Mathematics

Research article

The hunting cooperation of a predator under two prey's competition and fear-effect in the prey-predator fractional-order model

Ali Yousef[1]**, Ashraf Adnan Thirthar**[2]**, Abdesslem Larmani Alaoui**[3]**, Prabir Panja**[4] **and Thabet Abdeljawad**[5,6,*]

[1] Department of mathematics, Kuwait College of Science and Technology, 2723 Kuwait City, Kuwait

[2] Department of Studies and Planning, University of Fallujah, Anbar, Iraq

[3] Moulay Ismail university, FST Errachidia, MAIS Laboratory, MAMCS Group, Morocco

[4] Department of Applied Science, Haldia Institute of Technology, Purba Midnapore-721657, West Bengal, India

[5] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[6] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: This paper investigates a fractional-order mathematical model of predator-prey interaction in the ecology considering the fear of the prey, which is generated in addition by competition of two prey species, to the predator that is in cooperation with its species to hunt the preys. At first, we show that the system has non-negative solutions. The existence and uniqueness of the established fractional-order differential equation system were proven using the Lipschitz Criteria. In applying the theory of Routh-Hurwitz Criteria, we determine the stability of the equilibria based on specific conditions. The discretization of the fractional-order system provides us information to show that the system undergoes Neimark-Sacker Bifurcation. In the end, a series of numerical simulations are conducted to verify the theoretical part of the study and authenticate the effect of fear and fractional order on our model's behavior.

1. Introduction

In recent years, many studies have considered the ecological system's habitat from applied mathematics [1–4, 6–8, 10–26, 28–36, 39–42]. In these studeies, the fundamental research in mathematical modeling of the predator-prey model can be seen by Lotka and Volterra in [21, 35], which opened many new aspects in modeling predator-prey interaction and showed the possibility of studying and predicting their dynamics.

In describing the habitat of the predator-prey model, functional response represents one of the essential terms. It shows the manner of interaction between prey and predator. More precisely, it expresses the attack methods and the quantity of the predator to the prey. In the literature, there are three forms of functional response: Type I, Type II, and Type III, where the predator consumption rate increases linear, hyperbolic or sigmoidal, respectively. Several mathematical forms represent this dynamics of the prey-predator model, which is $f(x) = ax$ of Holling Type I and $f(x) = ax/b + x$ of Holling Type II, where a, b and x are respectively the maximum predation rate, the half-saturation constant, and the prey biomass. On the other hand, the multi-species form of prey and their competition might increase the chance of the predator to attack and hunt them if the predator species itself is also in cooperation during hunting. Recently, Alves and Hilker [1] included a cooperation term to the rate of attack of the predator population and proposed the functional response $P(x, y) = (p + by)x$, where x and $p > 0$ are respectively the prey density and the attack rate of the predator on the prey, while $b > 0$ describes the predator cooperation in hunting.

In this food chain cycle of the habitat, the fear of the prey started to become an important and realistic impact in the mathematical model to describe the predator-prey interaction, cooperation of the predators, and the fear of the prey. Many studies have been conducted to evaluate the effect of fear on prey population density [24, 31, 42] with different functions of response. However, it is new to study and examine the fear effect, the interaction of predator-prey, and the cooperation of predators in a single model.

Our following model considers an expanded form of Mukherjee [25] where we divide the ecological community into three compartments: $y(t)$ represents the predator, while there are two preys who are in competition and are denoted as $x_1(t)$ and $x_2(t)$, respectively. We consider a habitat where one predator compartment exists, which hunts $x_1(t)$ in cooperation, while species $x_2(t)$ is hunted individually. Also, we assume that both preys $x_1(t)$ and $x_2(t)$ are in competition which allows the predator to attack and hunt them easier. The collective hunting of predator $y(t)$ to species $x_1(t)$ activates a fear effect of the prey towards the hunter.

Therefore, this scenario is formulated as an ODE system as follows:

$$\begin{cases} \dfrac{dx_1}{dt} = \left(\eta + \dfrac{\kappa(1-\eta)}{\kappa + y}\right)r_1 x_1 - (d + a_1 x_1 + \beta_1 x_2)x_1 - (p + by)x_1 y \\ \dfrac{dx_2}{dt} = x_2(r_2 - \beta_2 x_1 - a_2 x_2) \\ \dfrac{dy}{dt} = c(p + by)yx_1 - my. \end{cases} \tag{1.1}$$

We equipped the system (1.1) with the initial condition

$$x_1(0) > 0, \quad x_2(0) > 0 \text{ and } y(0) > 0.$$

All the biological description of the parameters are given as below:

Parameter	Environmental Interpretation
r_1	Rate of the intrinsic growth of the prey
η	Minimum cost of fear
κ	Level of fear
r_2	Rate of intrinsic growth of the competitor for the prey
a_1	Coefficient of the intraspecific competition of the prey
a_2	Coefficient of the intraspecific competition of the predator
β_1	The interspecific competition coefficient of the competitor for the prey
β_2	The interspecific competition coefficient of the competitor for the predator
c	Efficiency of conversion of consumed prey into new predators
m	Rate of death of the predator
d	Death rate of the prey

The modeling of prey-predator systems through fractional-order differential equations has many advantages. The nonlocal property of fractional-order models not only depends on the current state but also depends on its prior historical states. The transformation of an integer-order model into a fractional-order model needs to be precise with respect to the order of differentiation α. However, a small change in α may cause a big change in the behavior of the solutions [41]. Fractional-order differential equations can model complex biological phenomena with non-linear behavior and long-term memory, which cannot be represented mathematically by integer-order differential equations (IDEs). For example, Bozkurt established the glioblastoma multiform (GBM)-immune system (IS) interaction using a fractional order differential equation system to include the delay time (memory effect) in [5]. Besides this, in many papers, researchers have proven the importance of fractional derivatives in modeling biological phenomena such as freedom towards ordering the derivative, dealing with species memory that has been achieved during their cycle life, genetic characteristics, and others (see for example [4, 6, 8, 9, 13, 27, 30, 37, 38, 41]). Based on this information, we modify model (1.1) in a fractional-order form to present the study in a more natural state that suits the food chain life cycle requirements. Thus, we will consider the fractional-order ecological model such as

$$\begin{cases} D^\alpha x_1(t) = \left(\eta + \dfrac{\kappa(1-\eta)}{\kappa + y}\right)r_1 x_1 - (d + a_1 x_1 + \beta_1 x_2)x_1 - (p + by)x_1 y \\ D^\alpha x_2(t) = x_2(r_2 - \beta_2 x_1 - a_2 x_2) \\ D^\alpha x_3(t) = c(p + by)x_1 y - my. \end{cases} \tag{1.2}$$

2. Preliminaries

In this part, we want to introduce some fundamental properties of fractional-order differential equations, which will be helpful in the main sections.

Definition 2.1. *[7] Caputo fractional derivatives can be given as follows*

$$D^\alpha f(t) = I^{d-\alpha} f^d(t), \quad \beta > 0,$$

where D^α is Caputo differential operator of order α, d is the least integer, which is not less than α, and I^θ is the Riemann-Liouville integral operator of order θ, which is given by

$$I^\theta g(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\tau)^{\theta-1} g(\tau) d\tau, \quad \theta > 0,$$

where $\Gamma(\theta)$ is the Euler's Gamma function.

