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Research article

An existence theorem for nonlinear functional Volterra integral equations via Petryshyn's fixed point theorem

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Abstract: Using the method of Petryshyn's fixed point theorem in Banach algebra, we investigate the existence of solutions for functional integral equations, which involves as specific cases many functional integral equations that appear in different branches of non-linear analysis and their applications. Finally, we recall some particular cases and examples to validate the applicability of our study.

Keywords: Petryshyn's fixed point theorem; measure of noncompactness (MNC); functional integral equation (FIE)
Mathematics Subject Classification: 47H10

1. Introduction

Functional integral equations (FIEs) have many application in mechanical vibrations, kinetic theory of gases, radiative transfer, mathematical physics, control theory, and engineering. The theory of FIEs is speedily growing with the help of different studies of fixed point theory, topology, and non-linear analysis (cf. [2–4, 7, 8, 10, 11, 14, 15, 18, 20, 25, 31]).

This article is dedicated to study the following FIEs.

$$z(\varphi) = q\left(\varphi, f(\varphi, z(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right), \tag{1.1}$$

for all $\varphi \in I_b = [0, b]$ in the Banach algebra $C(I_b)$.

Equation (1.1) contains many particular cases see for example [12,19,21–23,26]. Moreover, many previous studies examined the existence of the solutions for different FIEs by Darbo's fixed point theorem in different spaces (cf. [5, 6, 9, 16, 17, 24, 28, 30]). We generalize these results by using Petryshyn's fixed point theorem.

This article is motivated by studying non-linear FIE under a general set of assumptions by using the theory of MNC and Petryshyn's fixed point theorem. Moreover, the bounded condition explains that the "sublinear condition" that has been recognized in literature will not play a meaningful role here. Finally, we present some particular cases and examples that show the utilization of FIEs.

2. Preliminaries

In this article, let $\mathbb{R}$ be the set of real numbers, F be real Banach space and $B_\rho = B(z, \rho)$ be a closed ball centered at z with radius ρ.

Definition 2.1. [1] Let $G \in F$ and

$$\alpha(G) = \inf\left\{\sigma > 0 : G = \bigcup_{i=1}^{n} G_i \text{ with diam } G_i \leq \sigma,\ i = 1, 2, ..., n\right\}$$

is called the Kuratowski MNC.

Definition 2.2. [1] The Hausdroff MNC

$$\vartheta(G) = \inf\{\sigma > 0 : \exists \text{ a finite } \sigma \text{ net for G in } F\}, \tag{2.1}$$

where, by a finite σ net for G in F it involves, as a set $\{z_1, z_2, ..., z_n\} \subset F$ such that the ball $B_\sigma(F, z_1), B_\sigma(F, z_2), ..., B_\sigma(F, z_n)$ over G. Those MNC are commonly related that is

$$\vartheta(G) \leq \alpha(G) \leq 2\vartheta(G)$$

for any bounded set $G \subset F$.

Theorem 2.3. *Let $G, \hat{G} \in F$ and $\lambda \in \mathbb{R}$. Then*

(i) $\vartheta(G) = 0$ *if and only if G is pre-compact;*

(ii) $G \subseteq \hat{G} \implies \vartheta(G) \leq \vartheta(\hat{G})$;

(iii) $\vartheta(ConvG) = \vartheta(G)$;

(iv) $\vartheta(G \cup \hat{G}) = \max\{\vartheta(G), \vartheta(\hat{G})\}$;

(v) $\vartheta(\lambda G) = |\lambda|\vartheta(G)$, *where* $\lambda G = \{\lambda z : z \in G\}$;

(vi) $\vartheta(G + \hat{G}) \leq \vartheta(G) + \vartheta(\hat{G})$.

In the sequel, $C[0, b]$ consisting of all real valued continuous function defined on $I_b = [0, b]$ with the usual norm

$$\|z\| = \sup\{|z(\varphi)| : \varphi \in [0, b]\}.$$

The space $C[0, b]$ is also the structure of Banach algebra. The modulus of continuity of $z \in C[0, b]$ is defined as

$$\omega(z, \sigma) = \sup\{|z(\varphi) - z(\hat{\varphi})| : \varphi, \hat{\varphi} \in [0, b], |\varphi - \hat{\varphi}| \leq \sigma\}.$$

and

$$\omega(G, \sigma) = \sup\{\omega(z, \sigma) : z \in G\},$$

$$\omega_0(G) = \lim_{\sigma \to 0} \omega(G, \sigma).$$

Theorem 2.4. *[19] The Hausdorff MNC is similar to*

$$\vartheta(G) = \lim_{\sigma \to 0} sup\ \omega(z, \sigma) \tag{2.2}$$

for all bounded sets $G \subset C[0, b]$.

