



Research article

Analysis of nonlinear time-fractional Klein-Gordon equation with power law kernel

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Abstract: We investigate the nonlinear Klein-Gordon equation with Caputo fractional derivative. The general series solution of the system is derived by using the composition of the double Laplace transform with the decomposition method. It is noted that the obtained solution converges to the exact solution of the model. The existence of the model in the presence of Caputo fractional derivative is performed. The validity and precision of the presented method are exhibited with particular examples with suitable subsidiary conditions, where good agreements are obtained. The error analysis and its corresponding surface plots are presented for each example. From the numerical solutions, we observe that the proposed system admits soliton solutions. It is noticed that the amplitude of the wave solution increases with deviations in time, that concludes the factor ω considerably increases the amplitude and disrupts the dispersion/nonlinearity properties, as a result, may admit the excitation in the dynamical system. We have also depicted the physical behavior that states the advancement of localized mode excitations in the system.

Keywords: Klein-Gordon equation; Caputo operator; double Laplace transform; decomposition method; localized modes excitations

Mathematics Subject Classification: 35Bxx, 35Qxx, 37Mxx, 65Mxx, 41Axx

1. Introduction

Fractional-order calculus has received considerable attention in the engineering and physical sciences over the last few decades to model a number of diverse phenomena in robotic-technology, bio-engineering, control theory, viscoelasticity diffusion model, relaxation processes and signal processing [1, 2]. The order of derivatives, as well as integrals in the fractional-order calculus, is

arbitrary. Therefore, fractional-order NPDEs have developed a fundamental interest in generalising integer-order NPDEs to model complex systems in thermodynamics, engineering, fluid dynamics and optical physics [3].

The enormous advantage of using fractional differential equations (FDEs) in modeling real-world problems is their global behavior together with preserving memory [4] which is not present in integer-order differential equations. It has also been noted that FDEs fastly converge to ordinary differential equations (ODEs) in a case when fractional-order is equal to one. Moreover, fractional calculus can clarify the basic features of various models and processes them more precisely than integer-order [5]. Several techniques have been applied to study analytical as well as numerical solutions of FNPDEs such as, Variational Iteration Method (VIM) [6], Laplace transforms Method [7], the double Laplace transform [8], invariant subspace method [9], Integral transform [10], Sumudu transform Method (STM) [11], natural transform [12] and Adomian decomposition method (ADM) [13].

The Klein-Gordon equation (KGE) considered herein is a basic non-linear evolution equation that arises in relativistic quantum Mechanics. It was formulated by Erwin Schrödinger for the non-relativistic wave equation in quantum physics, while precisely studied by the famous physicists O. Klein and W. Gordon (as it is named after their work) in 1926 [14, 15]. The KGE has an extensive variety of applications in classical field theory [16] as well as in quantum field theory [17]. It has also been extensively used in numerous areas of physical phenomena such as in solid-state physics, dispersive wave-phenomena, nonlinear optics, elementary particles behavior, dislocations propagation in crystals, and different class of soliton solutions [18]. Here, we investigate equation of the form [19]

$$\frac{\partial^\omega \psi}{\partial t^\omega} - \frac{\partial^2 \psi}{\partial x^2} + p\psi + qg(\psi) = r(x, t), \quad 1 < \omega \leq 2, \quad (1.1)$$

together with

$$\psi(x, 0) = \mathcal{F}(x), \quad \psi_t(x, 0) = \mathcal{G}(x),$$

where $\psi = \psi(x, t)$, $g(\psi)$ and $r(x, t)$ represent nonlinear term and external function respectively.

The nonlinear differential equations involve numerous fractional differential operators, such as, Caputo, Hilfer, Riemann-Liouville (R-L), Atangana-Baleanu in Caputo's sense, and Caputo-Fabrizio, [20]. The above fractional operators are very useful in FC due to the complexities of fractional-PDEs/ODEs because standard operators cannot handle some equations to obtain explicit solutions. The Caputo fractional derivative is the basic idea of fractional derivatives. All the fractional derivatives will reduce in Caputo or Riemann-Liouville fractional derivatives after some parametric replacement. One can assume that the fractional derivative could provide a power-law of the local behavior of non-differentiable functions. The Caputo fractional derivative was introduced by Michele Caputo in 1967 [21] to study initial/boundary value problems in many areas of real-world phenomena. The Caputo's derivative has many advantages as it is the most important tool for dealing with integer order models in a fractional sense with suitable subsidiary conditions [23]. Most of the problems have been handled precisely using Caputo operator [24].

The integer order KGE has broadly studied by using a variety of methods [25]. Time-fractional Klein Gordon equations with Caputo's fractional operator have also been extensively studied using a variety of numerical and analytical techniques [26]. Here, we apply double Laplace transform with decomposition method to study the general solutions of the governing model with power law. Some

particular examples are also studied numerically with some physical analysis. For preliminaries and some basic definitions of Caputo's derivative, see [31] and the reference therein.

Definition 1. Let us suppose $\psi(x, t)$ lies in $x - t$ plane, the double Laplace transform (DLT) of $\psi(x, t)$ is defined by [32]

$$\mathcal{L}_x \mathcal{L}_t [\psi(x, t)] = \int_0^\infty e^{-rx} \int_0^\infty e^{-st} \psi \, dx \, dt,$$

where, $r, s \in (C)$.