Theorem 2.2. *[7] Consider the $N-$dimensional system*

$$\begin{cases} \dfrac{d^\alpha x}{dt^\alpha} = & Bx, \\ x(0) = & x_0, \end{cases}$$

where B is arbitrary constant $N \times N$ matrix and $\alpha \in (0, 1)$.

1) $x = 0$ *is asymptotically stable if and only if all eigenvalues λ_i, $i = 1, 2, \ldots, N$ of B satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.*
2) $x = 0$ *is stable if and only if all the eigenvalues of B satisfy $|arg(\lambda_i)| \geq \frac{\alpha\pi}{2}$ and eigenvalues with $|arg(\lambda_i)| = \frac{\alpha\pi}{2}$ have same geometric multiplicity and algebraic multiplicity.*

Theorem 2.3. *[7] Consider the fractional order system*

$$\begin{cases} \dfrac{d^\alpha x}{dt^\alpha} = f(x), \\ x(0) = x_0 \text{ with } \alpha \in (0, 1) \text{ and } x \in R^\alpha. \end{cases}$$

An equilibrium point x^* is locally asymptotically stable if all the eigenvalues λ_i of $J = \frac{\partial f(x^*)}{\partial x}$ satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

3. Non-negativity and boundedness of solution in system (1.2)

This part investigates the existence and uniqueness of the solution in (1.2) and the non-negativity and boundedness.

Theorem 3.1. *System* (1.2) *shows a unique solution for non-negative initial conditions.*

Proof. Let $E = \{(x_1, x_2, y) \in R^3; max\{|x_1|, |x_2|, |y| \leq K\}\}$. In this proof, we use Hong et al. approach's [13]. Let $X = (x_1, x_2, y)$ and $H(X) = (H_1(X), H_2(X), H_3(X))$, where we have

$$\begin{cases} H_1(X) = \left[\dfrac{\eta + \kappa(1-\eta)}{\kappa + y}\right] r_1 x_1 - (d + a_1 x_1 + \beta_1 x_2) x_1 - (p + by) x_1 y \\ H_2(X) = x_2(r_2 - \beta_2 x_1 - a_2 x_2) \\ H_3(X) = c(p + by) x_1 y - my. \end{cases} \tag{3.1}$$

For $X, \bar{X} \in E$, it obvious to see from (3.1) that we get

$$\begin{aligned} \left\|H(X) - H(\bar{X})\right\| = {} & |H_1(X) - H_1(\bar{X})| + |H_2(X) - H_2(\bar{X})| + |H_3(X) - H_3(\bar{X})| \\ \leq {} & [\eta r_1 + r_1 \kappa^2(1-\eta) + d + Kr_1\kappa(1-\eta) + 2Ka_1 + K\beta_1 \\ & + Kp + K\beta_2 + cpK + Kcb]|x_1 - \bar{x}_1| + [K\beta_1 + r_2 + K\beta_2 + 2Ka_2]|x_2 - \bar{x}_2| \\ & + [K(r_1\kappa(1-\eta)) + Kp + 2Ka_2 + Kcp + 2Kcb + m]|y - \bar{y}| \\ \leq {} & M\left\|X - \bar{X}\right\|, \end{aligned}$$

where

$$M = max\{\eta r_1 + r_1\kappa^2(1 - \eta) + d + K[r_1\kappa(1 - \eta) + 2a_1 + \beta_1 + p + \beta_2 + cp + cb],$$
$$K\beta_1 + r_2 + K\beta_2 + 2Ka_2, K[(r_1\kappa(1 - \eta)) + p + 2a_2 + cp + 2cb - m]\}.$$

From the analysis mentioned above, the Lipschitz condition of M satisfies. Thus, we can conclude that the solution of system (1.2) exists and shows a unique solution. □

Theorem 3.2. *The boundedness and non-negative behavior of all solutions of system* (1.2) *are permanent in* R_+^3.

Proof. Let L be the function defined as $L(t) = x_1(t) + x_2(t) + \frac{1}{c}y(t)$. Thus, for $\tau = min\{d\eta r_1 - \kappa(1 - \eta), m\}$, and $sup_{t\to\infty} x_2(t) = r_2/a_2$, we have

$$D^\alpha L(t) + \tau L(t) = \eta r_1 x_1 + \left[\frac{\kappa(1 - \eta)}{\kappa + y}\right] r_1 x_1 - dx_1 - a_1 x_1^2 - \beta_1 x_1 x_2 + r_2 x_2 - \beta_2 x_1 x_2$$
$$- a_2 x_2^2 - \frac{m}{c}y + \tau x_2(t) + \tau x_1(t) + \frac{\tau}{c}y(t)$$
$$= -a_2(x_2 - \frac{r_2}{2a_2})^2 + \frac{r_2^2}{4a_2} + (\tau - (d - \eta r_1 - \kappa(1 - \eta)))x_1$$
$$- (\beta_1 + \beta_2)x_1 x_2 + \tau x_2(t) + \frac{1}{c}(\tau - m)y(t)$$
$$< \frac{r_2}{a_2}\left(\frac{r_2}{4} + \tau\right).$$

By using the comparison theorem in [6], we obtain

$$L(t) \leq L(0)G_\alpha(-\tau(t)) + \frac{r_2}{a_2}\left(\frac{r_2}{4} + \tau\right)t^\alpha G_{\alpha,\alpha+1}(-\tau^\alpha(t)),$$

where G_α is the Mittag-Leffler function. Using Lemma 5 and Corollary 6 of [6], we have

$$L(t) \leq \frac{r_2}{\tau a_2}\left(\frac{r_2}{4} + \tau\right) \text{ as } \quad t \to \infty.$$

Hence, it is proven that all solutions of model (1.2) initiating in R_+^3 are in the region Γ, where

$$\Gamma = (x_1, x_2, y) \in R_+^3 : L \leq \frac{r_2}{\tau a_2}\left(\frac{r_2}{4} + \tau\right) + \epsilon, \epsilon > 0.$$

Now, we can see also the non-negativity of all solutions in system (1.2). From the first equation of (1.2) and Γ we obtain,

$$D^\alpha x_1(t) \geq qx_1,$$

where

$$P = \frac{r_2}{\tau a_2}\left(\frac{r_2}{4} + \tau\right),$$
$$Q = \eta + \left[\frac{\kappa(1 - \eta)}{\kappa + cp}\right] r_1 - (d + a_1 P + \beta_1 P) - c(p + bP)P.$$

By using the standard comparison theorem for fractional-order in Choi et al. [6] and the fact that $G_{\alpha,1}(t) > 0$ for any $\alpha \in (0, 1)$ in [37], it follows that

$$x_1(t) \geq x_{1_0}G_{(\alpha,1)}(qt^\alpha); x_1(t) \geq 0.$$

Similarly, we can get from the second and third equations of (1.2) the following

$$x_2(t) \geq x_{2_0} G_{(\alpha,1)}(-(\beta_2 P - a_2 P)t^\alpha),$$
$$y(t) \geq y_0 G_{(\alpha,1)}(-mt^\alpha)y.$$

