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Research article

Liapounoff type inequality for pseudo-integral of interval-valued function

Tatjana Grbić[1], Slavica Medić[1,*], Nataša Duraković[1], Sandra Buhmiler[1], Slaviša Dumnić[1] and Janja Jerebic[2]

[1] Faculty of Technical Sciences, University of Novi Sad, Novi Sad 21000, Serbia
[2] Faculty of Organizational Sciences, University of Maribor, Kranj 6230, Slovenia

* **Correspondence:** Email: slavicam@uns.ac.rs; Tel: +381216350770.

Abstract: In this paper, two new Liapounoff type inequalities in terms of pseudo-analysis dealing with set-valued functions are given. The first one is given for a pseudo-integral of set-valued function where pseudo-operations are given by a generator $g : [0, \infty] \to [0, \infty]$ and the second one is given for the semiring $([0, \infty], \sup, \odot)$ with generated pseudo-multiplication. The interval Liapounoff inequality is applied for estimation of interval-valued central g-moment of order n for interval-valued functions in a g-semiring.

Keywords: semiring; interval-valued function; pseudo-integral of an interval-valued function; Liapounoff type inequality; interval-valued central g-moment of order n
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1. Introduction

In mathematics, integral inequalities are a useful tool in solving various problems. Some of the best known are Chebyshev, Jensen, Hölder, Minkowski, Cauchy-Schwartz and Liapounoff inequalities. Many generalizations of these types of inequalities (and many others) for different classes of integrals have been investigated. In paper [6], are given for fuzzy integral and in [2, 7, 11] for Sugeno integral, in [1, 3, 8] for pseudo-integral.

The main topic of this paper is Liapounoff type inequality. One of its forms is

$$\left(\int_0^1 f(x)^s\,dx\right)^{r-t} \le \left(\int_0^1 f(x)^t\,dx\right)^{r-s} \left(\int_0^1 f(x)^r\,dx\right)^{s-t},$$

where $0 < t < s < r$ and $f : [0, 1] \to [0, \infty)$ is an integrable function (see [5]).

Pseudo-analysis is a part of mathematical analysis where the field of real numbers is replaced with a closed or semi-closed interval $[a,b] \subset [-\infty,\infty]$, and operations of addition and multiplication of real numbers are replaced with two new operations, pseudo-addition and pseudo-multiplication, defined on the considered interval $[a,b]$. The structure $([a,b],\oplus,\odot)$, where $\oplus$ and $\odot$ are pseudo-addition and pseudo-multiplications, respectively, is called semiring. There are three classes of semirings [20, 21]. In this paper, the focus is on two classes of semirings on the interval $[0,\infty]$. In the first part of the investigation the focus is on so-called g-semirings where pseudo-operations are given by a continuous function $g : [0.\infty] \to [0,\infty]$. The second part of the investigation is dedicated to idempotent semiring $([0,\infty], \sup, \odot)$ with generated pseudo-multiplication. Tools from pseudo-analysis have applications in various fields, such as game theory, nonlinear partial differential equations, probability theory, interval probability theory, etc. (see [13, 21, 27, 28]).

Two generalizations of the Liapounoff type inequality for pseudo-integral are given in [15]. The first generalization is given for the semiring $([0,1],\oplus,\odot)$ when both pseudo-operations are given by the generator $g : [0,1] \to [0,1]$ and the second generalization refers to the semiring $([0,1], \sup, \odot)$ from the first class when the pseudo-multiplication is given by an increasing generator $g : [0,1] \to [0,1]$.

Based on results from [15] it holds that

$$\left(\int_{[0,1]}^{\oplus} f^{(s)} \odot d\mu\right)^{(r-t)} \leq \left(\int_{[0,1]}^{\oplus} f^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} f^{(r)} \odot d\mu\right)^{(s-t)}, \tag{1.1}$$

where $0 < t < s < r$, $f : [0,1] \to [0,1]$ is a measurable function, $\oplus$ and $\odot$ are pseudo-operations given by generator $g : [0,1] \to [0,1]$, $\leq$ is the total order on the considered semiring from the second class and μ is an $\oplus$-measure.

In the case when the semiring belongs to the first class, pseudo-addition is sup and pseudo-multiplication is given by an increasing generator $g : [0,1] \to [0,1]$ the Liapounoff type inequality has the form

$$\left(\int_{[0,1]}^{\sup} f^{(s)} \odot d\mu\right)^{(r-t)} \leq \left(\int_{[0,1]}^{\sup} f^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\sup} f^{(r)} \odot d\mu\right)^{(s-t)}, \tag{1.2}$$

where $0 < t < s < r$, $f : [0,1] \to [0,1]$ is a measurable function and μ is a complete sup-measure.

One generalization of ordinary functions (single-valued functions) are set-valued functions. The theory of set-valued functions has application in the mathematical economy and optimal control [12]. The well-known terms for single-valued function, such as continuity, differentiation and integration are investigated in the field of set-valued functions. The first results of the integration of set-valued functions are given in [4]. Set-valued functions and their pseudo-integration has been investigated in [10]. The investigation of pseudo-integral of set-valued function is focused on a special case of the set-valued function, interval-valued function since the interval-valued function is suitable for applications. Different integral inequalities for interval-valued functions with respect to pseudo-integral are proven in [14, 29]. Jensen type and Hölder type inequality for interval-valued Choquet integrals are given in [14] and inequalities of Liapounoff and Stolarsky type for Choquet-like integrals with respect to nonmonotonic fuzzy measures are given in [29]. Interval Minkowski's inequality, interval Radon's inequality, and interval Beckenbach's inequality for Aumann integral are proven in [24].

The paper is organized in the following way. The second section contains preliminary notions needed for the investigation. It contains the definition of semiring and some illustrative examples. Definition of $\oplus$-measure and pseudo-integral are also the part of this section as well as definitions of operations on intervals and interval-valued function and its integration. The third section contains the main result of the paper, the Liapounoff type inequality for the pseudo-integral of an interval-valued function and some illustrative examples. The fourth section contains the definition of the interval-valued central moment of order n and an example where the Liapounoff inequality for pseudo-integral of interval-valued function is used for estimation of the interval-valued central moment of order n.

2. Preliminary notions

Some basic notions and definitions from pseudo-analysis are presented in this section.