Definition 2. Application of DLT on fractional-order operator in Caputo's sense gives

$$\mathcal{L}_x \mathcal{L}_t \left\{ {}^C D_x^\omega \psi(x, t) \right\} = r^\omega \bar{\psi}(r, s) - \sum_{k=0}^{n-1} r^{\omega-1-k} \mathcal{L}_t \left\{ \frac{\partial^k \psi(0, t)}{\partial x^k} \right\},$$

and

$$\mathcal{L}_x \mathcal{L}_t \left\{ {}^C D_t^\beta \psi(x, t) \right\} = s^\beta \bar{\psi}(p, s) - \sum_{k=0}^{m-1} s^{\beta-1-k} \mathcal{L}_x \left\{ \frac{\partial^k \psi(x, 0)}{\partial t^k} \right\},$$

where, $m = [\beta] + 1$ and $n = [\omega] + 1$. Hence, we infer that

$$\mathcal{L}_x \mathcal{L}_t \psi(x) v(t) = \bar{\psi}(p) \bar{v}(s) = \mathcal{L}_x \psi(x) \mathcal{L}_t v(t).$$

The inverse DLT $\mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \{\bar{\psi}\} = \psi$, is represented by

$$\mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \{\bar{\psi}(x, t)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \int_{d-i\infty}^{d+i\infty} e^{px} \bar{\psi}(p, s) \, dp \, ds,$$

where $Re(p) \geq c$ and $Re(s) \geq d$, and $c, d \in \mathbb{R}$ to be chosen appropriately.

2. Existence of the solution

It is often more challenging to find the closed form of series solution to a nonlinear FDEs due to their complexity. Therefore question arises about the existence of the solution to such FDEs. For this, we utilize the applications of fixed point theory to study whether the solution of our considered system exists. So far, in the literature there exists no such theory for the existence of our considered system. We use here for the first time the $\beta - l$ -Geraghty type contraction to show that there exists a solution to the considered model. So we progress as follows

$${}_a^C D_t^\omega \psi - \frac{\partial \psi}{\partial x} + p\psi + qg(\psi) = r, \quad 1 < \omega \leq 2, \quad (2.1)$$

with

$$\psi(x, 0) = \mathcal{F}(x), \quad \psi_t(x, 0) = \mathcal{G}(x). \quad (2.2)$$

The above equation can also be expressed in the form

$${}_a^C D_t^\omega \psi = \mathcal{H}(x, t, \psi), \quad 1 < \omega \leq 2, \quad (2.3)$$

where

$$\mathcal{H}(x, t, \psi) = \frac{\partial \psi}{\partial x} - p\psi - qg(\psi) + r. \quad (2.4)$$

For the existence of the above model, we use the following notions.

Let Ω be the family of continuous and increasing functions defined as $\mathbb{1} : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\mathbb{1}(qx) \leq q\mathbb{1}(x) \leq qx, \quad q > 1,$$

and the elements of Θ are non-decreasing functions, such that

$$\varepsilon : [0, \infty) \rightarrow [0, \frac{1}{\rho_1}), \quad \text{where } \rho_1 \geq 1.$$

Definition 3. Suppose that (\mathcal{M}, d) be a complete b -metric space: Let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ also consider that $\exists \mathbb{F} : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ with $\mathbb{F}(m, n)\mathbb{1}(\rho_1^3 d(\mathcal{T}_m, \mathcal{T}_n)) \leq \varepsilon(\mathbb{1}d(m, n))\mathbb{1}(d(m, n))$, for $m, n \in \mathcal{M}$, where $\rho_1 \geq 1$, $\varepsilon \in \Theta$ and $\mathbb{1} \in \Omega$. Then \mathcal{T} is known a generalized $\mathbb{F} - \mathbb{1}$ -Gergaghty type contraction mapping.

Definition 4. Consider $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$, where \mathcal{M} is non-empty and $\mathbb{F} : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$, where $\beta(m, n) \geq 1 \Rightarrow \beta(\mathcal{M}_m, \mathcal{M}_n) \forall m, n \in \mathcal{M}$, then \mathcal{T} is called β -admissible mapping.

First we show that there exists the fixed point for the considered model Eq (2.3), for this we apply the following theorem.

Theorem 1. [33] Let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ be a generalized $\mathbb{F} - \mathbb{1}$ -Gergaghty type contraction such that

(1) \mathcal{T} is β -admissible.

(2) There exists $v_0 \in \mathcal{M}$ with $\beta(v_0, \mathcal{T}v_0) \geq 1$.

(3) $\{v_n\} \subset \mathcal{M}$, $\lim_{n \rightarrow \infty} v_n = v$, where $v \in \mathcal{M}$ and $\beta(v_n, v_{n+1}) \geq 1 \Rightarrow \beta(v_n, v) \geq 1$,

then \exists a fixed point for \mathcal{T} . Let $\mathcal{M} = C(\pi, \mathcal{R})$ and $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$, where $\pi = [0, 1] \times [0, 1]$ given by

$$d(\mathbf{u}, \mathbf{v}) = \|(\mathbf{u} - \mathbf{v})^2\|_\infty = \sup_{m \in [0, M] \ t \in [0, T]} (\mathbf{u} - \mathbf{v})^2,$$

thus (\mathcal{M}, d) be a complete b -metric space. The following theorem shows the existence of solution of the considered model Eq (2.3).