Hence, obtain also $y(t) \geq 0$ and $x_2(t) \geq 0$. □

To analyse the stability conditions for the next section, we need to introduce the Jocabian matrix $J(x_1, x_2, y)$ associated to system (1.2), which is given by the following:

$$J(x_1, x_2, y) =$$
$$\begin{pmatrix} \frac{\eta+\kappa(1-\eta)}{\kappa+y} r_1 - (d + 2a_1 x_1 + \beta_1 x_2) - (p + by)y & -\beta_1 x_1 & -\left[\left(\frac{\kappa(1-\eta)}{(\kappa+y)^2}\right) r_1 + p + 2by\right] x_1 \\ -\beta_2 x_2 & r_2 - \beta_2 x_1 - 2a_2 x_2 & 0 \\ c(p + by)y & 0 & cpx_1 + 2cbx_1 y - m \end{pmatrix}.$$

4. Equilibria and their stability

System (1.2) has four equilibria:

1) Trivial equilibrium $P_0(0, 0, 0)$, which exists always.
2) Prey equilibrium $P_1(\frac{d-1}{a}, 0, 0)$, which exists provided that $d > 1$.
3) Predator-free equilibrium $P_2(\bar{x}_1, \bar{x}_2, 0)$ where $\bar{x}_1 = \frac{1}{\beta_2}(r_2 - a_2\bar{x}_2)$ and $\bar{x}_2 = \frac{a_1 r_1 + \beta_2 d - \beta_2 r_1}{a_1 a_2 - \beta_1 \beta_2}$, which exists provided that: $a_1 r_1 + \beta_2 d > \beta_2 r_1$, $a_1 a_2 > \beta_1 \beta_2$ and $r_2 > a_2 \bar{x}_2$.
4) The positive equilibrium point $P^*(x_1, \frac{1}{b}(\beta_2 x_1 - r_2), \frac{m-cpx_1}{cpx_1})$ exists, if $\beta_2 x_1 > r_2$ and $m > cpx_1$, where x_1 is a root of the following polynomial

$$A_4 x_1^4 + A_3 x_1^3 + A_2 x_1^2 + A_1 x_1 + A_0 = 0. \tag{4.1}$$

Here, we denote $A_i, i = 0, 1, 2, 3, 4$ such as

$$A_0 = -bm^3$$
$$A_1 = -(\kappa b - 2m^2)cpb$$
$$A_2 = -mc^2 b^2(d - \eta r_1) - mc^2 p^2(3b - 2) - \kappa pmbc^2(b - 2) - \beta_1 r_2 mc^2 b$$
$$A_3 = \eta r_1 c^3 b^3(\kappa - 1) + \kappa r_1 c^3 b^3(1 - \eta) - \kappa c^3 b^3(\kappa - 1) + \beta_1 r_2 c^3 b^3(p - \kappa) + \kappa p^2 c^3 b^2(1 - p)$$
$$\quad + mac^2 b^2 + \beta_1 \beta_2 mc^2 b^2$$
$$A_4 = ac^3 b^3(\kappa - 1) + \beta_1 \beta_2 c^3 b(\kappa b - p).$$

It is obvious to show that A_4 and A_1 has positive and negative signs, respectively, under the condition that we have

$$2p \leq \kappa b \leq bp.$$

The number of positive real roots of (4.1) can be determined from the signs of A_3 and A_2. The number of roots will be revealed using the Descartes Rule. A number of positive roots are illustrated in Table 1.

Table 1. Number of possible positive roots of the fourth-degree polynomial equation.

Cases	A4	A3	A2	A1	A0	Changes in Sign	Total Possible Positive Roots
1	+	+	+	-	-	1	1
2	+	+	-	-	-	1	1
3	+	-	-	-	-	1	1
4	+	-	+	-	-	3	0,3

Therefore, results can be determined by the following Lemma.

Lemma 4.1. *The fractional prey-predator model* (1.2) *has*

1) *A unique positive equilibrium point if cases 1–3 are satisfied.*
2) *More than one positive equilibrium when case 4 is satisfied.*

Theorem 4.2. *The trivial equilibrium point $P_0(0,0,0)$ is always unstable.*

Proof. The characteristic equation of J at the trivial equilibrium point $P_0(0,0,0)$ has the following form

$$(\lambda - r1 + d)(\lambda - r2)(\lambda + m) = 0.$$

The eigenvalues are $\lambda_1 = r_1 - d$; $\lambda_2 = r_2$ and $\lambda_3 = -m$. Thus $|arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$, whenever $r_1 > d$ and $|arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$, since $r_2 > 0$ and $|arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$. Hence, the convergence of the Mittag-Leffler function ensures that P_0 is always a saddle point. The instability of the trivial equilibrium point ensures that we do not expect a total extinction of the food chain cycle in the habitat. $\square$

Theorem 4.3. *The prey equilibrium point $P_1(\frac{r_1-d}{a1},0,0)$ is always unstable.*

Proof. The characteristic equation of J at the point P_1 has the following form

$$(\lambda - r_1 + d)\left(\lambda - r_2 + \frac{\beta_2(r_1 - d)}{a_1}\right)\left(\lambda - \frac{cp(r_1 - d)}{a_1} + m\right) = 0.$$

The eigenvalues are $\lambda_1 = r_1 - d$; $\lambda_2 = r_2 - \frac{\beta_2(r_1-d)}{a_1}$ and $\lambda_3 = \frac{cp(r_1-d)}{a_1} - m$. Now, if $r_1 < d$; $r_2 a_1 < \beta_2(r_1 - d)$ and $cp(r_1 - d) < a_1 m$, then $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$; $|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$. Hence the prey equilibrium P_1 is locally asymptotically stable if $r_1 < d$; $r_2 a_1 < \beta_2(r_1 - d)$ and $cp(r_1 - d) < a_1 m$, which is not possible because the condition of existence for this point is $r_1 > d$. Thus, P_1 is unstable. $\square$

Theorem 4.4. *The local asymptotic stability of $P_2(\bar{x}_1, \bar{x}_2, 0)$ holds if $C_1 > 0, C_3 > 0$ and $C_1 C_2 > C_3$.*

Proof. The characteristic equation of the Jacobian matrix J at $P_2(\bar{x}_1, \bar{x}_2, 0)$ is given by

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0,$$

where $C_1 = -J_{11} - J_{33} - J_{22}, C_2 = J_{22}J_{33} + J_{11}J_{33} + J_{12}J_{21} + J_{13}J_{31} + J_{11}J_{22}, C_3 = -J_{11}J_{22}J_{33} - J_{12}J_{21}J_{33} - J_{13}J_{31}J_{22}, J_{11} = r_1 - (d + 2ax_1 + \beta_1 x_2), J_{12} = -\beta_1 x_1, J_{13} = [(1 - \eta)/\kappa r1 + p]x_1, J_{21} = -\beta_2 x_2, J_{22} = r2 - \beta_2 x_1 - a_2 x_2, J_{31} = c(p + by)y$ and $J_{33} = cpx_1 - m$. The above equation has three values $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}, |arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}, |arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$, if $C_1 > 0, C_3 > 0$ and $C_1 C_2 > C_3$ holds. $\square$