2.1. Semiring

Let $[a,b]$ be a closed (or semi-closed) subinterval of $[-\infty,\infty]$ and let $\leq$ be the total order on the interval $[a,b]$. On the interval $[a,b]$ two operations are considered:

i) commutative and associative binary operation $\oplus$ called *pseudo-addition* which is non-decreasing with respect to $\leq$ and has a neutral element denoted by $\mathbf{0}$,
ii) commutative and associative binary operation $\odot$ called *pseudo-multiplication* which is positively non-decreasing with respect to $\leq$, i.e. $x \leq y$ implies $x \odot z \leq y \odot z$ for all $z \in [a,b]_+$ where $[a,b]_+ = \{z \in [a,b] : \mathbf{0} \leq z\}$. It has a neutral element denoted by $\mathbf{1}$.

The structure $([a,b],\oplus,\odot)$ is called a *semiring* if the following conditions are satisfied:

i) $\oplus : [a,b]^2 \to [a,b]$ and $\odot : [a,b]^2 \to [a,b]$ are pseudo-addition and pseudo-multiplication, respectively,
ii) pseudo-multiplication $\odot$ is distributive over pseudo-addition $\oplus$, i.e. $x \odot (y \oplus z) = x \odot y \oplus x \odot z$ for all $x,y,z \in [a,b]$,
iii) for all $x \in [a,b]$ it holds $\mathbf{0} \odot x = \mathbf{0}$.

More about pseudo-operations and semirings can be found in [20–22].

Depending on other properties that pseudo-operations possess, there are three classes of semirings (see [21, 22]).

The first class consists of semirings where pseudo-addition is an idempotent operation, and pseudo-multiplication is a non-idempotent operation. The semiring investigated in this paper is $([a,b],\sup,\odot)$, where pseudo-addition is $x \oplus y = \sup(x,y)$ and pseudo-multiplication is given by $x \odot y = g^{-1}(g(x)g(y))$ and $g : [a,b] \to [0,\infty]$ is a continuous increasing function. The function g is called *generator of pseudo-multiplication*. An example of this type of semiring is $([-\infty,\infty],\sup,+)$, where the generator for pseudo-multiplication is $g(x) = e^x$.

The second class consists semirings where both pseudo-operations are strict and defined by a monotone and continuous function $g : [a,b] \to [0,\infty]$, by

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x) \cdot g(y)).$$

The function g is called *generator* of pseudo-operations and the semiring is called *g-semiring*. In this case, the pseudo-operations are called *g-operations*.

The third class of semirings are semirings with both idempotent pseudo-operations.

In this investigation the pseudo-multiplication is always defined by a generator g, and in this case the *pseudo-power* $x^{(n)}$ is defined by

$$x^{(n)} = \underbrace{x \odot x \odot \cdots \odot x}_{n} = g^{-1}(g^n(x)),$$

for $x \in [a, b]$ and $n \in \mathbb{N}$. It can be shown (see [3]) that the pseudo-power $x^{(p)}$ is well defined for all $p \in (0, \infty) \cap \mathbb{Q}$ in the same way, i.e.

$$x^{(p)} = g^{-1}(g^p(x)), \quad p \in (0, \infty) \cap \mathbb{Q}.$$

Due to the continuity of $\odot$, it holds that (see [3])

$$x^{(p)} = \sup\{x^{(r)} | r \in (0, p), r \in \mathbb{Q}\}, \quad p \in \mathbb{R} \setminus \mathbb{Q}, \quad p > 0. \tag{2.1}$$

On the class of g-semirings, the total order on the interval $[a, b]$ is given by $x \preceq y$ if and only if $g(x) \leq g(y)$. If g is an increasing generator, the total order $\preceq$ is the usual order $\leq$ on the real line and if g is a decreasing generator, the total order $\preceq$ is opposite to the usual order on the real line. If pseudo-addition is $x \oplus y = \sup(x, y)$, then the total order is defined by $x \preceq y$ if and only if $\sup(x, y) = y$, and the total order is the usual order $\leq$ on the real line. Similarly, if $\oplus = \min$ the total order $\preceq$ is the order opposite to the usual order on the real line.

2.2. *Pseudo-integral*

One generalization of additive measure from the classical measure theory is so-called $\oplus$-measure.

Let X be a non-empty set, and let $\mathcal{A}$ be the σ-algebra of the subsets of X. A set function $\mu : \mathcal{A} \to [a, b]_+$ is an *$\oplus$-measure*, or a *pseudo-additive measure* if

i) $\mu(\emptyset) = \mathbf{0}$,

ii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \bigoplus_{i=1}^{n} \mu(A_i)$, where $\{A_i\}$ is a sequence of pairwise disjoint sets from $\mathcal{A}$.

If $\oplus$ is an idempotent operation, the first condition and disjointness of sets from the second condition can be omitted.

For the g-semiring, the second condition has the form

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} g^{-1}\left(\sum_{i=1}^{n} g \circ \mu(A_i)\right).$$

The focus of this paper is on two cases of pseudo-integral (see [20]).

Let $([a, b], \oplus, \odot)$ be a g-semiring. *g-integral* of a measurable function $f : X \to [a, b]$ is of the form

$$\int_X^{\oplus} f \odot d\mu = g^{-1}\left(\int_X (g \circ f) d(g \circ \mu)\right),$$

where the integral on the right-hand side is the Lebesgue integral. In the special case, when $X = [c, d]$, $\mathcal{A} = \mathcal{B}([c, d])$ is a Borel σ-algebra on $[c, d]$ and $g \circ \mu$ is a Lebesgue measure on $[c, d]$, then

$$\int_{[c,d]}^{\oplus} f \odot d\mu = g^{-1}\left(\int_{c}^{d} g(f(x))dx\right).$$

Let $([a, b], \sup, \odot)$ be a semiring where pseudo-multiplication is given by an increasing function $g : [a, b] \to [0, \infty]$. Pseudo-integral of a measurable function $f : X \to [a, b]$ is

$$\int_{X}^{\sup} f \odot d\mu = \sup_{x \in X} \left(g^{-1}(g(f(x))\ g(\Psi(x)))\right),$$

where $\Psi : [c, d] \to [a, b]$ is a continuous density which determines sup-decomposable measure μ.

