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Research article

Mahgoub transform and Hyers-Ulam stability of n^{th} order linear differential equations

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Abstract: The main aim of this paper is to investigate various types of Hyers-Ulam stability of linear differential equations of n^{th} order with constant coefficients using the Mahgoub transform method. We also show the Hyers-Ulam constants of these differential equations and give some main results.

Keywords: Mahgoub transform; Hyers-Ulam stability; n^{th} order linear differential equation; functional equation
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1. Introduction

In 1940 Ulam [25] proposed a very general Hyers-Ulam stability problem: When is the statement of the theorem still true or nearly true despite slight variations on the theorems hypothesis? In the following year Hyers [10] come up with the first two positive answer to Ulam's question by proving the stability of the additive functional equation in Banach spaces. Since then, Hyers result has been widely generalised in terms of the control conditions used to do define the concept of an approximate solution (see [5–9, 18, 19, 23, 26, 28]).

The generalization of Ulam's questions has been relatively recently proposed by replacing functional equations with differential equations.

Let I be a subinterval of $\mathbb{R}$, let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$ and let n be a positive integer. The differential equation

$$\rho(h, w, w^1, w^{''}, \cdots, w^{(n)}) = 0$$

has the Hyers-Ulam stability if there exists a constant $K > 0$ such that the following statement is true for any $\epsilon > 0$: if an n times continuously differentiable function

$$\tau : I \to \mathbb{K}$$

satisfies the inequality

$$|\rho(t, \tau, \tau^{'}, \tau^{''}, ..., \tau^{(n)})| \leq \epsilon,$$

for all $t \in I$, then there exists a solution $\zeta : I \to \mathbb{K}$ of the differential equation that satisfies the inequality

$$|\tau(t) - \zeta(t)| \leq K\epsilon,$$

for all $t \in I$.

Obloza seems to the first author who has investigated the Hyers-Ulam stability of linear differential equation (see [16, 17]). In 1998, Alsina and Ger continued the study of Obloza's Hyers-Ulam stability of differential equations. Indeed they proved [4] in the following theorem.

Theorem 1.1. *Let I be a non-empty open subinterval of $\mathbb{R}$. If a differentiable function $w : I \to \mathbb{R}$ satisfies the differential inequality*

$$||w^{'}(t) - w(t)|| \leq \epsilon,$$

for any $t \in I$ and for some $\epsilon > 0$, then there exists a differentiable function $\zeta : I \to \mathbb{R}$ satisfying $\zeta'(t) = w(t)$ and

$$||w(t) - \zeta(t)|| \leq K\epsilon$$

for all $t \in I$.

This result of Alisina and Ger has been generalized by Takahashi et al. [22]. They proved that the Hyers-Ulam stability holds true for the Banach space valued differential equation $w^{'}(t) = \chi w(t)$. Indeed the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings (see [11, 12, 15]).

In 2006 Jung [13] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2008, Wang et al. [27] studied the Hyers-Ulam stability of linear differential equations of first order using the integral factor method. Meanwhile, Rus [21] discussed various types of Hyers-Ulam stability of the ordinary differential equations

$$w^{'}(t) = pw(t) + h.$$

In 2014, Alqifiary and Jung [3] proved the Hyers-Ulam stability of linear differential equation of the form

$$w^{(n)}(t) + \sum_{k=0}^{n-1} \gamma_k w^{(k)}(t) = h(t)$$

by using the Laplace transform method, where γ_k scalars and $w(t)$ is an n times continuously differentiable function of the exponential order (see [20]).

In 2016, Mahgoub [14] introduced a Mahgoub transform for solving linear ordinary differential equations with constant coeffcient. Aggarwal et al. [2] used Mahgoub transform for solving linear Volterra integral equations. Kumar and Viswanathan [24] used Mahgoub transform to mechanics and electrical circuit problems. Recently, Aggarwal [1] introduced a comparative study of Mohand and Mahgoub transform.

Based on the above results, our main goal is to more efficiently prove the Hyers-Ulam stability of the n^{th} order linear differential equations

$$w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) = 0 \tag{1.1}$$

and

$$w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) = m(t) \tag{1.2}$$

by using Mahgoub integral transform method, where $a_{n-1}, \cdots, a_2, a_1, a_0$ are scalars and $w(t)$ is a continuously differentiable function in exponential order.