Theorem 2.5. *[27] Assume $T : F \to F$ be a continuous mapping of Banach space F. T is called a k-set contraction if for all $H \subset F$ with H bounded, $T(H)$ is bounded and*

$$\alpha(TH) \leq k\alpha(H),\ for\ k \in (0, 1).$$

Moreover, if

$$\alpha(TH) < \alpha(H),\ \ \forall\ \alpha(H) > 0,$$

then T is called densifying or condensing map.

Theorem 2.6. *[29] Suppose that $T : B_\rho \to F$ be a condensing mapping which satisfying the boundary condition,*

$$if\ \ T(z) = kz, \quad for\ some\ z \in \partial B_\rho\ \ then\ k \leq 1,$$

then the set of fixed points in B_ρ is non-empty. This is called Petryshyn's fixed point theorem.

3. Main results

Now, we investigate the existence of the Eq (1.1) under the following assumptions;

(1) $q \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f, h \in C(I_b \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, D] \times \mathbb{R}, \mathbb{R})$, and $\theta : I_b \to \mathbb{R}^+$, $\gamma : [0, D] \to I_b$, $\alpha,\beta : I_b \to I_b$, are continuous such that $\theta(\varphi) \leq D$, $\forall\ \varphi \in I_b,\ D \geq 0$.

(2) There exist non-negative constants $p_i,\ i = 1, \cdots 5$, such that

$$|q(\varphi, u, v, w) - q(\varphi, \hat{u}, \hat{v}, \hat{w}| \leq p_1|u - \hat{u}| + p_2|v - \hat{v}| + p_3|w - \hat{w}|;$$

$$|f(\varphi, z) - f(\varphi, \hat{z})| \leq p_4|z - \hat{z}|;$$

$$|h(\varphi, z) - h(\varphi, \hat{z})| \leq p_5|z - \hat{z}|.$$

(3) There exists $\rho > 0$ such that q satisfy the inequality

$$sup\{|q(\varphi, u, v, w)| : \varphi \in I_b, u, v \in [-\rho, \rho], w \in [-DH_1, DH_1]\} \leq \rho,$$

where

$$H_1 = sup\{|p(\varphi, s, z)| : \forall\varphi \in I_b, s \in [0, D] \ \ and \ \ z \in [-\rho, \rho]\}.$$

Theorem 3.1. *Under the assumptions* (1)–(3) *and if* $p_1 p_4 + p_2 p_5 < 1$, $\forall z \in I_b$. *Then Eq (1.1) has at least one solution in* $F = C(I_b)$.

Proof. Define the operator $T : B_\rho \to F$, where $B_\rho = \{z \in C(I_b) : \|z\| \leq \rho\}$ in the following form

$$(Tz)(\varphi) = q\left(\varphi, f(\varphi, z(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right).$$

Now, we show that T is continuous on B_ρ. Choose $\sigma > 0$ and any $z, x \in B_\rho$ such that $\|z - x\| < \sigma$. We get