Theorem 4.5. *The positive equilibrium point P^* is conditionally locally asymptotically stable.*

Proof. The characteristic equation associated with $J = [\rho_{ij}]_{i:j=1,2,3}$ around the positive equilibrium point P^* is

$$\psi(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$A_1 = -(\rho_{11} + \rho_{22} + \rho_{33})$$

$$A_2 = \rho_{11}\rho_{33} + \rho_{22}\rho_{33} + \rho_{11}\rho_{33} + \rho_{11}\rho_{22} - \rho_{12}\rho_{21} - \rho_{31}\rho_{13}$$

$$A_3 = \rho_{12}\rho_{21}\rho_{33} + \rho_{31}\rho_{13}\rho_{22} - \rho_{11}\rho_{22}\rho_{33}.$$

Let $D(\psi)$ be the discriminant of $\psi(\lambda)$, it can be given as

$$\begin{pmatrix} 1 & A_1 & A_2 & A_3 & 0 \\ 0 & 1 & A_1 & A_2 & A_3 \\ 3 & 2A_1 & A_2 & 0 & 0 \\ 0 & 3 & 2A_1 & A_2 & 0 \\ 0 & 0 & 3 & 2A_1 & A_2 \end{pmatrix}.$$

Then, we have

$$D(\psi) = 18A_1A_2A_3 + (A_1A_2)^2 - 4A_3A_1^2 - 4A_2^3 - 27A_3^2.$$

The positive equilibrium point P^* is locally asymptotically stable provided one of the following satisfy:

1) $D(\psi) > 0, A_1 > 0, A_1A_2 > A_3$ and $A_3 > 0$.
2) $D(\psi) < 0, A_1 \geq 0, A_3 > 0, A_2 \geq 0$ and $\theta < 2/3$.
3) $D(\psi) < 0, A_1 > 0, A_1A_2 = A_3, A_2 > 0$ and for all $\theta \in (0, 1)$.

$\square$

Theorem 4.6. *The positive equilibrium point P^* is conditionally globally asymptotically stable.*

Proof. Let V be the Lyapunov function defined such as

$$V(t) = \left(x_1 - x_1^* - x_1^* ln\frac{x_1}{x_1^*}\right) + w_1\left(x_2 - x_2^* - x_2^* ln\frac{x_2}{x_2^*}\right) + w_2\left(y - y^* - y^* ln\frac{y}{y^*}\right),$$

where w_1 and w_2 are non-negative constants. Taking fractional-order derivative on both sides, we have

$$D^\alpha V(t) = (x_1 - x_1^*)\left[\alpha(1 - \eta)r_1\left\{\frac{1}{y + \alpha} - \frac{1}{\alpha + y^*}\right\} - a(x_1 - x_1^*) - \beta_1(x_2 - x_2^*)\right.$$

$$\left. - b(y^2 - y^{*2}) - p(y - y^*)\right] + w_1(x_2 - x_2^*)[-a_2(x_2 - x_2^*) - \beta_2(x_1 - x_1^*)]$$

$$+ w_2(y - y^*)[cp(x_1 - x_1^*) + cb(x_1y - x_1^*y^*)].$$

Calculations show that we obtain

$$D^\alpha V(t) = -w_1b(x_2 - x_2^*)^2 - a(x_1 - x_1^*)^2 - w_2cbx_1^*(y - y^*)^2 - \frac{\alpha(1 - \eta)r_1(x_1 - x_1^*)}{(\alpha + y^*)(\alpha + y)}$$

$$+ (\beta_1 - \beta_2w_1)(x_2 - x_2^*)(x_1 - x_1^*) - p(1 - cw_2)(x_1 - x_1^*)(y - y^*)$$

$$- b(y - y^*)(x_1 - x_1^*)\{y(1 - cw_2) + y^*\}.$$

If we choose now $w_1 = \frac{\beta_1}{\beta_2}$ and $w_2 = \frac{1}{c}$, then it is seen that we have

$$D^\alpha V(t) = -\frac{b\beta_1}{\beta_2}(x_2 - x_2^*)^2 - a(x_1 - x_1^*)^2 - bx_1^*(y - y^*)^2 - \frac{\alpha(1 - \eta)r_1(x_1 - x_1^*)}{(\alpha + y^*)(\alpha + y)}$$
$$- b(y - y^*)(x_1 - x_1^*).$$

Considering the ecological, environmental event, we assumed that all the parameters are positive and in addition, if $\eta < 1$, $x_1 > x_1^*$ and $y > y_1^*$, then we have $D^\alpha V(t) \leq 0$. This completes the proof. $\qquad\square$

5. Bifurcation analysis

5.1. Neimark-Sacker bifurcation

In this part, the Neimark-Sacker bifurcation conditions of the Caputo fractional order model are investigated. Now, let us take the following Caputo fractional-order model

$$D^\alpha y = g(a, y), \text{ where } \alpha \in (0, 1), y \in R^3.$$

We suppose that E^* is an equilibrium point of system (1.2). Then (1.2) undergoes a Neimark-Sacker bifurcation around the point E^* concerning the parameter a at $a = a^*$ provided that the following conditions are satisfied:

(i) the Jacobian matrix of system (1.2) at the equilibrium point E^* has a pair of complex conjugate eigenvalues $\lambda_{1,2} = \alpha_j \pm i\omega_j$ become purely imaginary at $a = a^*$.

(ii) $\xi_{1,2}(\alpha, a^*) = 0$.

(iii) $\left.\frac{\partial \xi_{1,2}}{\partial a}\right|_{a=a^*} \neq 0$.

where $\xi_i(\alpha, a) = -\min_{i=1,2}\left|\arg(\lambda_i(a))\right| + \frac{\alpha\pi}{2}$.

The fractional-order derivative has an important role in increasing the stability of the considered model. Therefore, the conditions for which the system (1.2) undergoes Neimark-Sacker bifurcation concerning α is established as follows:

(i) the Jacobian matrix of system (1.2) at E^* has a pair of complex conjugate eigenvalues $\lambda_{1,2} = \alpha_j \pm i\omega_j$ become purely imaginary at $\alpha = \alpha^*$.

(ii) $\phi_{1,2}(\alpha^*) = 0$.

(iii) $\left.\frac{\partial \phi_{1,2}}{\partial \alpha}\right|_{\alpha=\alpha^*} \neq 0$ where $\phi_i(\alpha) = \frac{\alpha\pi}{2} - \min_{i=1,2}\left|\arg(\lambda_i(\alpha))\right|$.