Theorem 2. Suppose that $\exists \mathcal{J} : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

(1) $|\mathcal{H}(x, t, \psi(x, t)) - \mathcal{H}(x, t, \phi(x, t))| \leq \frac{\sqrt{\alpha+1}}{3\sqrt{3}} \sqrt{\varepsilon(\mathbb{1}|\mathbf{u} - \mathbf{v}|^2)\mathbb{1}(|\mathbf{u} - \mathbf{v}|^2)}$, for $x \in [0, X]$, $t \in [0, T]$, and $\mathbf{u}, \mathbf{v} \in \mathcal{M}$ with $\mathcal{J}(\mathbf{u}, \mathbf{v}) \geq 0$.

(2) there exists $u_1 \in \mathcal{M}$ with $\mathcal{J}(u_1, \mathcal{T}u_1) \geq 0$, where $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ is defined by $\mathcal{T}u_j = u_0u_0 + u_1u_t + I_t^\omega \mathcal{H}(x, t, \psi(x, t))$.

(3) for $u, v \in \mathcal{M}$, $\mathcal{J}(u, v) \geq 0 \Rightarrow \mathcal{J}(\mathcal{T}u_n, \mathcal{T}v)$.

(4) $\{u_n\} \subset \mathcal{M}$, $u_n \rightarrow u$ where $u \in \mathcal{M}$ and $\mathcal{J}(u_n, u_{n+1}) \Rightarrow \mathcal{J}(u_n, u) \geq 0$, for $n \in \mathbb{N}$. Then there exists a solution of the model Eq (2.3).

Proof. Applying the fractional integral to Eq (2.3), we obtain

$$\psi(x, t) = C_0\psi(x, 0) + C_1\psi_t(x, 0) + I_t^\omega \mathcal{H}(x, t, \psi(x, t)) = \mathcal{T}\psi(x, t).$$

Here we prove that \mathcal{T} has a fixed point using the above technique, thus

$$\begin{aligned} |\mathcal{T}\psi(x, t) - \mathcal{T}\phi(x, t)|^2 &= |I_t^\omega \mathcal{H}(x, t, \psi(x, t)) - I_t^\omega \mathcal{H}(x, t, \phi(x, t))|^2 \\ &\leq I_t^\omega \{|\mathcal{H}(x, t, \psi(x, t)) - \mathcal{H}(x, t, \phi(x, t))|\}^2 \\ &= \left\{ \frac{1}{\sqrt{\omega}} |\mathcal{H}(x, t, \psi(x, t)) - \mathcal{H}(x, t, \phi(x, t))| \right\}^2 \\ &\leq \left\{ \frac{\omega \sqrt{\omega}}{3 \sqrt{3} \sqrt{\omega} T^\omega} \int_0^t (t-s)^{\omega-1} \sqrt{\varepsilon(1(|u-v|^2)1(|u-v|^2))} \right\}^2 \\ &\leq \left\{ \omega T^\omega \int_0^t (t-s)^{\omega-1} \sqrt{\varepsilon(1(\sup_{x \in [0, X]} \sup_{t \in [0, T]} |u-v|^2)1(\sup_{x \in [0, X]} \sup_{t \in [0, T]} |u-v|^2))} \right\}^2 \\ &\leq \frac{1}{3 \sqrt{3}} \varepsilon(1(d(u, v))1(d(u, v))). \end{aligned}$$

Hence for $u, v \in C(\pi)$ with $\mathcal{J}(u, v) \geq 0$ we have $27\|(\mathcal{T}u - \mathcal{T}v)^2\| \leq \varepsilon(1(d(u, v))1(d(u, v)))$. Now $\mathbb{F} : C([0, X] \times [0, T], \mathbb{R}) \times C([0, X] \times [0, T], \mathbb{R}) \rightarrow [0, \infty)$ by

$$\beta(u, v) = \begin{cases} 1 & \text{if } \mathcal{J}(u, v) \geq 0, \\ 0 & \text{else,} \end{cases}$$

and

$$\begin{aligned} \beta(u, v)1(27d(\mathcal{T}u, \mathcal{T}v)) &\leq 27d(\mathcal{T}u, \mathcal{T}v) \\ &\leq \varepsilon(1d(u, v)1d(u, v)). \end{aligned}$$

Thus, \mathcal{T} is an $\beta - 1$ -contraction. Now to show that \mathcal{T} is β -admissible, we have from condition (iii)

$$\beta(u, v) \geq 1 \Rightarrow \mathcal{J}(u, v) \geq 0 \Rightarrow \mathcal{J}(\mathcal{T}u, \mathcal{T}v) \geq 0 \Rightarrow \beta(\mathcal{T}u, \mathcal{T}v) \geq 1.$$

For $u, v \in C(\pi, \mathbb{R})$, from condition (ii) we have $u \in C(\pi, \mathbb{R})$. Such that $\beta(u_0, \mathcal{T}u_0) \geq 1$. Similarly from (iv) and Theorem 1, there exists $u^* \in C(\pi, \mathbb{R})$, such that $u^* = \mathcal{T}u^*$. Therefore we proved that the model Eq (2.3) has a solution. \square

3. Modified double Laplace decomposition method (MDLDM)

Here, we study the above technique, which is a composition of DLT with the decomposition method. This method can be applied to find the general series solutions for various PDEs/ODEs. This is an efficient technique to study the analytical solutions of several nonlinear systems [34]. Let us consider the general non-linear system

$$L\psi + R\psi + N\psi = r(x, t). \quad (3.1)$$

Here, L and R is linear and nonlinear operators, $r(x, t)$ is some particular external function and N is nonlinearity in the system. The convergence analysis of the considered technique can be seen in [35].