$$\begin{aligned}
&|(Tz)(\varphi) - (Tx)(\varphi)| \\
= &\left|q\left(\varphi, f(\varphi, z(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right)\right. \\
&\left.-q\left(\varphi, f(\varphi, x(\alpha(\varphi))), h(\varphi, x(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, x(\gamma(s)))ds\right)\right| \\
\leq &\left|q\left(\varphi, f(\varphi, z(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right)\right. \\
&\left.-q\left(\varphi, f(\varphi, x(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right)\right| \\
&+\left|q\left(\varphi, f(\varphi, x(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right)\right. \\
&\left.-q\left(\varphi, f(\varphi, x(\alpha(\varphi))), h(\varphi, x(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right)\right| \\
&+\left|q\left(\varphi, f(\varphi, x(\alpha(\varphi))), h(\varphi, x(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right)\right. \\
&\left.-q\left(\varphi, f(\varphi, x(\alpha(\varphi))), h(\varphi, x(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, x(\gamma(s)))ds\right)\right| \\
\leq &\ p_1|f(\varphi, z(\alpha(\varphi))) - f(\varphi, x(\alpha(\varphi)))| \\
&+p_2|h(\varphi, z(\alpha(\varphi))) - h(\varphi, x(\alpha(\varphi)))| \\
&+p_3\int_0^{\theta(\varphi)} |p(\varphi, s, z(\gamma(s))) - p(\varphi, s, x(\gamma(s)))|ds \\
\leq &\ p_1 p_4|z(\alpha(\varphi)) - x(\alpha(\varphi))| + p_2 p_5|z(\beta(\varphi)) - x(\beta(\varphi))| + p_3 D\omega(p, \sigma) \\
\leq &\ p_1 p_4\|z - x\| + p_2 p_5\|z - x\| + p_3 D\omega(p, \sigma),
\end{aligned}$$

where

$$\omega(p,\sigma) = sup\{|p(\varphi,s,z) - p(\varphi,s,x)| : \varphi \in I_b, s \in [0,D], z,x \in [-\rho,\rho], |z-x| \leq \sigma\}.$$

From the uniform continuity of $p(\varphi,s,z)$ on the subset $I_b \times [0,D] \times \mathbb{R}$, we infer that $\omega(p,\sigma) \to 0$ as $\sigma \to 0$. Thus, we prove that the operator T is continuous on B_ρ.

Next, we show that T satisfy the densifying map. Take arbitrary $\sigma > 0$ and $z \in G$, where G is bounded subset of F, $\varphi_1, \varphi_2 \in I_b$ with $|\varphi_1 - \varphi_2| \leq \sigma$, we have

$$\begin{aligned}
& |(Tz)(\varphi_2) - (Tz)(\varphi_1)| \\
= & \left|q\left(\varphi_2, f(\varphi_2, z(\alpha(\varphi_2))), h(\varphi_2, z(\beta(\varphi_2))), \int_0^{\theta(\varphi_2)} p(\varphi_2, s, z(\gamma(s)))ds\right)\right. \\
& \left. -q\left(\varphi_1, f(\varphi_1, z(\alpha(\varphi_1))), h(\varphi_1, z(\beta(\varphi_1))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right| \\
\leq & \left|q\left(\varphi_2, f(\varphi_2, z(\alpha(\varphi_2))), h(\varphi_2, z(\beta(\varphi_2))), \int_0^{\theta(\varphi_2)} p(\varphi_2, s, z(\gamma(s)))ds\right)\right. \\
& \left. -q\left(\varphi_2, f(\varphi_2, z(\alpha(\varphi_2))), h(\varphi_2, z(\beta(\varphi_2))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right| \\
& +\left|q\left(\varphi_2, f(\varphi_2, z(\alpha(\varphi_2))), h(\varphi_2, z(\beta(\varphi_2))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right. \\
& \left. -q\left(\varphi_2, f(\varphi_2, z(\alpha(\varphi_2))), h(\varphi_1, z(\beta(\varphi_2))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right| \\
& +\left|q\left(\varphi_2, f(\varphi_2, z(\alpha(\varphi_2))), h(\varphi_1, z(\beta(\varphi_2))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right. \\
& \left. -q\left(\varphi_2, f(\varphi_1, z(\alpha(\varphi_1))), h(\varphi_1, z(\beta(\varphi_1))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right| \\
& +\left|q\left(\varphi_2, f(\varphi_1, z(\alpha(\varphi_1))), h(\varphi_1, z(\beta(\varphi_1))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right. \\
& \left. -q\left(\varphi_1, f(\varphi_1, z(\alpha(\varphi_1))), h(\varphi_1, z(\beta(\varphi_1))), \int_0^{\theta(\varphi_1)} p(\varphi_1, s, z(\gamma(s)))ds\right)\right| \\
\leq & \; p_1|f(\varphi_2, z(\alpha(\varphi_2)) - f(\varphi_2, z(\alpha(\varphi_1))| + p_1|f(\varphi_2, z(\alpha(\varphi_1)) - f(\varphi_1, z(\alpha(\varphi_1))| \\
& +p_2|h(\varphi_2, z(\beta(\varphi_2)) - h(\varphi_2, z(\beta(\varphi_1))| + p_1|h(\varphi_2, z(\beta(\varphi_1)) - h(\varphi_1, z(\beta(\varphi_1))| \\
& +p_3\left|\int_0^{\theta(\varphi_1)} (p(\varphi_2, s, z(\gamma(s)) - p(\varphi_1, s, z(\gamma(s)))ds + \int_{\theta(\varphi_1)}^{\theta(\varphi_2)} p(\varphi_2, s, z(\gamma(s)))ds\right| \\
& +\omega_p(I_b, \sigma),
\end{aligned}$$