Due to the fact that every semiring $([a, b], \sup, \odot)$ of the first class can be obtained as a limit of a family of g-semirings of the second class generated by g^{λ}, i.e.,

$$\lim_{\lambda \to \infty} x \oplus_{\lambda} y = \lim_{\lambda \to \infty} (g^{\lambda})^{-1}(g^{\lambda}(x) + g^{\lambda}(y)) = \sup(x, y)$$

and

$$x \odot_{\lambda} y = (g^{\lambda})^{-1}(g^{\lambda}(x)\ g^{\lambda}(y)) = g^{-1}(g(x)\ g(y)) = x \odot y,$$

where g is an increasing generator of pseudo-multiplication $\odot$ (see [19]), this research is focused on the class of g-semirings.

Also, in [19] it is shown that for the semiring $([0, \infty], \sup, \odot)$ with generated pseudo-multiplication it holds that

$$\int_{X}^{\sup} f \odot d\mu = \lim_{\lambda \to +\infty} (g^{\lambda})^{-1}\left(\int_{X} (g^{\lambda} \circ f)d(g \circ \mu)\right), \tag{2.2}$$

where μ is a sup-decomposable measure on $[0, \infty]$ and $f : [0, \infty] \to [0, \infty]$ is a continuous function.

2.3. Pseudo-operations on intervals

As this paper deals with interval-valued functions, for further work it is necessary to define pseudo-multiplication on the class

$$\mathcal{I} = \{[x, y] : x \leq y \ \text{ and } \ [x, y] \subseteq [a, b]_{+}\}$$

of closed sub-intervals of $[a, b]_{+}$.

The pseudo-product of two intervals $A = [l_1, r_1]$ and $B = [l_2, r_2]$ is defined in [16]. Since this paper only deals with generated pseudo-multiplication, $A \odot B$ when $x \odot y = g^{-1}(g(x)\ g(y))$ is given below.

If the pseudo-multiplication is given by an increasing generator g, then

$$A \odot B = [g^{-1}\left(g(l_1)g(l_2)\right), g^{-1}\left(g(r_1)g(r_2)\right)], \tag{2.3}$$

and when the generator g is a decreasing function, it holds that

$$A \odot B = [g^{-1}\left(g(r_1)g(r_2)\right), g^{-1}\left(g(l_1)g(l_2)\right)]. \tag{2.4}$$

For every family $\{[l_i, r_i] : [l_i, r_i] \in \mathcal{I}, i \in I\}$ of closed sub-intervals of $[a, b]_+$, where the index set I is a countable set, based on results from [9] and [25] it holds that

$$\sup_{i \in I}[l_i, r_i] = [\sup_{i \in I} l_i, \sup_{i \in I} r_i]. \tag{2.5}$$

Also,

$$\lim_{n\to\infty}[l_n, r_n] = [\lim_{n\to\infty} l_n, \lim_{n\to\infty} r_n], \tag{2.6}$$

if $\lim_{n\to\infty} l_n$ and $\lim_{n\to\infty} r_n$ exist.

The pseudo-power $x^{(p)}$, $p \in (0, \infty)$, $x \in [0, \infty]$ is extended to the *pseudo-power* $A^{(p)}$ of a set $A \subset [0, \infty]$ (see [16]) as

$$A^{(p)} = \{x^{(p)} : x \in A\}.$$

In the special case, when $A = [c, d]$, the next lemma is shown in [16].

Lemma 1. *Let $n, m \in \mathbb{N}$, $p \in \mathbb{R}^+ \setminus \mathbb{Q}$ and pseudo-multiplication $\odot$ is given by a generator g. Then it holds that*

i) $[c, d]^{(n)} = \left[c^{(n)}, d^{(n)}\right]$,

ii) $[c, d]^{(\frac{1}{n})} = \left[c^{(\frac{1}{n})}, d^{(\frac{1}{n})}\right]$,

iii) $[c, d]^{(\frac{m}{n})} = \left[c^{(\frac{m}{n})}, d^{(\frac{m}{n})}\right]$.

iv) $[c, d]^{(p)} = \sup\left\{\left[c^{(r)}, d^{(r)}\right] : r \in (0, p) \cap \mathbb{Q}\right\}$.

Based on (2.1), (2.5) and *iv*) from Lemma 1 it holds that

$$[c, d]^{(p)} = [c^{(p)}, d^{(p)}], \quad p \in \mathbb{R} \setminus \mathbb{Q}. \tag{2.7}$$

2.4. The relation "less or equal" on $\mathcal{I}$

The relation "less or equal" for the intervals from $\mathcal{I}$ is denoted by $\leq_S$.

Definition 1. *Let $A, B \in \mathcal{I}$. $A \leq_S B$ if and only if for all $x \in A$ there exists $y \in B$ such that $x \leq y$ and for all $y \in B$ there exists $x \in A$ such that $x \leq y$.*

In paper [17], the relation $\leq_S$ was defined in a more general manner, on the set of all non-empty subsets of the interval $[a, b]$.

The necessity of introducing the relation $\leq_S$ was illustrated in [17] by an example - if the usual subset is used instead of $\leq_S$, for $x, y \in [a, b]_+$ such that $x \leq y$ and $x \neq y$, for elements $A = [x, x]$ and $B = [y, y]$ neither $A \subseteq B$ nor $B \subseteq A$ holds, but it holds that $A \leq_S B$.

2.5. Pseudo-integral of an interval-valued function

Let X be a non-empty set, $([a, b], \oplus, \odot)$ a semiring, $\mathcal{I}$ a class of closed sub-intervals of $[a, b]_+$ and $F : X \to \mathcal{I}$ an interval-valued function.

An interval-valued function F is pseudo-integrably bounded if there exists a function $h \in L^1_\oplus(\mu)$ such that

i) $\bigoplus_{\alpha \in F(x)} \alpha \leq h(x)$, for the non-idempotent pseudo-addition,

ii) $\sup_{\alpha \in F(x)} \alpha \leq h(x)$, for the pseudo-addition given by an increasing generator g,

iii) $\inf_{\alpha \in F(x)} \alpha \leq h(x)$, for the pseudo-addition given by a decreasing generator g

holds, where $L^1_{\oplus}(\mu)$ is the family of functions which are integrable with respect to the pseudo-integral in the sense of the considered semiring.