2. Preliminaries

In this section, we introduce some standard notations and definitions which will be useful to prove our main results.

Throughout this paper, $\mathbb{K}$ denotes either the real field $\mathbb{R}$ or complex field $\mathbb{C}$. A function $h : [0, \infty) \to \mathbb{K}$ is called of exponential order if there exist constants $P, Q \in R$ such that

$$|h(t)| \leq Pe^{Qt}$$

for all $t \geq 0$. Similarly, a function $j : (-\infty, 0] \to \mathbb{K}$ is called of exponential order if there exist constants $P, Q \in R$ such that

$$|j(t)| \leq Pe^{Qt}$$

for all $t \leq 0$.

Definition 2.1. The Mahgoub (integral) transform of the function $h : [0, \infty) \to \mathbb{K}$ is defined by

$$\psi(h(t)) = a \int_0^{\infty} h(s)e^{-as} ds := H(a),$$

where ψ is the Mahgoub integral transform operator.

The Mahgoub integral transform for the function $h : [0, \infty) \to \mathbb{K}$ exist if $h(t)$ is piecewise continuous and of exponential order.

These conditions are the only sufficient conditions for the existence of Mahgoub transform of the function $h(t)$.

Linearity: If h and j are have Mahgoub transform as $\psi(h)$ and $\psi(j)$, then $\psi(\alpha h(s) + \beta j(s)) = \alpha\psi(h) + \beta\psi(j)$, where $\alpha\beta \in R_+$.

Proof.

$$\begin{aligned}\psi(\alpha h(s)+\beta j(s)) &= a\int_0^\infty (\alpha h(s)+\beta j(s))\cdot e^{-as}ds \\ &= a\int_0^\infty \alpha h(s)\cdot e^{-as}ds + a\int_0^\infty \beta j(s)\cdot e^{-as}ds \\ &= \alpha\int_0^\infty ah(s)\cdot e^{-as}ds + \beta\int_0^\infty aj(s)\cdot e^{-as}ds \\ &= \alpha\psi(h)+\beta\psi(j).\end{aligned}$$

□

Definition 2.2 (Convolution of two functions)**.** The convolution is, the change in the form of a value or expression without change in the value.

The convolutionn of two functions $h(t)$ and $j(t)$ is denoted by $h(t) * j(t)$ and is defined by

$$h(t) * j(t) = (h * j)(t) = \int_0^t h(s)j(t-s)ds = \int_0^t h(t-s)j(s)ds.$$

The convolution theorem of two functions h and j of Mahgoub transform is given by

$$\begin{aligned}\psi(h * j)(a) &= \frac{1}{a}\psi(h)\cdot\psi(j), \\ \psi(h * j)(a) &= \int_0^\infty (h * j)(t)[ae^{-at}]dt \\ &= \int_0^\infty\int_0^\infty h(s)j(t-s)ds(ae^{-at})dt \\ &= \int_0^\infty h(s)ds\int_0^\infty j(t-s)(ae^{-at})dt.\end{aligned}$$

Letting $t-s=y$, we have

$$\begin{aligned}\psi(h * j)(a) &= \int_0^\infty h(s)ds\int_0^\infty j(y)(ae^{-a(y+s)})dt \\ &= \int_0^\infty h(s)e^{-au}ds\int_0^\infty j(y)(ae^{-ay})dt \\ &= \frac{1}{a}\left\{a\int_0^\infty h(s)e^{-au}ds\cdot a\int_0^\infty j(y)e^{-ay}dt\right\} \\ &= \frac{1}{a}\psi(h)\cdot\psi(j).\end{aligned}$$

Theorem 2.3 (Convolution theorem for Mahgoub transform)**.** *Let* $h, j \in H'(R)$*. Then*

$$\psi(h * j)(v) = \frac{\psi(h)\psi(j)}{v}.$$

Proof.

$$\begin{aligned}\psi(h*j)(a) &=< (h*j), ae^{-as} >=< h(s), < j(t), ae^{-a(s+t)} >> \\ &=< h(s), < j(t), ae^{-as}\cdot e^{-at} >>=< h(s), ae^{-as} >< j(t), e^{-at} > \\ &= \frac{1}{a} < h(s), ae^{-as} > \cdot < j(t), ve^{-at} > \\ &= \frac{1}{a}\psi(h)\cdot\psi(j)\end{aligned}$$