5.2. Neimark-Sacker bifurcation with discretization

The discretization of system (1.2) is as follows

$$\left.\begin{aligned}
D^\alpha x_1 &= (\eta + \frac{\kappa(1-\eta)}{\kappa+y([\frac{t}{x}]x)})r_1 x_1([\frac{t}{x}]x) - (d + a_1 x_1([\frac{t}{x}]x) + \beta_1 x_2([\frac{t}{x}]x))x_1([\frac{t}{x}]x) \\
&\quad -(p + by([\frac{t}{x}]x))x_1([\frac{t}{x}]x)y([\frac{t}{x}]x) \\
D^\alpha x_2 &= x_2([\frac{t}{x}]x)(r_2 - \beta_2 x_1([\frac{t}{x}]x) - a_2 x_2([\frac{t}{x}]x)) \\
D^\alpha y &= c(p + by([\frac{t}{x}]x))y([\frac{t}{x}]x)x_1([\frac{t}{x}]x) - my([\frac{t}{x}]x).
\end{aligned}\right\} \qquad (5.1)$$

Then the solution of system (1.2) for $t \in [0, h)$, $\frac{t}{h} \in [0, 1)$ is given by

$$
\begin{aligned}
x_1(1) &= x_1(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}[(\eta + \frac{\kappa(1-\eta)}{\kappa+y(0)})r_1 x_1(0) - (d + a_1 x_1(0) + \beta_1 x_2(0))x_1(0) \\
&\quad -(p + by(0))x_1(0)y(0)] \\
x_2(1) &= x_2(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}[x_2(0)(r_2 - \beta_2 x_1(0) - a_2 x_2(0))] \\
y(1) &= y(0) + \frac{t^\alpha}{\Gamma(\alpha+1)}[c(p + by(0))y(0)x_1(0) - my(0)].
\end{aligned}
\tag{5.2}
$$

If we repeat the discretization process n times, it is obtained that

$$
\begin{aligned}
x_1(n+1) &= x_1(n) + \frac{(t-nh)^\alpha}{\Gamma(\alpha+1)}[(\eta + \frac{\kappa(1-\eta)}{\kappa+y(n)})r_1 x_1(n) - (d + a_1 x_1(n) + \beta_1 x_2(n))x_1(n) \\
&\quad -(p + by(n))x_1(n)y(n)] \\
x_2(n+1) &= x_2(n) + \frac{(t-nh)^\alpha}{\Gamma(\alpha+1)}[x_2(n)(r_2 - \beta_2 x_1(n) - a_2 x_2(n))] \\
y(n+1) &= y(n) + \frac{(t-nh)^\alpha}{\Gamma(\alpha+1)}[c(p + by(n))y(n)x_1(n) - my(n)].
\end{aligned}
\tag{5.3}
$$

Now for $t \in [nh, (n+1)h)$ and $t \to (n+1)h$ while $\alpha \to 1$, it is obtained that

$$
\begin{aligned}
x_1(n+1) &= x_1(n) + \frac{h^\alpha}{\Gamma(\alpha+1)}[(\eta + \frac{\kappa(1-\eta)}{\kappa+y(n)})r_1 x_1(n) - (d + a_1 x_1(n) + \beta_1 x_2(n))x_1(n) \\
&\quad -(p + by(n))x_1(n)y(n)] \\
x_2(n+1) &= x_2(n) + \frac{h^\alpha}{\Gamma(\alpha+1)}[x_2(n)(r_2 - \beta_2 x_1(n) - a_2 x_2(n))] \\
y(n+1) &= y(n) + \frac{h^\alpha}{\Gamma(\alpha+1)}[c(p + by(n))y(n)x_1(n) - my(n)].
\end{aligned}
\tag{5.4}
$$

The Jacobian matrix of system (5.1) around the interior equilibrium point E^* is given by

$$
J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & 0 \\ J_{31} & 0 & J_{33} \end{pmatrix},
\tag{5.5}
$$

where

$$
J_{11} = 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}[(\eta + \frac{\kappa(1-\eta)}{\kappa+y^*})r_1 - d - 2a_1 x_1^* - \beta_1 x_2^* - py^* - by^{*2}], J_{12} = -\frac{h^\alpha \beta_1 x_1^*}{\Gamma(\alpha+1)}
$$

$$
J_{13} = -\frac{h^\alpha}{\Gamma(\alpha+1)}\left[\frac{r_1\kappa(1-\eta)x_1^*}{(\kappa+y^*)^2} + px_1^* + 2bx_1^* y^*\right], J_{21} = -\frac{h^\alpha \beta_2 x_2^*}{\Gamma(\alpha+1)}
$$

$$
J_{22} = 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}[r_2 - \beta_2 x_1^* - 2a_2 x_2^*], J_{31} = \frac{h^\alpha}{\Gamma(\alpha+1)}[c(p + by^*)y^*]
$$

$$
J_{33} = 1 + \frac{h^\alpha}{\Gamma(\alpha+1)}[cpx_1^* + 2cbx_1^* y^* - m].
$$

While the characteristic equation of the Jacobian matrix (5.5) is given by

$$
\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0,
\tag{5.6}
$$

where

$$
B_1 = -J_{11} - J_{22} - J_{33}, B_2 = J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{12}J_{21} - J_{13}J_{31},
$$

$$
B_3 = J_{12}J_{21}J_{33} + J_{13}J_{31}J_{22} - J_{11}J_{22}J_{33}.
$$

Now, using the Jury condition [9], the unique positive equilibrium (x_1^*, x_2^*, y^*) is locally asymptotically stable if the following conditions are satisfied

$$|B_1 + B_3| < 1 + B_2, |B_1 + 3B_3| < 3 - B_2 \text{ and } B_3^2 + B_2 - B_1 B_3 < 1.$$

To study the Neimark-Sacker bifurcation in the system (5.1), we need the following explicit criterion of Hopf bifurcation.

Lemma 5.3. *(See [38]) Consider an n-dimensional discrete dynamical system* $Y_{k+1} = f_\mu(Y_k)$ *where* $\mu \in R$ *is the bifurcation parameter. Let* Y^* *be a fixed point of* f_μ *and the characteristic polynomial for Jacobian matrix* $J(Y^*) = (b_{ij})_{n \times n}$ *of the n-dimensional map* f_μ *is given by*

$$Q_\mu(\lambda) = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + ... + b_{n-1}\lambda + b_n. \tag{5.7}$$

where $b_i = b_i(\mu, u), i = 1, 2, ..., n$ *and u is a control parameter to be determined. Let* $\Delta_0^\pm(\mu, u) = 1, \Delta_1^\pm(\mu, u), ..., \Delta_n^\pm(\mu, u)$ *be a sequence of determinants defined by* $\Delta_i^\pm(\mu, u) = det(N_1 \pm N_2), i = 1, 2, ..., n$ *where*

$$N_1 = \begin{bmatrix} 1 & b_1 & b_2 & ... & b_{i-1} \\ 0 & 1 & b_1 & ... & b_{i-2} \\ 0 & 0 & 1 & ... & b_{i-3} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \text{ and } N_2 = \begin{bmatrix} b_{n-i+1} & b_{n-i+2} & ... & b_{n-1} & b_n \\ b_{n-i+2} & b_{n-i+3} & ... & b_n & 0 \\ ... & ... & ... & ... & ... \\ b_{n-1} & b_n & ... & 0 & 0 \\ b_n & 0 & ... & 0 & 0 \end{bmatrix}.$$