where

$$\omega_f(I_b,\sigma) = sup\{|f(\varphi,z) - f(\hat{\varphi},z)| : |\varphi - \hat{\varphi}| \leq \sigma, \varphi, \hat{\varphi} \in I_b, z \in [-\rho,\rho]\},$$

$$\omega_h(I_b,\sigma) = sup\{|h(\varphi,z) - h(\hat{\varphi},z)| : |\varphi - \hat{\varphi}| \leq \sigma, \varphi, \hat{\varphi} \in I_b, z \in [-\rho,\rho]\},$$

$$\omega_q(I_b,\sigma) = sup\{|q(\varphi,u,v,w) \quad - \quad q(\hat{\varphi},u,v,w)| : |\varphi-\hat{\varphi}| \le \sigma, \varphi,\hat{\varphi} \in I_b, \\ u,v \in [-\rho,\rho], w \in [-DH_1, DH_1]\},$$

$$\omega_p(I_b,\sigma) = sup\{|p(\varphi,s,z) - p(\hat{\varphi},s,z)| : |\varphi-\hat{\varphi}| \le \sigma, \varphi,\hat{\varphi} \in I_b, s \in [0,D], z \in [-\rho,\rho]\},$$

$$\omega(\theta,\sigma) = \sup\{|\theta(\varphi) - \theta(\hat{\varphi})| : \varphi,\hat{\varphi} \in I_b \ \ and \ \ |\varphi-\hat{\varphi}| < \sigma\}.$$

From above relations, we have

$$\begin{aligned} |(Tz)(\varphi_2) - (Tz)(\varphi_1)| \ \le \ & p_1p_4|z(\alpha(\varphi_2) - z(\alpha(\varphi_1)| + p_1\omega_f(I_b,\sigma) + p_2p_5|z(\beta(\varphi_2) \\ & -z(\beta(\varphi_1)| + p_2\omega_h(I_b,\sigma) + k_3D\omega_p(I_b,\sigma) \\ & +k_3H_1\omega(\alpha,\sigma) + \omega_q(I_b,\sigma). \end{aligned}$$

Taking limit as $\sigma \to 0$, we get

$$\omega(Tz,\sigma) \le (p_1p_4 + p_2p_5)\omega(z,\sigma).$$

This provide the following inequality

$$\vartheta(TG) \le (p_1p_4 + p_2p_5)\vartheta(G).$$

Hence T is a condensing map. Now, let $z \in \partial B_\rho$ and if $Tz = kz$ then $\|Tz\| = k\|z\| = k\rho$ and by (3), then

$$|Tz(\varphi)| = \left|q\left(\varphi, f(\varphi, z(\alpha(\varphi))), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi,s,z(\gamma(s)))ds\right)\right| \le \rho.$$

for all $\varphi \in I_b$, hence $\|Tz\| \le \rho$ i.e $k \le 1$. This completes the proof. □

Corollary 3.2. *Assume that*

(1) $q \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $h \in C(I_b \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0,D] \times \mathbb{R}, \mathbb{R})$,
and $\theta : I_b \to \mathbb{R}^+, \gamma : [0,D] \to I_d, \beta : I_b \to I_b$, *are continuous such that* $\theta(\varphi) \le D, \ \forall\, \varphi \in I_b, D \ge 0$.