Based on results from [10], the *pseudo-integral of a pseudo-integrably bounded interval-valued function* $F : X \to \mathcal{I}$ represented by its *border functions* $F_l,\ F_r : X \to [a, b]_+$ by $F(x) = [F_l(x), F_r(x)]$ is defined by

$$\int_X^{\oplus} F \odot d\mu = \left[\int_X^{\oplus} F_l \odot d\mu, \int_X^{\oplus} F_r \odot d\mu \right]. \tag{2.8}$$

If F is a pseudo-integrably bounded function, then F is a pseudo-integrable function. More about the pseudo-integral of an interval-valued function, its basic properties and application can be found in [10].

3. Liapounoff type pseudo-integral inequality of interval-valued function

The main result of this paper, the Liapounoff type inequality for pseudo-integral of an interval-valued function is presented in this section.

Obviously, the Liapounoff type inequalities from [15] hold for a g-semiring $([0,\infty], \oplus, \odot)$ with an increasing generating function $g : [0,\infty] \to [0,\infty]$ and a semiring $([0,\infty], \sup, \odot)$ and pseudo-multiplication given by an increasing generator $g : [0,\infty] \to [0,\infty]$. In those cases, Liapounoff type inequalities deal with a measurable function $f : [0,1] \to [0,\infty)$ and then $g \circ f : [0,1] \to [0,\infty]$, i.e. (1.1) and (1.2) hold. Based on this fact, in this investigation the interval $[0,\infty]$ is considered, instead of the interval $[0,1]$ observed in [15].

Let us consider a g-semiring $([0,\infty], \oplus, \odot)$ with generator $g : [0,\infty] \to [0,\infty]$ or a semiring $([0,\infty], \sup, \odot)$ where $\odot$ is given by an increasing generator $g : [0,\infty] \to [0,\infty]$ and an interval-valued function $F : [0,1] \to \mathcal{I}$ represented by its border functions $F_l,\ F_r : [0,1] \to [0,\infty)$ as $F(x) = [F_l(x), F_r(x)]$.

Lemma 2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ or $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then it holds that*

$$\left(\int_{[0,1]}^{\oplus} F^{(\alpha)} \odot d\mu \right)^{(\beta)} = \left[\left(\int_{[0,1]}^{\oplus} F_l^{(\alpha)} \odot d\mu \right)^{(\beta)}, \left(\int_{[0,1]}^{\oplus} F_r^{(\alpha)} \odot d\mu \right)^{(\beta)} \right].$$

Proof. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ or $\beta \in \mathbb{R} \setminus \mathbb{Q}$, based on (2.5), (2.7) and (2.8) it holds that

$$\begin{aligned} \left(\int_{[0,1]}^{\oplus} F^{(\alpha)} \odot d\mu \right)^{(\beta)} &= \left(\int_{[0,1]}^{\oplus} [F_l, F_r]^{(\alpha)} \odot d\mu \right)^{(\beta)} = \left(\int_{[0,1]}^{\oplus} [F_l^{(\alpha)}, F_r^{(\alpha)}] \odot d\mu \right)^{(\beta)} \\ &= \left[\int_{[0,1]}^{\oplus} F_l^{(\alpha)} \odot d\mu, \int_{[0,1]}^{\oplus} F_r^{(\alpha)} \odot d\mu \right]^{(\beta)} = \left[\left(\int_{[0,1]}^{\oplus} F_l^{(\alpha)} \odot d\mu \right)^{(\beta)}, \left(\int_{[0,1]}^{\oplus} F_r^{(\alpha)} \odot d\mu \right)^{(\beta)} \right]. \end{aligned}$$

□

Theorem 1. *Let* $([0,\infty],\oplus,\odot)$ *be a g-semiring with generator* $g:[0,\infty]\to[0,\infty]$. *For a pseudo-integrably bounded interval-valued function* $F(x)=[F_l(x),F_r(x)]$, *where the border functions* F_l, F_r : $[0,1]\to[0,\infty)$ *are measurable,* $t,s,r\in\mathbb{R}$ *and* $0<t<s<r$, *holds*

$$\left(\int_{[0,1]}^{\oplus} F^{(s)}\odot d\mu\right)^{(r-t)} \preceq_S \left(\int_{[0,1]}^{\oplus} F^{(t)}\odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F^{(r)}\odot d\mu\right)^{(s-t)}. \tag{3.1}$$

Proof. Let all pseudo-powers in (3.1) be rational numbers.

From (2.8) and Lemma 1 for the left-hand side of inequality (3.1) it holds that

$$\left(\int_{[0,1]}^{\oplus} F^{(s)}\odot d\mu\right)^{(r-t)} = \left(\int_{[0,1]}^{\oplus} [F_l,F_r]^{(s)}\odot d\mu\right)^{(r-t)} = \left(\int_{[0,1]}^{\oplus} [F_l^{(s)},F_r^{(s)}]\odot d\mu\right)^{(r-t)}$$

$$= \left[\int_{[0,1]}^{\oplus} F_l^{(s)}\odot d\mu, \int_{[0,1]}^{\oplus} F_r^{(s)}\odot d\mu\right]^{(r-t)} = \left[\left(\int_{[0,1]}^{\oplus} F_l^{(s)}\odot d\mu\right)^{(r-t)}, \left(\int_{[0,1]}^{\oplus} F_r^{(s)}\odot d\mu\right)^{(r-t)}\right].$$

Similarly, for the right-hand side of inequality (3.1) from Lemma 1, definition of pseudo-multiplication on $[0,\infty]$ and definition of pseudo-multiplication on $\mathcal{I}$ it follows that

$$\left(\int_{[0,1]}^{\oplus} F^{(t)}\odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F^{(r)}\odot d\mu\right)^{(s-t)}$$

$$= \left[\int_{[0,1]}^{\oplus} F_l^{(t)}\odot d\mu, \int_{[0,1]}^{\oplus} F_r^{(t)}\odot d\mu\right]^{(r-s)} \odot \left[\int_{[0,1]}^{\oplus} F_l^{(r)}\odot d\mu, \int_{[0,1]}^{\oplus} F_r^{(r)}\odot d\mu\right]^{(s-t)}$$

$$= \left[\left(\int_{[0,1]}^{\oplus} F_l^{(t)}\odot d\mu\right)^{(r-s)}, \left(\int_{[0,1]}^{\oplus} F_r^{(t)}\odot d\mu\right)^{(r-s)}\right] \odot \left[\left(\int_{[0,1]}^{\oplus} F_l^{(r)}\odot d\mu\right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_r^{(r)}\odot d\mu\right)^{(s-t)}\right].$$

Let the generator g be an increasing function. Based on (2.3) it holds that

$$\left(\int_{[0,1]}^{\oplus} F^{(t)}\odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F^{(r)}\odot d\mu\right)^{(s-t)}$$

$$= \left[\left(\int_{[0,1]}^{\oplus} F_l^{(t)}\odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)}\odot d\mu\right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_r^{(t)}\odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)}\odot d\mu\right)^{(s-t)}\right].$$

Since the generator g is an increasing function the total order $\preceq$ is the usual order on the real line.