This completes the proof. □

Lemma 2.4. *If Mahgoub transform of function $F(s)$ is $\psi(a)$, then Mahgoub transform of function $F(bs)$ is given by $\psi\left(\frac{a}{b}\right)$.*

Proof. By the definition of Mahgoub transform, we have

$$\psi(F(bs)) = a\int_0^\infty F(bs)e^{-as}ds. \tag{2.1}$$

Putting $bs = p \Rightarrow bds = dp$ in (2.1), we have

$$\begin{aligned}\psi(F(bs)) &= \frac{a}{b}\int_0^\infty F(p)e^{-\frac{ap}{b}}dp, \\ \psi(F(bs)) &= \psi\left(\frac{a}{b}\right).\end{aligned}$$

□

Lemma 2.5. *If Mahgoub transform of function $F(s)$ is $\psi(a)$, then Mahgoub transform of function $e^{bs}F(s)$ is given by $\frac{a}{a-b}\psi(a-b)$.*

Proof. By the definition of Mahgoub transform, we have

$$\begin{aligned}\psi(e^{bs}F(s)) &= a\int_0^\infty e^{bs}F(s)e^{-as}ds = a\int_0^\infty F(s)e^{-(a-b)s}ds = \frac{a}{(a-b)}(a-b)\int_0^\infty F(s)e^{-(a-b)s}ds \\ &= \frac{a}{(a-b)}\psi(a-b).\end{aligned}$$

□

Lemma 2.6. *If $\psi(F(s)) = \psi(a)$, then $\psi(sF(s)) = \left[\frac{1}{a} - \frac{d}{da}\right]\psi(a)$.*

Proof. By the definition of Mahgoub transform, we have

$$\psi(F(s)) = a\int_0^\infty F(s)e^{-as}ds = \psi(a).$$

Then

$$\frac{d}{da}\psi(a) = \int_0^\infty F(s)e^{-as}ds + a\int_0^\infty (-s)F(s)e^{-as}ds = \frac{1}{a}a\int_0^\infty F(s)e^{-as}ds - a\int_0^\infty sF(s)e^{-as}ds$$

$$
\begin{aligned}
&= \frac{1}{a}\psi(a) - \psi(sF(s)), \\
\psi(sF(s)) &= \frac{1}{a}\psi(a) - \frac{d}{da}\psi(a), \\
\psi(sF(s)) &= \left[\frac{1}{a} - \frac{d}{da}\right]\psi(a).
\end{aligned}
$$

□

Definition 2.7 (Inverse Mahgoub transform)**.** If $\psi(h(t)) = H(a)$, then $h(t)$ is called the inverse Mahgoub transform of $H(a)$ and is denoted as $h(t) = \psi^{-1}(H(a))$, where ψ^{-1} is the inverse Mahgoub transform operator.

Definition 2.8. The Mitteg-Liffler function of one parameter is denoted by $E_\beta(t)$ and defined by

$$E_\beta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + 1)},$$

where $t, \beta \in C$ and $R(\beta) > 0$. If we put $\beta = 1$, then the above equation becomes

$$E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

Now, we give the different definitions of Hyers-Ulam stability and Hyers-Ulam σ-stability of the differential equations (1.1) and (1.2).

Throughout this section, consider

$$H = \{h : [0, \infty) \to \mathbb{K} | h \text{ is a continuously differentiable function of exponential order}\}.$$

Definition 2.9. (i) The linear differential equation (1.1) is said to have the Hyers-Ulam stability (for class H) if there exists a constant $K > 0$ such that the following statement is true: For every $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w''(t) + a_1 w'(t) + a_0 w(t)| \le \epsilon \tag{2.2}$$

for all $t \ge 0$, then there exists a solution $\tau : [0, \infty) \to \mathbb{K}$ of the differential equation (1.1) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \le K\epsilon$$

for all $t \ge 0$.