General solution of proposed model in Caputo's sense

Using the technique defined above and expressing Eq (1.1) in the form

$${}^C D_t^\omega \psi(x, t) - \frac{\partial^2 \psi}{\partial x^2} + p\psi + qg(\psi) = r(x, t), \quad 1 < \omega \leq 2, \quad (3.2)$$

with

$$\psi(x, 0) = \mathcal{F}(x), \quad \psi_t(x, 0) = \mathcal{G}(x). \quad (3.3)$$

Applying DLT to above equation, we obtain

$$\mathcal{L}_x \mathcal{L}_t \{ {}^C D_t^\omega \psi \} - \mathcal{L}_x \mathcal{L}_t \left\{ \frac{\partial^2 \psi}{\partial x^2} \right\} + p \mathcal{L}_x \mathcal{L}_t \{ \psi \} + q \mathcal{L}_x \mathcal{L}_t \{ g\psi \} = \mathcal{L}_x \mathcal{L}_t \{ r(x, t) \}. \quad (3.4)$$

Applying DLT on fractional order, gives

$$\begin{aligned} \mathcal{L}_x \mathcal{L}_t \{ \psi \} &= \frac{1}{s} \mathcal{L}_x \{ \psi(x, 0) \} + \frac{1}{s^2} \mathcal{L}_x \{ \psi_t(x, 0) \} + \frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \left\{ \frac{\partial^2 \psi}{\partial x^2} \right\} + p \frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi \} \\ &+ q \frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ g(\psi) \} + \mathcal{L}_x \mathcal{L}_t \{ r(x, t) \}. \end{aligned} \quad (3.5)$$

Similarly, applying Laplace transform on Eq (3.3), gives

$$\mathcal{L}_x \{ \psi(x, 0) \} = \overline{\mathcal{F}}(p), \quad \mathcal{L}_x \{ \psi_t(x, 0) \} = t \overline{\mathcal{G}}(p). \quad (3.6)$$

Now consider

$$\psi = \sum_{n=0}^{\infty} \psi_n, \quad (3.7)$$

where the non-linear term can be degraded as

$$g(\psi) = \sum_{i=0}^{\infty} A_i, \quad (3.8)$$

where A_n , is given by [36]

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\sum_{k=0}^n \lambda^k g(\psi_k) \right]_{\lambda=0}. \quad (3.9)$$

Finally, applying inverse DLT to Eq (3.2), using Eq (3.6) and Eq (3.9), gives

$$\begin{aligned} \psi_0 &= \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s} \overline{\mathcal{F}}(p, 0) \right] + t \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^2} \overline{\mathcal{G}}(p, 0) \right] = \psi(x, 0), \\ \psi_1 &= \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi_{0xx} \} \right] - p \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi_0 \} \right] - q \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ A_0 \} \right] \\ &\quad + \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ r(x, t) \} \right], \\ \psi_2 &= \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi_{1xx} \} \right] - p \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi_1 \} \right] - q \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ A_1 \} \right], \\ \psi_3 &= \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi_{2xx} \} \right] - p \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ \psi_2 \} \right] - q \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^\omega} \mathcal{L}_x \mathcal{L}_t \{ A_2 \} \right]. \end{aligned}$$

In a similar manner, other terms can be computed. Final result can be obtained as

$$\psi(x, t) = \sum_{n=0}^{\infty} \psi_n(x, t). \quad (3.10)$$

which is the general solution of Eq (3.2) in series form by using the proposed method as discussed above.

4. Applications of MDLDM

Here, we present numerical examples on the TFKG equation in Caputo's sense given as Eq (3.2) and discuss the behaviour of each example. We apply the aforesaid technique discussed in Section 3, to obtain the approximate solution of the problems.

Example 1. Consider the nonlinear TFKG equation

$$D_t^\omega \psi - \frac{\partial^2 \psi}{\partial x^2} + \frac{3}{4} \psi - \frac{3}{2} \psi^3 = 0, \quad 1 < \omega \leq 2, \quad p = \frac{3}{4}, \quad q = \frac{3}{2}, \quad (4.1)$$

$$g(\psi) = \psi^3, \quad r(x, t) = 0, \quad (4.2)$$

with

$$\psi(x, 0) = -\operatorname{sech}(x), \quad \psi_t(x, 0) = \frac{1}{2} \operatorname{sech}(x) \tanh(x). \quad (4.3)$$

For $\alpha = 2$, the exact solution of Eq (4.1) can be obtained in the form [19]

$$\psi(x, t) = -\operatorname{sech}\left(x + \frac{t}{2}\right). \quad (4.4)$$

Consider TFKG Eq (4.1) in Caputo's sense

$${}^c D_t^\omega \psi - \frac{\partial^2 \psi}{\partial x^2} + \frac{3}{4} \psi - \frac{3}{2} \psi^3 = 0, \quad 1 < \omega \leq 2. \quad (4.5)$$