For every $x \in \left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu\right)^{(r-t)}$ it holds that $x \leq \left(\int_{[0,1]}^{\oplus} F_r^{(s)} \odot d\mu\right)^{(r-t)}$. From inequality (1.1) applied to the function $F_r : [0,1] \to [0,\infty)$ it follows that

$$\left(\int_{[0,1]}^{\oplus} F_r^{(s)} \odot d\mu\right)^{(r-t)} \leq \left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu\right)^{(s-t)}.$$

Let

$$y = \left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu\right)^{(s-t)}.$$

It holds that

$$y \in \left[\left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu\right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu\right)^{(s-t)}\right].$$

Therefore, for every $x \in \left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu\right)^{(r-t)}$ holds $x \leq y$.

For every

$$y \in \left[\left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu\right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu\right)^{(s-t)}\right]$$

holds

$$\left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu\right)^{(s-t)} \leq y.$$

From (1.1) applied to the function $F_l : [0,1] \to [0,\infty)$ it follows that

$$\left(\int_{[0,1]}^{\oplus} F_l^{(s)} \odot d\mu\right)^{(r-t)} \leq \left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu\right)^{(s-t)},$$

so that for $x = \left(\int_{[0,1]}^{\oplus} F_l^{(s)} \odot d\mu\right)^{(r-t)}$ holds $x \in \left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu\right)^{(r-t)}$ and $x \leq y$.

Now the inequality (3.1) holds by Definition 1.

Let the generator g be a decreasing function. Based on (2.4) it holds that

$$\left(\int_{[0,1]}^{\oplus} F^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F^{(r)} \odot d\mu \right)^{(s-t)}$$

$$= \left[\left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu \right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu \right)^{(s-t)} \right].$$

Since the generator g is a decreasing function the total order $\preceq$ is the order opposite to the usual order on the real line.

For every $x \in \left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu \right)^{(r-t)}$ it holds that $\left(\int_{[0,1]}^{\oplus} F_l^{(s)} \odot d\mu \right)^{(r-t)} \preceq x$. Based on inequality (1.1) applied to the function $F_l : [0,1] \to [0,\infty)$ it follows that

$$\left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu \right)^{(s-t)} \preceq \left(\int_{[0,1]}^{\oplus} F_l^{(s)} \odot d\mu \right)^{(r-t)}.$$

Let

$$y = \left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu \right)^{(s-t)}.$$

Then it holds that

$$y \in \left[\left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu \right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu \right)^{(s-t)} \right]$$

and for every $x \in \left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu \right)^{(r-t)}$ it holds that $y \preceq x$.

For every

$$y \in \left[\left(\int_{[0,1]}^{\oplus} F_r^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_r^{(r)} \odot d\mu \right)^{(s-t)}, \left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu \right)^{(s-t)} \right]$$

holds

$$y \preceq \left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu \right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu \right)^{(s-t)}.$$

From inequality (1.1) applied to the function $F_l : [0,1] \to [0,\infty)$ it holds that

$$\left(\int_{[0,1]}^{\oplus} F_l^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F_l^{(r)} \odot d\mu\right)^{(s-t)} \leq \left(\int_{[0,1]}^{\oplus} F_l^{(s)} \odot d\mu\right)^{(r-t)},$$

so for $x = \left(\int_{[0,1]}^{\oplus} F_l^{(s)} \odot d\mu\right)^{(r-t)}$ holds $x \in \left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu\right)^{(r-t)}$ and $y \leq x$.

Now, inequality (3.1) holds by Definition 1.

If at least one of the pseudo-powers in (3.1) is not a rational number, the proof of the inequality (3.1) is similar, using Lemma 2 and *iv*) from Lemma 1. □

Remark 1. *For $g(x) = x$ Theorem 1 is Liapounoff type inequality for Aumann integral.*

Since the proof of Theorem 1 is based on pseudo-multiplication of sub-intervals from $\mathcal{I}$ and pseudo-power of elements from $\mathcal{I}$, based on results from [15] and [19] it is obvious that the next theorem holds.

Theorem 2. *Let $([0,\infty], \sup, \odot)$ be a semiring from the first class where pseudo-multiplication $\odot$ is given by an increasing generator $g : [0,\infty] \to [0,\infty]$ and μ is a* sup*-decomposable measure on $\mathcal{B}([0,1])$, given by $\mu(A) = \operatorname{ess\,sup}(\Psi(x) : x \in A)$, where $\Psi : [0,1] \to [0,\infty]$ is a continuous density. For a pseudo-integrably bounded interval-valued function $F = [F_l, F_r]$, where the border functions F_l, $F_r : [0,1] \to [0,\infty)$ are measurable, $t, s, r \in \mathbb{R}$ and $0 < t < s < r$ holds*

$$\left(\int_{[0,1]}^{\sup} F^{(s)} \odot d\mu\right)^{(r-t)} \preceq_S \left(\int_{[0,1]}^{\sup} F^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\sup} F^{(r)} \odot d\mu\right)^{(s-t)}. \tag{3.2}$$

Remark 2. *Theorem 1 holds if any g-semiring $([a,b], \oplus, \odot)$ is considered, where $[a,b] \subseteq [0,\infty]$, for $F(x) = [F_l(x), F_r(x)]$ holds $\mathrm{Range}(F_l) \subseteq [a,b]$ and $\mathrm{Range}(F_r) \subseteq [a,b]$. The similar holds for Theorem 2, for the semirings of the third class with generated pseudo-multipplication.*