(2) *There exist non-negative constants* $p_i, \ i = 1,\cdots 4$ *with* $p_2p_4 < 1$ *such that*

$$|q(\varphi,u,v,w) - q(\varphi,\hat{u},\hat{v},\hat{w}| \le p_1|u-\hat{u}| + p_2|v-\hat{v}| + p_3|w-\hat{w}|;$$

$$|h(\varphi,z) - h(\varphi,\hat{z})| \le p_4|z-\hat{z}|.$$

(3) *There exists* $\rho > 0$ *such that* q *satisfy the inequality*

$$sup\{|q(\varphi,u,v,w)| : \varphi \in I_b, u,v \in [-\rho,\rho], w \in [-DH_1, DH_1]\} \le \rho,$$

where,

$$H_1 = sup\{|p(\varphi,s,z)| : \textit{for all}\ \varphi \in I_b, s \in [0,D] \ \ and \ \ z \in [-\rho,\rho]\}.$$

Then

$$z(\varphi) = q\left(\varphi, z(\alpha(\varphi)), h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi,s,z(\gamma(s)))ds\right), \tag{3.1}$$

has at least one solution in $C(I_b)$.

Corollary 3.3. *Let*

(1) $q \in C(I_b \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f, h \in C(I_b \times \mathbb{R}, \mathbb{R})$, $p \in C(I_b \times [0, D] \times \mathbb{R}, \mathbb{R})$,
and $\theta : I_b \to \mathbb{R}^+, \gamma : [0, D] \to I_d, \beta : I_b \to I_b$, *are continuous such that* $\theta(\varphi) \leq D,\ \forall\, \varphi \in I_b,\ D \geq 0.$

(2) *There exist non-negative constants* $p_i, i = 1, \ldots 4$ *with* $p_4 + p_1 p_3 < 1$ *such that*

$$|q(\varphi, v, w) - q(\varphi, \hat{v}, \hat{w}| \leq p_1|v - \hat{v}| + p_2|w - \hat{w}|;$$

$$|h(\varphi, z) - h(\varphi, \hat{z})| \leq p_3|z - \hat{z}|;$$

$$|f(\varphi, z) - f(\varphi, \hat{z})| \leq p_4|z - \hat{z}|.$$

(3) *There exists* $\rho > 0$ *such that* q *satisfy the inequality*

$$sup\{|f(\varphi, u) + q(\varphi, v, w)| : \varphi \in I_b, u, \in [-\rho, \rho], v \in [-\rho, \rho], w \in [-DH_1, DH_1]\} \leq \rho,$$

where,

$$H_1 = sup\{|p(\varphi, s, z)| : \text{for all } \varphi \in I_b, s \in [0, D] \ \text{ and } \ z \in [-\rho, \rho]\}.$$

Then

$$z(\varphi) = f(\varphi, z(\alpha(\varphi))) + q\left(\varphi, h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right) \tag{3.2}$$

has at least one solution in $C(I_b)$.

4. Particular cases and examples

Example 4.1. If $q(\varphi, u, v, w) = 1 + vw, \beta(\varphi) = \theta(\varphi) = \gamma(\varphi) = \varphi, h(\varphi, z(\beta(\varphi))) = z(\varphi)$ and $p(\varphi, s, z) = \frac{\varphi}{\varphi+s}\phi(s)z$. Then Eq (1.1) has the Chandrasekhar integral equation type in radiative transfer [3].

$$z(\varphi) = 1 + z(\varphi) \int_0^{\varphi} \frac{\varphi}{\varphi + s}\phi(s)z(s)ds, \varphi \in I_b = [0, b].$$

Example 4.2. Let $q(\varphi, u, v, w) = q(\varphi, v, w)$, then the Eq (1.1) takes the form

$$z(\varphi) = q\left(\varphi, h(\varphi, z(\beta(\varphi))), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right), \varphi \in I_b = [0, b],$$

which is studied in [13].

Example 4.3. Let $q(\varphi, u, v, w) = q(\varphi, vw),\ \gamma(\varphi) = \theta(\varphi) = \varphi$, then Eq (1.1) takes the form

$$z(\varphi) = h(\varphi, z(\beta(\varphi))) \int_0^{\varphi} p(\varphi, s, z(s))ds, \varphi \in I_b = [0, b],$$

which is examined in [22].

Example 4.4. Let $q(\varphi, u, v, w) = q(\varphi, u, v, w),\ f(\varphi, z) = h(\varphi, z) = z$, then Eq (1.1) takes the form

$$z(\varphi) = q\left(\varphi, z(\alpha(\varphi)), z(\beta(\varphi)), \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds\right), \varphi \in I_b = [0, b],$$

which is examined in [19].