Applying MDLDM scheme discussed in Section 3, we obtain

$$\begin{aligned} \psi_0 &= -\operatorname{sech}(x) + \frac{t}{2} \operatorname{sech}(x) \tanh(x), \\ \psi_1 &= \frac{t^\omega}{\Gamma(\omega + 1)} \left[-\frac{1}{4} - \frac{3}{2} \operatorname{sech}^2(x) \right] \operatorname{sech}(x) + \frac{t^{\omega+1}}{\Gamma(\omega + 2)} \left[\frac{11}{8} - \frac{7}{4} \operatorname{sech}^2(x) \right] \operatorname{sech}(x) \tanh(x) \\ &\quad - \frac{2! t^{\omega+2}}{\Gamma(\omega + 3)} \left[\frac{9}{8} \operatorname{sech}^3(x) \tanh^2(x) \right] + \frac{3! t^{\omega+3}}{\Gamma(\omega + 4)} \left[\frac{3}{16} \operatorname{sech}^3(x) \tanh^3(x) \right]. \end{aligned}$$

In a similar manner, other terms can be computed. Final result can be obtained as

$$\psi(x, t) = \sum_{n=0}^{\infty} \psi_n(x, t). \quad (4.6)$$

Discussion

The error analysis between series solution Eq (4.6) and the exact solution Eq (4.4) is shown in Table 1, while the surface behaviour is shown in Figure 1 reveals that Eq (4.1) is depends on time (t). It should be noted that when the time (t) is small enough, there is less extent of error exist amongst the approximate and exact solutions obtained by the MDLDM method. Figure 2 [left panel] shows the absolute of wave solution Eq (4.6) with deviations in (α) with $t = 0.65$ in comparison with exact solution Eq (4.4). Notice that numerical result, Eq (4.6) exactly matches to the exact solution Eq (4.4). This shows that the governing equation admits a soliton solution. Figure 2 [right panel] represents Eq (4.6) reveals that the amplitude of the solitary potentials blow-up as t increases. The 3D profiles for Eq (4.6) is shown in Figure 3 versus x for $t = 0.65$. It reveals the progression of localized mode in the governing system. The solution obtained as Eq (4.6) versus x for $t = 1$ (*dashed line*), 0.8 (*solid curve*), 0.6 (*dotted curve*) Figure 4, when $\omega = 2$ and 1.7 are also depicted. Clearly, one can see the wave amplitude enhancement with variations in t that concludes that coefficient (ω) considerably increases the wave amplitudes.

Table 1. Comparison between the exact solution (4.4) with approximate solution obtained in the form (4.6).

(x,t)	Exact	$ \psi $	Exact- ψ	(x,t)	Exact	$ \psi $	Exact- ψ
(-6,0.6)	0.0067	0.0067	2.4083×10^{-5}	(-4,0.6)	0.0494	0.0492	1.7609×10^{-4}
(-2,0.6)	0.3536	0.3529	6.2194×10^{-4}	(0,0.6)	0.9566	0.9550	1.6000×10^{-3}
(2,0.6)	0.1985	0.1992	6.7883×10^{-4}	(4,0.6)	0.0271	0.0273	1.5198×10^{-4}
(6,0.6)	0.0037	0.0037	2.0728×10^{-5}	(-6,0.2)	0.0055	0.0055	8.4718×10^{-7}
(-4,0.2)	0.0405	0.0405	6.2013×10^{-6}	(-2,0.2)	0.2926	0.2926	2.4219×10^{-5}
(0,0.2)	0.9950	0.9950	2.0749×10^{-5}	(2,0.2)	0.2413	0.2413	2.4874×10^{-5}
(4,0.2)	0.0331	0.0331	5.9043×10^{-6}	(6,0.2)	0.0045	0.0045	8.0587×10^{-7}
(-6,0.05)	0.0051	0.0051	1.2989×10^{-8}	(-4,0.05)	0.0375	0.0375	9.5114×10^{-8}
(-2,0.05)	0.2723	0.2723	3.8310×10^{-7}	(0,0.05)	0.9997	0.9997	8.1360×10^{-8}
(2,0.05)	0.2595	0.2595	3.8564×10^{-7}	(4,0.05)	0.0357	0.0357	9.3954×10^{-8}
(6,0.05)	0.0048	0.0048	1.2828×10^{-8}				

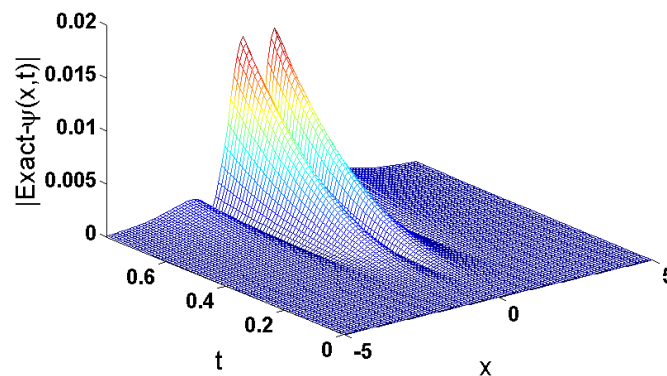


Figure 1. The surface plot of the error analysis given in Table 1.