Examples

For the interval-valued function $F(x) = [F_l(x), F_r(x)]$, from (2.8), the definition of g-integral and properties of interval-valued pseudo-integral (see [10]), for any generator g follows

$$\int_{[0,1]}^{\oplus} F \odot d\mu = \left[g^{-1}\left(\int_{[0,1]} (g \circ F_l) d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g \circ F_r) d(g \circ \mu)\right)\right]. \tag{3.3}$$

Now, the left-hand side of inequality (3.1) has the form

$$\left(\int_{[0,1]}^{\oplus} F^{(s)} \odot d\mu\right)^{(r-t)} = g^{-1} \circ g^{r-t}\left[g^{-1}\left(\int_{[0,1]} (g^s \circ F_l)\, d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g^s \circ F_r)\, d(g \circ \mu)\right)\right].$$

Similarly, for the right-hand side of inequality (3.1) it holds that

$$\left(\int_{[0,1]}^{\oplus} F^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} F^{(r)} \odot d\mu\right)^{(s-t)}$$

$$= g^{-1}\left(g^{r-s}\left(\left[g^{-1}\left(\int_{[0,1]} (g^t \circ F_l)\, d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g^t \circ F_r)\, d(g \circ \mu)\right)\right]\right)\right.$$

$$\left.\cdot\, g^{s-t}\left(\left[g^{-1}\left(\int_{[0,1]} (g^r \circ F_l)\, d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g^r \circ F_r)\, d(g \circ \mu)\right)\right]\right)\right).$$

Therefore, the inequality (3.1) has the form

$$g^{-1} \circ g^{r-t}\left[g^{-1}\left(\int_{[0,1]} (g^s \circ F_l)\, d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g^s \circ F_r)\, d(g \circ \mu)\right)\right]$$

$$\leq_S\; g^{-1}\left(g^{r-s}\left(\left[g^{-1}\left(\int_{[0,1]} (g^t \circ F_l)\, d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g^t \circ F_r)\, d(g \circ \mu)\right)\right]\right)\right.$$

$$\left.\cdot\, g^{s-t}\left(\left[g^{-1}\left(\int_{[0,1]} (g^r \circ F_l)\, d(g \circ \mu)\right), g^{-1}\left(\int_{[0,1]} (g^r \circ F_r)\, d(g \circ \mu)\right)\right]\right)\right).$$

Example 1. *Since indefinite integral $\int \sin x^2 dx$ cannot be expressed in terms of elementary functions, $\int_0^1 \sin x^2 dx$ will be estimated using the Liapounoff type inequality for the pseudo-integral of an interval-valued function.*

Let $([0,\infty], \oplus, \odot)$ be the g-semiring with generator $g(x) = x^2$. The interval-valued function $F(x) = [\sqrt{\sin x^2}, x]$ will be used for estimation of integral $\int_0^1 \sin x^2 dx$. In this case the left-hand side of the inequality (3.1) has the form

$$\left[\left(\int_{[0,1]}^{\oplus} (\sqrt{\sin x^2})^{(s)} d(g \circ \mu)\right)^{(r-t)}, \left(\int_{[0,1]}^{\oplus} x^{(s)} d(g \circ \mu)\right)^{(r-t)}\right]$$

and the right-hand side of the inequality (3.1) has the form

$$\left[\left(\int_{[0,1]}^{\oplus}(\sqrt{\sin x^2})^{(t)}d(g\circ\mu)\right)^{(r-s)}\odot\left(\int_{[0,1]}^{\oplus}(\sqrt{\sin x^2})^{(r)}d(g\circ\mu)\right)^{(s-t)},\left(\int_{[0,1]}^{\oplus}x^{(t)}d(g\circ\mu)\right)^{(r-s)}\odot\left(\int_{[0,1]}^{\oplus}x^{(r)}d(g\circ\mu)\right)^{(s-t)}\right].$$

From Definition 1 and Theorem 1 follows

$$\left(\int_{[0,1]}^{\oplus}(\sqrt{\sin x^2})^{(s)}d(g\circ\mu)\right)^{(r-t)}\leq\left(\int_0^1 x^{(t)}\,dx\right)^{(r-s)}\odot\left(\int_0^1 x^{(r)}\,dx\right)^{(s-t)}.$$

Now, for the chosen parameters $t=\frac{1}{2}$, $s=1$ *and* $r=\frac{3}{2}$ *from the fact that g is an increasing function it follows that*

$$\int_0^1 \sin x^2 dx\leq\frac{\sqrt{2}}{4}.$$

Example 2. *Let* $([0,\infty],\oplus,\odot)$ *be the g-semiring with generator* $g(x)=\ln(1+x)$. *For the interval-valued function* $F(x)=[F_l(x),F_r(x)]$, *the left side of inequality (3) has the form*

$$\left[e^{\left(\int_0^1(\ln^s(1+F_l(x))dx\right)^{r-t}}-1,e^{\left(\int_0^1(\ln^s(1+F_r(x))dx\right)^{r-t}}-1\right],$$

and the right side of inequality (3) has the form

$$\left[e^{\left(\int_0^1\ln^t(1+F_l(x))dx\right)^{r-s}\cdot\left(\int_0^1(\ln^r(1+F_l(x))dx\right)^{s-t}}-1,\ e^{\left(\int_0^1\ln^t(1+F_r(x))dx\right)^{r-s}\cdot\left(\int_0^1(\ln^r(1+F_r(x))dx\right)^{s-t}}-1\right].$$

Therefore,

$$\alpha_1\cdot e^{\left(\int_0^1(\ln^s(1+F_l(x))dx\right)^{r-t}}+\beta_1\cdot e^{\left(\int_0^1(\ln^s(1+F_r(x))dx\right)^{r-t}}$$

$$\leq\alpha_2\cdot e^{\left(\int_0^1\ln^t(1+F_l(x))dx\right)^{r-s}\cdot\left(\int_0^1(\ln^r(1+F_l(x))dx\right)^{s-t}}+\beta_2\cdot e^{\left(\int_0^1\ln^t(1+F_r(x))dx\right)^{r-s}\cdot\left(\int_0^1(\ln^r(1+F_r(x))dx\right)^{s-t}},$$

for every $\alpha_1,\beta_1,\alpha_2,\beta_2\in[0,1]$ *such that* $\alpha_1+\beta_1=1$ *and* $\alpha_2+\beta_2=1$.