Moreover, the following conditions hold:

(H1) *Eigenvalue assignment:* $\Delta_{n-1}^-(\mu_0, u) = 0, \Delta_{n-1}^+(\mu_0, u) > 0, Q_{\mu_0}(1) > 0, (-1)^n Q_{\mu_0}(-1) > 0, \Delta_i^\pm(\mu_0, u) > 0, i = n - 3, n - 5, ..., 1(or2)$ *when n is even or odd respectively.*

(H2) *Transversality condition:* $[(\frac{d}{d\mu})(\Delta_{n-1}^-(\mu, u))]_{\mu=\mu_0} \neq 0.$

(H3) *Nonresonance condition:* $cos(2\pi/j) \neq \phi$ *or resonance condition* $cos(2\pi/j) = \phi$ *for* $j = 3, 4, 5, ...$ *and* $\phi = -1 + 0.5 Q_{\mu_0}(1)\Delta_{n-3}^-(\mu_0, u)/\Delta_{n-2}^+(\mu_0, u).$ *Then Neimark-Sacker bifurcation happen if we take* μ *as a bifurcation parameter.*

Theorem 5.4. *The unique interior equilibrium point of model (5.4) undergoes Neimark-Sacker bifurcation if the following conditions hold:*

$$1 - B_2 + B_3(B_1 - B_3) = 0,$$
$$1 + B_2 - B_3(B_1 + B_3) > 0,$$
$$1 + B_1 + B_2 + B_3 > 0,$$
$$1 - B_1 + B_2 - B_3 > 0,$$

where B_1, B_2, B_3 *are described in Eq (5.6).*

Proof. According to Lemma 1 for $n = 3$ we have in Eq (5.6), the characteristic polynomial of system (5.4) evaluated at this positive interior equilibrium. In this case, we obtain the following conclusions:

$$\Delta_2^-(\mu) = 1 - B_2 + B_3(B_1 - B_3) = 0, \qquad \Delta_2^+(\mu) = 1 + B_2 - B_3(B_1 + B_3) > 0,$$
$$Q_\mu(1) = 1 + B_1 + B_2 + B_3 > 0, \qquad (-1)^3 Q_\mu(-1) = 1 - B_1 + B_2 - B_3 > 0.$$

□

6. Numerical simulations

In this part, we plot some simulations to support the previous results. We take $\alpha = 1.0, 0.9$ and 0.7, to show the effect of the fractional-order derivatives on the dynamics of the proposed model. The system's initial conditions are chosen as $(0.14, 0.25, 0.37)$. The system parameter values are summarized in Table 2.

By analyzing the obtained figures, we reached some numerical fndings. In Figures 1–6, we have implemented the dynamics of the three species concerning time t. The behaviors of x_1, x_2 and y populations are investigated with different parameter values listed in Table 2. We have also plotted the evolution of x_1, x_2 and y concerning the fractional-order derivatives $\alpha = 1.0, 0.9$ and 0.7. From Figure 1, we observe the stability of the prey equilibrium point for the parameter values of 2. 2 shows the existence of oscillations in the dynamics of the three species. In Figures 3 and 4, we take the fair parameter $k = 0.2$ and $k = 2$. It was obvious that the fair changed the behaviors of the species. In Figures 5 and 6, we choose the order of the fractional derivative $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.7$. We noticed that the approach of the fractional-order derivative α to 0 showed stability in the dynamical behavior of the three species.

Table 2. Parameter values of the system.

| Parameter	Value
r_1	0.200
η	0.700
r_2	0.100
a_1	0.010
a_2	0.040
β_1	0.381
β_2	0.300
c	0.040
m	0.500
p	0.050
b	0.020
d	0.100
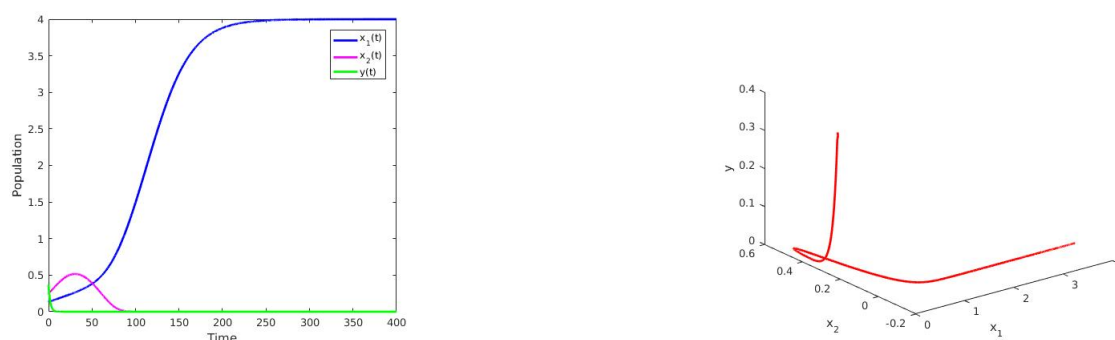
**Figure 1.** Stability of prey equilibrium point with parameter values of Table 2 and $k = 0$.
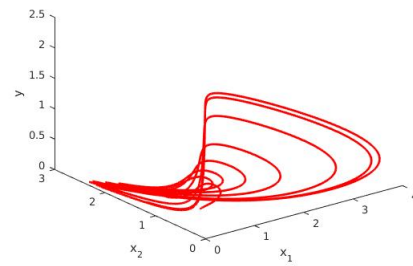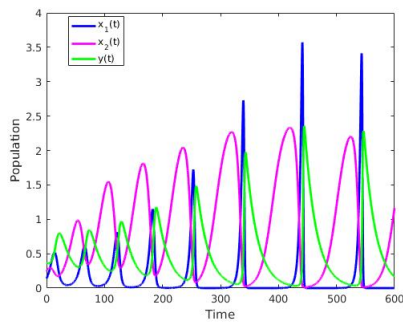
Figure 2. Unstable dynamics of the system equilibrium point with parameter values of the Table 2 except $r_1 = 0.3$, $d = 0.01$, $c = 0.4$, $p = 0.5$ and for $\alpha = 1$ and $k = 0$ (without fear effect).
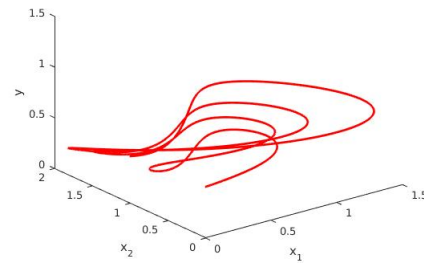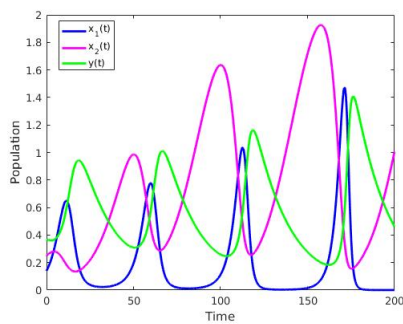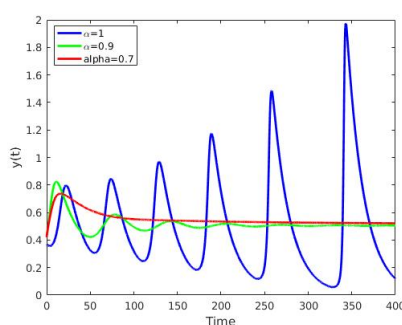