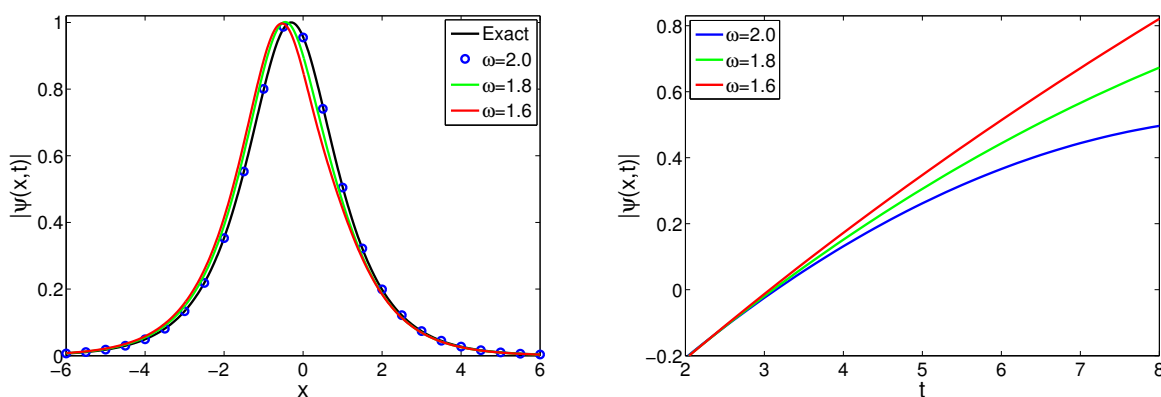


Figure 2. The left plot portrays comparison between Eq (4.4) and Eq (4.6) for various values of ω , while the right panel portrays solution profiles of $\psi(x, t)$ vs t for various values of ω .

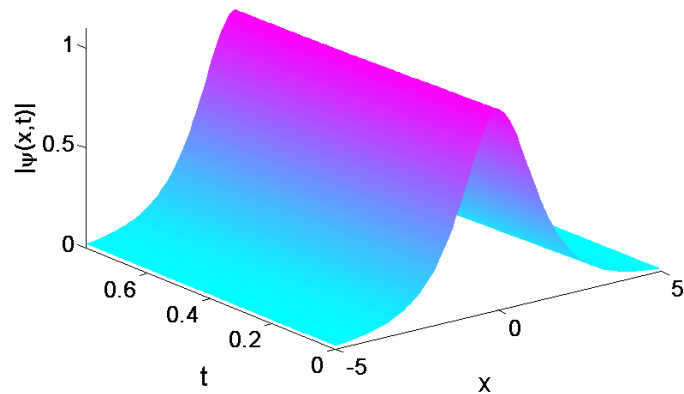


Figure 3. The surface plot of Eq (4.6) for the parameters used in the Figure 2's left panel.

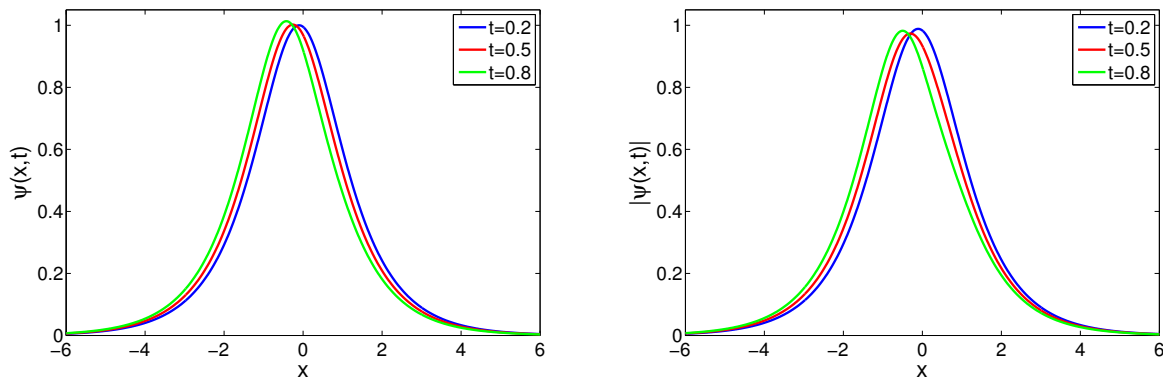


Figure 4. The solution profiles of Eq (4.6) for desperate values of ω with desperate values of t .

Example 2. Consider the time-fractional nonlinear KGE in the form

$$D_t^\omega \phi - \frac{\partial^2 \phi}{\partial x^2} + q\phi^3 = 0, \quad 1 < \omega \leq 2, \quad p = 0 \quad q = 1, \quad (4.7)$$

$$y(\phi) = \phi^3, \quad r(x, t) = 0, \quad (4.8)$$

with

$$\phi(x, 0) = R \tan(\lambda x), \quad \phi_t(x, 0) = R\eta \lambda \sec^2(\lambda x), \quad (4.9)$$

where

$$R = \sqrt{\frac{\rho}{\kappa}} \quad \text{and} \quad \lambda = \sqrt{\frac{-\rho}{2(\sigma + \eta^2)}}.$$