The following example is based on an example from [26].

Example 3. *Let* $([0,\infty],\sup,\odot)$ *be the semiring from the first class where pseudo-multiplication is generated by* $g(x)=x$. *Let* μ *be a* sup-*measure on* $([0,\infty],\mathcal{B}([0,\infty]))$ *with density function* $\Psi(x)=x$.

Let us consider the interval-valued function $F(x)=[F_l(x),2x]$, *where* $F_l(x)\leq 2x$, $x\in[0,1]$.

$$\left(\int_{[0,1]}^{\sup} (2x)^{(s)} \odot d\mu\right)^{(r-t)} = \left(\lim_{n\to\infty} \int_{[0,1]}^{\oplus_n} (2x)^{(s)} \odot d\mu_n\right)^{(r-t)} = \left(\lim_{n\to\infty} \int_0^1 (2x)^s \cdot x \cdot dx\right)^{r-t}$$

$$= \left(2^s \lim_{n\to\infty} \left(\frac{1}{sn+n+1}\right)^{\frac{1}{n}}\right)^{r-t} = 2^{s(r-t)}.$$

Similarly,

$$\left(\int_{[0,1]}^{\sup} F_l^{(s)} \odot d\mu\right)^{(r-t)} = \lim_{n\to\infty} \left(\int_0^1 F_l^{sn} x^n dx\right)^{\frac{r-t}{n}},$$

and

$$\left(\int_{[0,1]}^{\sup} F_l^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\sup} F_l^{(r)} \odot d\mu\right)^{(s-t)} = \lim_{n\to\infty} \left(\int_0^1 F_l^{tn} x^n dx\right)^{\frac{r-s}{n}} \cdot \left(\int_0^1 F_l^{rn} x^n dx\right)^{\frac{s-t}{n}},$$

and

$$\left(\int_{[0,1]}^{\sup} (2x)^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\sup} (2x)^{(r)} \odot d\mu\right)^{(s-t)} = 2^{s(r-t)}.$$

For the interval-valued function $F(x) = [F_l(x), 2x]$, *where* $F_l(x) \leq 2x$, $x \in [0,1]$, *based on (3.2) and (2.6) it holds that*

$$\lim_{n\to\infty} \left[\left(\int_0^1 F_l^{sn} x^n dx\right)^{\frac{r-t}{n}}, 2^{s(r-t)}\right] \leq_S \lim_{n\to\infty} \left[\left(\int_0^1 F_l^{tn} x^n dx\right)^{\frac{r-s}{n}} \cdot \left(\int_0^1 F_l^{rn} x^n dx\right)^{\frac{s-t}{n}}, 2^{s(r-t)}\right].$$

4. Liapounoff inequality for interval-valued central g-moments of order n

In this part, the interval Liapounoff inequality is applied for estimation of interval-valued central g-moment of order n for interval-valued functions in a g-semiring.

Let $([a,b], \oplus, \odot)$ be a g-semiring such that $\mu([a,b]) = \mathbf{1}$, where $\mathbf{1}$ is neutral element for the given pseudo-multiplication and μ is an $\oplus$-measure. It is known that in the case when $\mu([a,b]) = \mathbf{1}$, pseudo-additive measure μ is called pseudo-probability measure and it is given by $\mu = g^{-1} \circ P$, where g is a generator of pseudo-operations and P is a probability measure (see [18]).

Based on the result from [18] the *central g-moment of order* $n > 0$ for a measurable function $f : [0,1] \to [a,b]$ is given by

$$E^{g,n}[f] = g^{-1}\left(\int_0^1 g^n \circ f(x)\, dx\right). \tag{4.1}$$

One representation of the Liapounoff inequality for pseudo-integral (1.1) in terms of the central g-moment of order n is given in the following Lemma.

Lemma 3. *Let* $([a,b],\oplus,\odot)$ *be a g-semiring such that* $\mu([a,b]) = \mathbf{1}$. *For a measurable function* $f : [0,1] \to [0,\infty)$ *and central g-moment of order n it holds that*

$$g^{-1} \circ g^{r-t} \left(E^{g,s}[f]\right) \leq g^{-1} \left(g^{r-s} \left(E^{g,t}[f]\right) \cdot g^{s-t} \left(E^{g,r}[f]\right)\right). \tag{4.2}$$

Proof. For a measurable function f, the left-hand side of the Liapounoff inequality for pseudo-integral given in (1.1) is

$$\left(\int_{[0,1]}^{\oplus} f^{(s)} \odot d\mu\right)^{(r-t)} = g^{-1} \circ g^{r-t} \left(g^{-1}\left(\int_0^1 (g^s \circ f)\, d(g\circ\mu)\right)\right) = g^{-1} \circ g^{r-t} \left(E^{g,s}[f]\right).$$

The right-hand side of the same inequality is

$$\left(\int_{[0,1]}^{\oplus} f^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int_{[0,1]}^{\oplus} f^{(r)} \odot d\mu\right)^{(s-t)}$$

$$= g^{-1}\left(g^{r-s} \circ g^{-1}\left(\int_0^1 (g^t \circ f)\, d(g\circ\mu)\right) \cdot g^{s-t} \circ g^{-1}\left(\int_0^1 (g^r \circ f)\, d(g\circ\mu)\right)\right)$$

$$= g^{-1} \left(g^{r-s} \left(E^{g,t}[f]\right) \cdot g^{s-t} \left(E^{g,r}[f]\right)\right),$$

so that the inequality (4.2) holds. □

The following definition is one new generalization of the central g-moment of order n in the sense of the interval-valued functions.

