Theorems 1–4 hold for $B^A_{\delta,*}$ and $\tilde{B}^A_{\delta,*}$.

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Conflict of interest

No conflict of interest.

References

1. J. Alvarez, R. J. Babgy, D. S. Kurtz, C. Pérez, Weighted estimates for commutators of linear operators, *Studia Math.*, **104** (1993), 195–209.
2. M. Bownik, Boundedness of operators on Hardy spaces via atomic decompositions, *Proc. Amer. Math. Soc.*, **133** (2005), 3535–3542. https://doi.org/10.1090/S0002-9939-05-07892-5
3. W. G. Chen, G. E. Hu, Weak type (H^1, L^1) estimate for multilinear singular integral operator, *Adv. Math.*, **30** (2001), 63–69.
4. J. Cohen, A sharp estimate for a multilinear singular integral on R^n, *Indiana U. Math. J.*, **30** (1981), 693–702.
5. J. Cohen, J. Gosselin, A BMO estimate for multilinear singular integral, *Illinois J. Math.*, **30** (1986), 445–464.
6. R. R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.*, **103** (1976), 611–635. https://doi.org/10.2307/1970954
7. J. Garcia-Cuerva, M. L. Herrero, A theory of Hardy spaces associated to the Herz spaces, *P. Lond. Math. Soc.*, **69** (1994), 605–628. https://doi.org/10.1112/plms/s3-69.3.605
8. J. Garcia-Cuerva, J. R. de Francia, *Weighted norm inequalities and related topics*, 1985.
9. E. Harboure, C. Segovia, J. L. Torrea, Boundedness of commutators of fractional and singular integrals for the extreme values of p, *Illinois J. Math.*, **41** (1997), 676–700.
10. E. Hernandez, G. Weiss, D. C. Yang, The ϕ-transform and wavelet characterizations of Herz-type spaces, *Collect. Math.*, **47** (1996), 285–320.
11. E. Hernandez, D. C. Yang, Interpolation of Herz-type Hardy spaces, *Illinois J. Math.*, **42** (1998), 564–581.
12. L. Z. Liu, Weighted weak type (H^1, L^1) estimates for commutators of Littlewood-Paley operator, *Indian J. Math.*, **45** (2003), 71–78.

13. L. Z. Liu, Weighted endpoint estimates for multilinear Littlewood-Paley operators, *Acta Math. Univ. Comenianae*, **73** (2004), 55–67.

14. S. Z. Lu, *Four lectures on real H^p spaces*, NI: World Scientific, River Edge, 1995.

15. S. Z. Lu, D. C. Yang, The decomposition of the weighted Herz spaces on R^n and its applications, *Chinese Sci. Abstr. Ser. A*, **38** (1995), 147–158.

16. S. Z. Lu, D. C. Yang, The weighted Herz type Hardy spaces and its applications, *Chinese Sci. Abstr. Ser. A*, **38** (1995), 662–673.

17. B. H. Qui, Weighted Hardy spaces, *Math. Nachr.*, **103** (1981), 45–62.

18. E. M. Stein, T. S. Murphy, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton: Princeton University Press, 1993.

19. A. Torchinsky, *Real variable methods in harmonic analysis*, New York: Academic Press, 1986.

20. A. Torchinsky, S. Wang, A note on the Marcinkiewicz integral, *Colloq. Math.*, **60/61** (1990), 235–243.