The parameters ρ, κ, σ , and $\eta \in \mathbb{R}$. It should be noted that, for $\alpha = 2$, an exact solution of Eq (4.7) can be obtained in the form [37]

$$\phi(x, t) = R \tan[\lambda(x + \eta t)]. \quad (4.10)$$

Writing Eq (4.7) in Caputo's sense gives

$${}^c D_t^\omega \phi - \frac{\partial^2 \phi}{\partial x^2} + \phi^3 = 0, \quad 1 < \omega \leq 2. \quad (4.11)$$

The series solution of Eq (4.11) with conditions (4.9) gives

$$\begin{aligned} \phi_0 &= R \tan(\lambda x) + t R \eta \lambda \sec^2(\lambda x), \\ \phi_1 &= \left(\frac{t^\omega}{\Gamma(\omega + 1)} \right) [2R\lambda^2 \sec^2(\lambda x) \tan(\lambda x) - R^2 \tan^3(\lambda x)] \\ &\quad + R \eta \lambda \left(\frac{t^{\omega+1}}{\Gamma(\omega + 2)} \right) [4\lambda^2 \sec^2(\lambda x) \tan^2(\lambda x) + 2\lambda^2 \sec^4(\lambda x) - 3R^2 \tan^2(\lambda x) \sec^2(\lambda x)] \\ &\quad - 3R^3 \lambda^2 \eta^2 \left(\frac{2! t^{\omega+2}}{\Gamma(\omega + 3)} \right) [\sec^4(\lambda x) \tan(\lambda x)] - R^3 \lambda^3 \eta^3 \left(\frac{3! t^{\omega+3}}{\Gamma(\omega + 4)} \right) [\operatorname{sech}^6(\lambda x)]. \end{aligned}$$

In a similar manner, other terms can be computed. The final result can be written in the form

$$\phi = \sum_{n=0}^{\infty} \phi_n. \quad (4.12)$$

Discussion

The parameters as $\kappa = 1$, $\sigma = -8.5$, $\eta = 0.05$, and $\rho = -1$ are considered for numerical illustration. The error amongst the approximate and exact solutions of Eq (4.7) is shown in Table 2 and its corresponding surface plot is presented in Figure 5. The numerical solution, Eq (4.12) and exact solution Eq (4.10) is depicted in Figure 6 [left panel], for $t = 7$ and for different values of time-fractional coefficient (ω). It is noted that TFKG Eq (4.12) may admits the excitation in the system. This amount enrichment in ω overturned the wave amplitude as it interrupt the dispersion/nonlinearity effects. To see the effect of a temporal variable (t) on the wave solution, Eq (4.12) is displayed in Figure 6 [right panel] which shows that $\phi(x, t)$ increases with time. Further, the 3D profiles for Eq (4.12) is shown versus x with $t = 7$ in Figure 7 for ($\omega = 2$), which represents the physical behaviour of Eq (4.12). It shows the advancement of localized mode excitations in the governing equation. The solution of Eq (4.12) versus x with $t = 7$ (dashed curve), 6 (solid green curve), 5 (dotted curve), in Figure 8, with $\omega = 2$ and 1.5 respectively is depicted. Clearly, the wave amplitude increases with deviations in t . It infers that the fractional order (ω) significantly increases the wave amplitudes.

Table 2. Comparison between approximate solution obtained in the form (4.12) with exact solution (4.7).

(x,t)	Exact	$ \phi $	Exact- ϕ	(x,t)	Exact	$ \phi $	Exact- ϕ
(-10,0.02)	0.9845	0.9845	4×10^{-7}	(-8,0.02)	0.9596	0.9596	4×10^{-6}
(-6,0.02)	0.8968	0.8965	3×10^{-5}	(-4,0.02)	0.7488	0.7486	2×10^{-5}
(-2,0.02)	0.4504	0.4503	1×10^{-5}	(0,0.02)	0	0	0
(2,0.02)	0.4504	0.4503	1×10^{-5}	(4,0.02)	0.7488	0.7486	2×10^{-5}
(6,0.02)	0.8968	0.8965	3×10^{-5}	(8,0.02)	0.9596	0.9596	4×10^{-6}
(10,0.02)	0.9845	0.9845	4×10^{-7}	(-10,0.05)	0.9845	0.9845	1.8751×10^{-5}
(-8,0.05)	0.9595	0.9596	4.8543×10^{-5}	(-6,0.05)	0.8966	0.8968	1.2196×10^{-4}
(-4,0.05)	0.7486	0.7488	2.8545×10^{-4}	(-2,0.05)	0.4499	0.4499	5.8734×10^{-4}
(0,0.05)	0.0012	0.0012	1.1300×10^{-6}	(2,0.05)	0.4508	0.4505	3.7952×10^{-4}
(4,0.05)	0.7491	0.7488	2.4730×10^{-4}	(6,0.05)	0.8969	0.8968	1.1554×10^{-4}
(8,0.05)	0.9596	0.9596	4.7525×10^{-5}	(10,0.05)	0.9845	0.9845	1.8587×10^{-5}

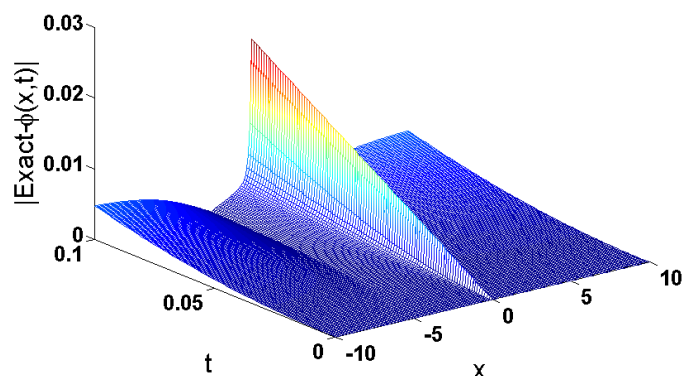


Figure 5. The surface plot for Table 2.