Definition 2. *Let* $([0,\infty],\oplus,\odot)$ *be a g-semiring and* $F = [F_l, F_r]$ *be an interval-valued function where the border functions* $F_l,\ F_r : [0,1] \to [0,\infty)$ *are measurable. The* interval-valued central g-moment of order $n > 0$ *for the interval-valued function* $F = [F_l, F_r]$, *is*

$$E_I^{g,n}[F] = [E^{g,n}[F_l], E^{g,n}[F_r]].$$

Theorem 3. *Let* $([0,\infty],\oplus,\odot)$ *be a g-semiring and let* $F = [F_l, F_r]$ *be an interval-valued function where the border functions* $F_l,\ F_r : [0,1] \to [0,\infty)$ *are measurable. For interval-valued central g-moment of order n it holds that*

$$g^{-1}\left(g^{r-t}\left(E_I^{g,s}[F]\right)\right) \preceq_S g^{-1}\left(g^{r-s}(E_I^{g,t}[F]) \cdot g^{s-t}\left(E_I^{g,r}[F]\right)\right). \tag{4.3}$$

Proof. From (4.1) $0 < t < s < r$, for the left-hand side of inequality (3.1) holds

$$\left(\int\limits_{[0,1]}^{\oplus} F^{(s)} \odot d\mu\right)^{(r-t)} = g^{-1} \circ g^{r-t}\left(\left[g^{-1}\left(\int_0^1 (g^s \circ F_l)\, d(g\circ\mu)\right),\ g^{-1}\left(\int_0^1 (g^s \circ F_r)\, d(g\circ\mu)\right)\right]\right)$$

$$= g^{-1}\left(g^{r-t}\left([E^{g,s}[F_l], E^{g,s}[F_r]]\right)\right)$$

$$= g^{-1}\left(g^{r-t}\left(E_I^{g,s}[F]\right)\right),$$

and for the right-hand side of the same inequality holds

$$\left(\int\limits_{[0,1]}^{\oplus} F^{(t)} \odot d\mu\right)^{(r-s)} \odot \left(\int\limits_{[0,1]}^{\oplus} F^{(r)} \odot d\mu\right)^{(s-t)}$$

$$= g^{-1}\left(g^{r-s}\left(\left[g^{-1}\left(\int_0^1 (g^t \circ F_l)\, d(g\circ\mu)\right), g^{-1}\left(\int_0^1 (g^t \circ F_r)\, d(g\circ\mu)\right)\right]\right)\right.$$

$$\left.\cdot\, g^{s-t}\left(\left[g^{-1}\left(\int_0^1 (g^r \circ F_l)\, d(g\circ\mu)\right), g^{-1}\left(\int_0^1 (g^r \circ F_r)\, d(g\circ\mu)\right)\right]\right)\right)$$

$$= g^{-1}\left(g^{r-s}\left([E^{g,t}[F_l], E^{g,t}[F_r]]\right) \cdot g^{s-t}\left([E^{g,r}[F_l], E^{g,r}[F_r]]\right)\right)$$

$$= g^{-1}\left(g^{r-s}(E_I^{g,t}[F]) \cdot g^{s-t}\left(E_I^{g,r}[F]\right)\right).$$

Now, from (3) follows the inequality (4.3). □

Example 4. *Let* $([0,\infty), \oplus, \odot)$ *be a g-semiring with generator* $g(x) = x^{\frac{1}{n}}$, $n > 1$. *The inverse function is* $g^{-1}(x) = x^n$, *and the pseudo operation are given by* $x \oplus y = (\sqrt[n]{x} + \sqrt[n]{y})^n$ *and* $x \odot y = xy$.

Let $F = [F_l, F_r]$ *be an interval-valued function with measurable border functions,* $t = n-1$, $s = n$ *and* $r = n+1$, $n > 1$.

Since g is an increasing generator from (4.2) for function F_l *holds*

$$g^2\left(E^{g,n}[F_l]\right) \le g\left(E^{g,n-1}[F_l]\right) \cdot g\left(E^{g,n+1}[F_l]\right)$$

$$\left(\sqrt[n]{E^{g,n}[F_l]}\right)^2 \le \sqrt[n]{E^{g,n-1}[F_l]} \cdot \sqrt[n]{E^{g,n+1}[F_l]}$$

$$E^{g,n}[F_l] \le \sqrt{E^{g,n-1}[F_l] \cdot E^{g,n+1}[F_l]}. \tag{4.4}$$

Analogously, for function F_r *it follows that*

$$E^{g,n}[F_r] \le \sqrt{E^{g,n-1}[F_r] \cdot E^{g,n+1}[F_r]}. \tag{4.5}$$

For all $x \in [E^{g,n}[F_l], E^{g,n}[F_r]]$ *holds* $x \leq E^{g,n}[F_r]$. *From (4.5) holds*

$$x \leq \sqrt{E^{g,n-1}[F_r] \cdot E^{g,n+1}[F_r]}.$$

Also, for all $y \in \left[\sqrt{E^{g,n-1}[F_l] \cdot E^{g,n+1}[F_l]}, \sqrt{E^{g,n-1}[F_r] \cdot E^{g,n+1}[F_r]}\right]$ *holds the inequality* $\sqrt{E^{g,n-1}[F_l] \cdot E^{g,n+1}[F_l]} \leq y$, *and from (4.4) it follows that*

$$E^{g,n}[F_l] \leq y.$$

From Definition 1 it follows that

$$[E^{g,n}[F_l], E^{g,n}[F_r]] \leq_S \left[\sqrt{E^{g,n-1}[F_l] \cdot E^{g,n+1}[F_l]}, \sqrt{E^{g,n-1}[F_r] \cdot E^{g,n+1}[F_r]}\right],$$

so one estimation of interval-valued central g-moment of order n is

$$E_I^{g,n}[F] \leq_S \sqrt{[E^{g,n-1}[F_l] \cdot E^{g,n+1}[F_l], E^{g,n-1}[F_r] \cdot E^{g,n+1}[F_r]]}. \tag{4.6}$$

Note that in inequality (4.6), the estimation of interval-valued central g-moment of order n is obtained using interval-valued central g-moment of order $n-1$ *and interval-valued central g-moment of order* $n+1$.

5. Conclusions

In this paper, we have proven two generalizations of the Liapounoff inequality for pseudo-integral of interval-valued function. Also, the Liapounoff inequality for central g-moment of order n for a function f and the Liapounoff inequality for interval-valued central g-moment of order n for an interval-valued function F are proven.

The first step in the future investigation will be the generalization of theorems about the convergence of a sequence of random variables using the inequality (4.2) for the central g-moment of order n in the pseudo-probability space. The second step will be the generalization of theorems about the convergence of a sequence of interval-valued random sets using the inequality (4.3) for interval-valued central g-moment of order n, in the pseudo-probability space.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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