Figure 3. Behavior of the three species $x_2(t)$, $x_1(t)$ and $y(t)$ with the same set of parameters used in Figure 2 except $k = 0.2$ (with fear effect).
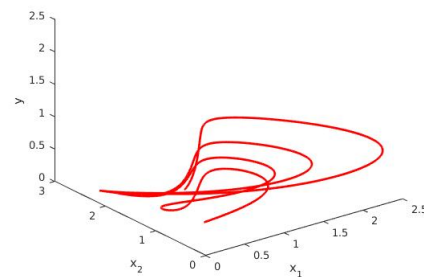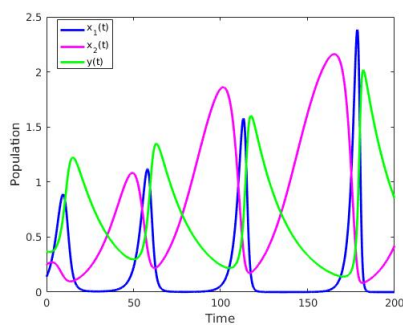
Figure 4. Trajectories of system (1.1) with the same set of parameters used in Figure 2 except $k = 2$ (with fear effect).
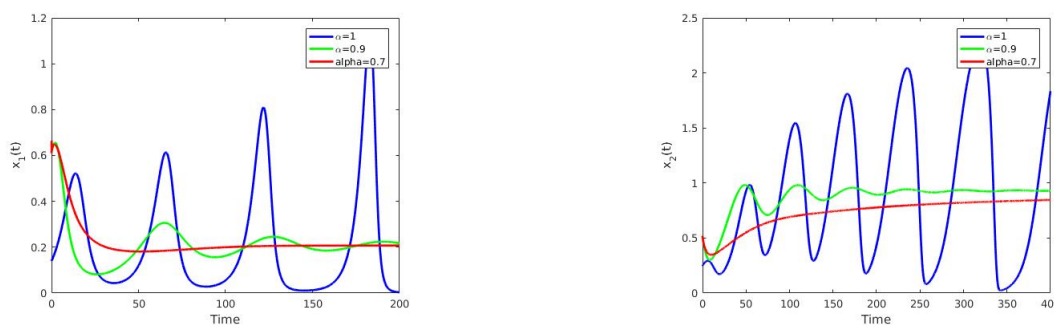
Figure 5. Fractional-order derivative impact on the behavior of x_1 and x_2.

Figure 6. Fractional-order derivative impact on the dynamics of y.

7. Conclusions

This work dealt with an ecological food chain cycle of a fractional-order predator-prey system. Both preys competed, while the single predator species showed cooperation in hunting species x_1. This collective strategy leads to a fear effect in the x_1 compartment. The second prey species x_2 was hunted individually so that an extreme change could not be noticed on this site. We used the Caputo fractional-order derivative considering the historical state of memory effect in the system. This study shows that fractional-order derivatives had a crucial role in controlling the stability of solutions in the three species. Furthermore, it affected the dynamics of solution, as can be seen in Figures 5 and 6. We observe from these figures that the approach to zero of the fractional-order derivative increases stability in the system of the three species x_1, x_2 and y. Moreover, the change of fair parameter k showed infuence in temporal behaviors of the three species. In Figures 1 and 4, one can see that the fear of x_1 is effective when the population density of the predator increases. In this stage, the density of x_1 decreases. The case of predator-free equilibrium point shows that the compartment of x_1 increases the population's carrying capacity without any fear. However, x_2 is below the threshold of its compartment, and therefore it is extinct. y doesn't have enough food in the habitat and shrinks.

Acknowledgments

Conflict of interest

The authors declare no conflict of interest.

References

1. M. T. Alves, F. M. Hilker, Hunting cooperation and Allee effects in predators, *J. Theor. Biol.*, **419** (2017), 13–22. https://doi.org/10.1016/j.jtbi.2017.02.002

2. A. Atangana, J. F. Gomez-Aguilar, Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena, *Eur. Phys. J. Plus.*, **133** (2018), 166. https://doi.org/10.1140/epjp/i2018-12021-3

3. C. M. Baker, F. Diele, D. Lacitignola, C. Marangi, A. Martiradonna, Optimal control of invasive species through a dynamical systems approach, *Nonlinear Anal.-Real*, **49** (2019), 45–70. https://doi.org/10.1016/j.nonrwa.2019.02.007

4. D. Barman, J. Roy, H. Alrabaiah, P. Panja, S. P. Mondal, S. Alam, Impact of predator incited fear and prey refuge in a fractional order prey predator model, *Chaos Soliton. Fract.*, **142** (2021), 110420. https://doi.org/10.1016/j.chaos.2020.110420

5. F. Bozkurt, Stability analysis of a fractional order differential equation system of a GBM-IS interaction depending on the density, *Appl. Math. Inf. Sci.*, **8** (2014), 1021–1028. http://dx.doi.org/10.12785/amis/080310

6. S. K. Choi, B. Kang, N. Koo, Stability for Caputo fractional differential systems, *Abstr. Appl. Anal.*, **2014** (2014), 631419. https://doi.org/10.1155/2014/631419

7. K. Diethelm, *The analysis of fractional differenial equations*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2

8. S. Djilali, B. Ghanbari, The influence of an infectious disease on a prey-predator model equipped with a fractional-order derivative, *Adv. Differ. Equ.*, **2021** (2021), 20. https://doi.org/10.1186/s13662-020-03177-9

9. L. Edelstein-Keshet, *Mathematical models in biology*, Philadelphia, Pa, USA: Society for Industrial and Applied Mathematics, 2005. https://doi.org/10.1137/1.9780898719147

10. B. Ghanbari, S. Djilali, Mathematical and numerical analysis of a three-species predator-prey model with herd behavior and time fractional-order derivative, *Math Method. Appl. Sci.*, **43** (2020), 1736–1752. https://doi.org/10.1002/mma.5999

11. B. Ghanbari, S. Djilali, Mathematical analysis of a fractional-order predator-prey model with prey social behavior and infection developed in predator population, *Chaos Soliton. Fract.*, **138** (2020), 109960. https://doi.org/10.1016/j.chaos.2020.109960

12. M. E. Gilpin, M. L. Rosenzweig, Enriched predator-prey systems: Theoretical stability, *Science*, **177** (1972), 902–904. http://doi.org/10.1126/science.177.4052.902