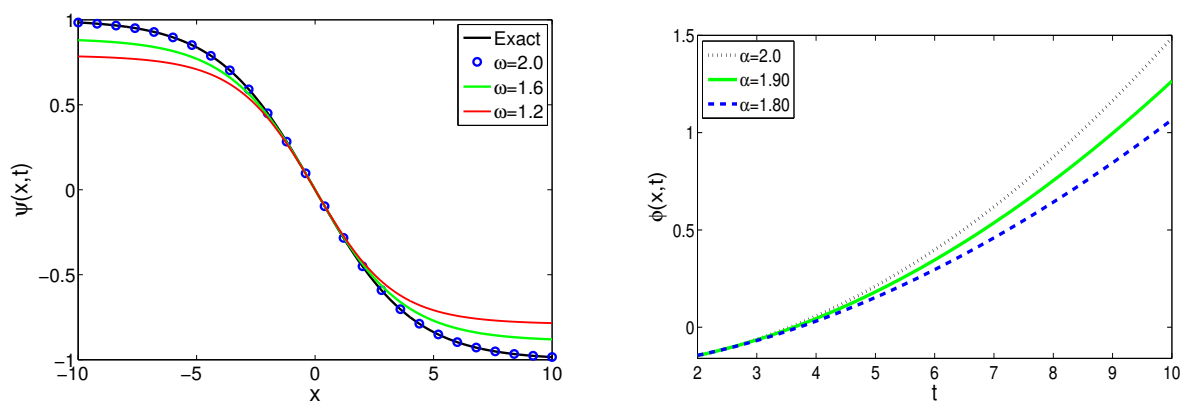


Figure 6. Comparison between Eq (4.10) and Eq (4.12) for different values of ω [left panel]. The solution profiles of $\psi(x, t)$ against time (t) interval with various values of ω [right panel].

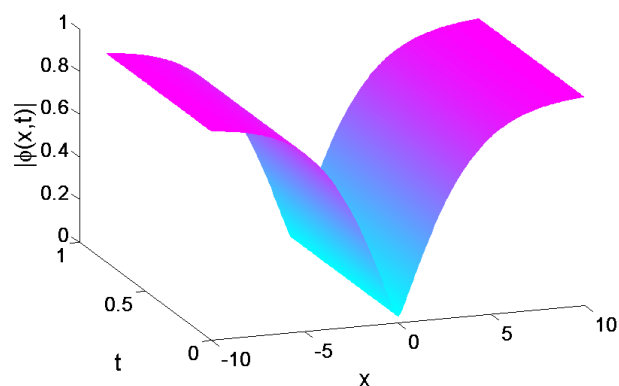


Figure 7. The surface plot for the parameters used for the left panel of Figure 6.

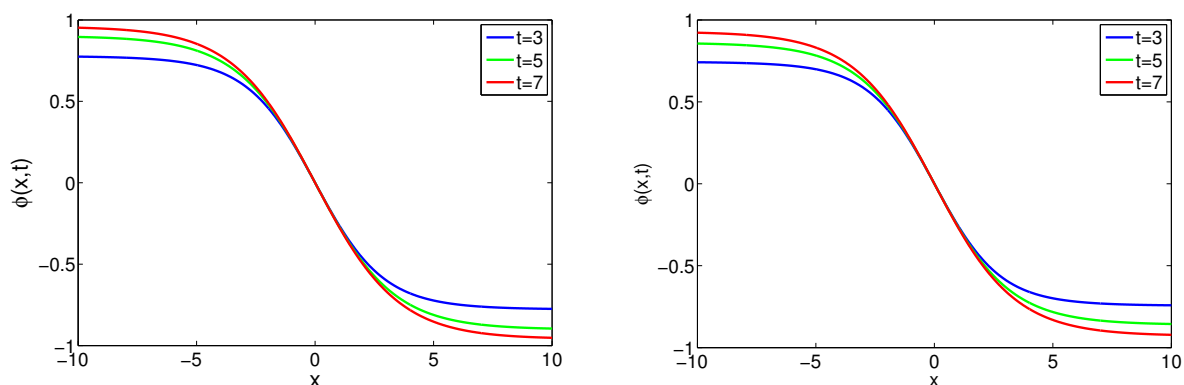


Figure 8. The solution profiles of Eq (4.12) for different ω with different values of time(t).

5. Conclusions

We have studied the TFKG equation using double Laplace transforms with the decomposition method. The general solution of proposed system is obtained as a class of general series solution. It is relevant to note that following only two iterations, fairly precise results are obtained that converges to the exact solution of the governing equation. The proposed method offers perfect numerical results without any alteration and complicated numerical methods for the governing equation in the fractional case. The numerical results obtained for particular examples are compared with the exact solutions at the classical order. The result profiles with physical interpretations for different fraction orders were revealed explicitly.

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Conflict of interest

It is declared that all the authors have no conflict of interest regarding this manuscript.

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