Example 4.5. Let $q(\varphi, u, v, w) = u + q(\varphi, w, v),\ h(\varphi, z) = z$ and $\gamma(\varphi) = \theta(\varphi) = \alpha(\varphi) = \varphi$, then Eq (1.1) takes the form

$$z(\varphi) = f(\varphi, z(\varphi)) + q\left(\varphi, \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds, z(\beta(\varphi))\right), \varphi \in I_b = [0, b],$$

which is studied in [21].

Example 4.6. Let $q(\varphi, u, v, w) = u + vw$, then Eq (1.1) takes the form

$$z(\varphi) = f(\varphi, z(\alpha(\varphi))) + h(\varphi, z(\beta(\varphi))) \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds, \varphi \in I_b = [0, b],$$

which is studied in [28].

Example 4.7. Let the following Volterra non-linear FIE:

$$\begin{aligned} z(\varphi) &= e^{-\varphi^2} + \left(\frac{e^{-\sqrt{\varphi}}\varphi^2}{3 + 3\varphi^4}\right)\ln(1 + |z(\varphi)|) + \left(\frac{\varphi^4}{4 + 4\varphi^4}\right)\arctan(|z(\sqrt{\varphi})|) \\ &\quad + \frac{1}{2}\int_0^{\sqrt{\varphi}} e^{-3\sqrt{s}}\left(\frac{e^{\sqrt{s}}}{4} + \sqrt{\varphi}\ \cos(s^2) + \frac{1}{2}z(s^2)\right)ds, \quad \varphi \in [0, 1]. \end{aligned} \tag{4.1}$$

Equation (4.1) is special case of Eq (1.1) with

$$\alpha(\varphi) = \varphi, \beta(\varphi) = \theta(\varphi) = \gamma(\varphi) = \sqrt{\varphi}, \forall\ \varphi \in [0, 1]$$

and

$$q(\varphi, u, v, w) = q_1(\varphi, u, v) + q_2(\varphi, w),$$

where

$$q_1(\varphi, u, v) = \frac{1}{3}u + \frac{1}{4}v, \quad u = \left(\frac{e^{-\sqrt{\varphi}}\varphi^2}{1 + \varphi^4}\right)\ln(1 + |z(\varphi)|),$$

$$v = \left(\frac{\varphi^4}{1 + \varphi^4}\right)\arctan(|z(\sqrt{\varphi})|), \quad q_2(\varphi, w) = \frac{w}{2},$$

$$w = \int_0^{\theta(\varphi)} p(\varphi, s, z(\gamma(s)))ds, \quad p(\varphi, s, z) = e^{-3\sqrt{s}}\left(\frac{e^{\sqrt{s}}}{4} + \sqrt{\varphi}\ \cos(s^2) + \frac{1}{2}z(s^2)\right).$$

It is obvious that assumptions (1) and (2) of Theorem 3.1 are satisfied. We need to check that assumption (3) holds true. So, choose $\rho = 3 + \frac{3}{4}e$, then $H_1 \le \frac{5e}{8} + \frac{5}{2}$ and

$$\begin{aligned} &\sup\{|q(\varphi, u, v, w)| : \varphi \in [0, 1], u, v \in [-\rho, \rho], w \in [-DH_1, DH_1]\} \\ &\le \sup\{\left|\frac{1}{3}u(\varphi) + \frac{1}{4}v + \frac{1}{2}w\right|; \varphi \in [0, 1], -\left(\frac{5e}{8} + \frac{5}{2}\right) \le w \le \left(\frac{5e}{8} + \frac{5}{2}\right)\} \\ &\le 3 + \frac{3}{4}e. \end{aligned}$$

All conditions of Theorem 3.1 are satisfied, Hence Eq (4.1) has at least one solution in $C[0, 1]$.

5. Conclusions

In this article, we have examined the existence of the solutions of non-linear functional integral equations in Banach algebra by utilizing a strategy, which is distinguishable from different authors technique (see [12, 16, 17, 21–23, 26]). The advantage of Theorem 2.6 among the others (Darbo and Schauder fixed point theorems) lies in that in using the theorem, one does not require to confirm the involved operator maps a closed convex subset onto itself.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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