13. L. L. Hong, L. Zhang, C. Hu, Y. L. Jiang, Z. D. Teng, Dynamical analysis of a fractional order predator-prey model incorporating a prey refuge, *J. Appl. Math. Comput.*, **54** (2017), 435–449. https://doi.org/10.1007/s12190-016-1017-8

14. S. B. Hsu, J. P. Shi, Relaxation oscillation profile of limit cycle in predator-prey system, *Discrete Cont. Dyn. Syst. B*, **11** (2009), 893–911. https://doi.org/10.3934/dcdsb.2009.11.893

15. S. B. Hsu, T. W. Hwang, Y. Kuang, Global dynamics of a predator-prey model with Hassell-Varley type functional response, *Discrete Cont. Dyn. Syst. B*, **10** (2008), 857–871. https://doi.org/10.3934/dcdsb.2008.10.857

16. M. Kot, *Elements of mathematical ecology*, Cambridge: Cambridge University Press, 2001. http://doi.org/10.1017/CBO9780511608520

17. S. Kumar, R. Kumar, R. P. Agarwal, B. Samet, A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods, *Math. Method. Appl. Sci.*, **43** (2020), 5564–5578. https://doi.org/10.1002/mma.6297

18. S. Kumar, A. Ahmadian, R. Kumar, D. Kumar, J. Singh, D. Baleanu, et al., An efficient numerical method for fractional SIR epidemic model of infectious disease by using Bernstein wavelets, *Mathematics*, **8** (2020), 558. https://doi.org/10.3390/math8040558

19. S. Kumar, R. Kumar, C. Cattani, B. Samet, Chaotic behavior of fractional predator-prey dynamical system, *Chaos Soliton. Fract.*, **135** (2020), 109811. https://doi.org/10.1016/j.chaos.2020.109811

20. Y. Kuznetsov, S. Rinaldi, Remarks on food chain dynamics, *Math. Biosci.*, **134** (1996), 1–33. http://doi.org/10.1016/0025-5564(95)00104-2

21. A. J. Lotka, *Elements of physical biology*, *Nature*, **461** (1925), 116. http://dx.doi.org/10.1038/116461b0

22. S. J. Majeed, R. K. Naji, A. A. Thirthar, The dynamics of an Omnivore-predator-prey model with harvesting and two different nonlinear functional responses, *AIP Conf. Proc.*, **2096** (2019), 020008. https://doi.org/10.1063/1.5097805

23. S. Mondal, G. P. Samanta, Dynamics of an additional food provided predator-prey system with prey refuge dependent on both species and constant harvest in predator, *Physica A*, **534** (2019), 122301. https://doi.org/10.1016/j.physa.2019.122301

24. S. Mondal, A. Maiti, G. P. Samanta, Effects of fear and additional food in a delayed predator-prey model, *Biophys. Rev. Lett.*, **13** (2018), 157–177. https://doi.org/10.1142/S1793048018500091

25. D. Mukherjee, Study of fear mechanism in predator-prey system in the presence of competitor for the prey, *Ecol. Genet. Genomics*, **15** (2020), 100052. https://doi.org/10.1016/j.egg.2020.100052

26. J. D. Murray, *Mathematical biology*, Springer, New York, 1993. https://doi.org/10.1007/978-3-662-08542-4

27. Z. Odibat, S. Momani, An algorithm for the numerical solution of differential equations of fractional order, *J. Appl. Math. Inform.*, **26** (2008), 15–27.

28. K. M. Owolabi, E. Pindza, A. Atangana, Analysis and pattern formation scenarios in the superdiffusive system of predation described with Caputo operator, *Chaos Soliton. Fract.*, **152** (2021), 111468. https://doi.org/10.1016/j.chaos.2021.111468

29. K. M. Owolabi, Analysis and simulation of herd behaviour dynamics based on derivative with non local and non-singular kernel, *Results Phys.*, **22** (2021), 103941. https://doi.org/10.1016/j.rinp.2021.103941

30. I. Ozturk, F. Ozkose, Stability analysis of fractional order mathematical model of tumor-immune system interaction, *Chaos Soliton. Fract.*, **133** (2020), 109614. https://doi.org/10.1016/j.chaos.2020.109614

31. S. Pal, S. Majhi, S. Mandal, N. Pal, Role of fear in a predator-prey model with Beddington-DeAngelis functional response, *Zeitschrift für Naturforschung A*, **74** (2019), 581–585. https://doi.org/10.1515/zna-2018-0449

32. P. Panja, S. Jana, S. K. Mondal, Dynamics of a stage structure prey-predator model with ratio-dependent functional response and anti-predator behavior of adult prey, *Numer. Algebra Control. Optim.*, **11** (2021), 391–405. https://doi.org/10.3934/naco.2020033

33. S. Saha, G. P. Samanta, Analysis of a predator-prey model with herd behavior and disease in prey incorporating prey refuge, *Int. J. Biomath.*, **12** (2019), 1950007. https://doi.org/10.1142/S1793524519500074

34. P. Veeresha, D. G. Prakasha, S. Kumar, A fractional model for propagation of classical optical solitons by using nonsingular derivative, *Math. Method. Appl. Sci.*, 2020. https://doi.org/10.1002/mma.6335

35. V. Volterra, Fluctuations in the abundance of species considered mathematically, *Nature*, **118** (1926), 558–560. https://doi.org/10.1038/118558a0

36. X. H. Wang, Z. Wang, J. W. Xia, Stability and bifurcation control of a delayed fractional-order eco-epidemiological model with incommensurate orders, *J. Franklin I.*, **356** (2019), 8278–8295. https://doi.org/10.1016/j.jfranklin.2019.07.028

37. Z. L. Wei, Q. D. Li, J. L. Che, Initial value problems for fractional differential equations involving Riemann-Liouville sequential derivative, *J. Math. Anal. Appl.*, **367** (2010), 260–272. https://doi.org/10.1016/j.jmaa.2010.01.023

38. G. L. Wen, Criterion to identify Hopf bifurcations in maps of arbitrary dimension, *Phys. Rev. E*, **72** (2005), 026201. https://doi.org/10.1103/PhysRevE.72.026201

39. Y. K. Xie, Z. Wang, B. Meng, X. Huang, Dynamical analysis for a fractional-order prey-predator model with Holling III type functional response, *Appl. Math. Lett.*, **16** (2020), 106342. https://doi.org/10.1016/j.aml.2020.106342

40. Y. Xie, Z. Wang, H. Shen, Y. Li, The dynamics of a delayed generalized fractional-order biological networks with predation behavior and material cycle, *Nonlinear Anal. Model. Control*, **25** (2020), 745–765. https://doi.org/10.15388/namc.2020.25.18391

41. A. Yousef, F. Bozkurt, Bifurcation and stability analysis of a system of fractional-order differential equations for a Plant-Herbivore model with Allee effect, *Mathematics*, **7** (2019), 454. https://doi.org/10.3390/math7050454

42. H. S. Zhang, Y. L. Cai, S. M. Fu, W. M. Wang, Impact of the fear effect in a prey-predator model incorporating a prey refuge, *Appl. Math. Comput.*, **356** (2019), 328–337. https://doi.org/10.1016/j.amc.2019.03.034