(ii) We say that the nonhomogeneous linear differential equation (1.2) has the Hyers-Ulam stability if there exists a constant $K > 0$ such that the following statement is true: For each $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w''(t) + a_1 w'(t) + a_0 w(t) - m(t)| \le \epsilon \tag{2.3}$$

for all $t \ge 0$, then there exists a solution $\tau : [0, \infty) \to \mathbb{K}$ of the differential equation (1.2) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \le K\epsilon,$$

for all $t \ge 0$, where the constant K is called as Hyers-Ulam constant.

Definition 2.10. Let $\sigma : [0,\infty) \to (0,\infty)$ be a function.

(i) We say that the homogeneous linear differential equation (1.1) has the Hyers-Ulam σ-stability (for the class H) if there exists a constant $K > 0$ such that the following statement is true: For every $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)| \leq \sigma(t)\epsilon \tag{2.4}$$

for all $t \geq 0$, then there exists a solution $\tau : [0,\infty) \to \mathbb{K}$ of the differential equation (1.1) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \leq K\sigma(t)\epsilon$$

for all $t \geq 0$.

(ii) We say that the nonhomogeneous linear differential equation (1.2) has the Hyers-Ulam stability if there exists a constant $K > 0$ such that the following statement is true: For each $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) - m(t)| \leq \sigma(t)\epsilon \tag{2.5}$$

for all $t \geq 0$, then there exists a solution $\tau : [0,\infty) \to \mathbb{K}$ of the differential equation (1.2) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \leq K\sigma(t)\epsilon,$$

for all $t \geq 0$, where the constant K is called as Hyers-Ulam constant.

Finally, we introduce the definitions of Mitteg-Liffler-Hyers-Ulam stability and Mitteg-Liffler-Hyers-Ulam σ-stability of the differential equations (1.1) and (1.2).

Definition 2.11. Let $E_\beta(t)$ be the Mitteg-Liffler function.

(i) We say that the differential equation (1.1) has the Mitteg-Liffler-Hyers-Ulam stability if their exists a constant $K > 0$ such that the following statement holds true: For every $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)| \leq E_\beta(t)\epsilon \tag{2.6}$$

for all $t \geq 0$, then there exists a solution $\tau : [0,\infty) \to \mathbb{K}$ of the differential equation (1.1) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \leq KE_\beta(t)\epsilon$$

for all $t \geq 0$.

(ii) We say that the nonhomogeneous linear differential equation (1.2) has the Mitteg-Liffler-Hyers-Ulam stability if there exists a constant $K > 0$ such that the following statement is true: For each $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) - m(t)| \leq E_\beta(t)\epsilon \tag{2.7}$$

for all $t \geq 0$, then there exists a solution $\tau : [0,\infty) \to \mathbb{K}$ of the differential equation (1.2) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \leq KE_\beta(t)\epsilon,$$

for all $t \geq 0$, where the constant K is called as Mitteg-Liffler-Hyers-Ulam constant.

Definition 2.12. Let $E_\beta(t)$ be the Mitteg-Liffler function.

(i) We say that the differential equation (1.1) has the Mitteg-Liffler-Hyers-Ulam σ-stability, if their exists a constant $K > 0$ such that the following statement holds true: For every $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)| \le \sigma(t)E_\beta(t)\epsilon \tag{2.8}$$

for all $t \ge 0$, then there exists a solution $\tau : [0, \infty) \to \mathbb{K}$ of the differential equation (1.1) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \le K\sigma(t)E_\beta(t)\epsilon,$$

for all $t \ge 0$.

(ii) We say that the nonhomogeneous linear differential equation (1.2) has the Mitteg-Liffler-Hyers-Ulam σ-stability if there exists a constant $K > 0$ such that the following statement is true: For each $\epsilon > 0$, if a function $w \in H$ satisfies the inequality

$$|w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) - m(t)| \le \sigma(t)E_\beta(t)\epsilon \tag{2.9}$$

for all $t \ge 0$, then there exists a solution $\tau : [0, \infty) \to \mathbb{K}$ of the differential equation (1.2) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \le K\sigma(t)E_\beta(t)\epsilon,$$

for all $t \ge 0$, where the constant K is called as Mitteg-Liffler-Hyers-Ulam σ-constant.

3. Hyers-Ulam stability of (1.1)

In this section, we prove several types of Hyers-Ulam stability of homogeneous n^{th} order linear differential equation (1.1) by using Mahgoub transform.

It should be noted that in this and the next section we investigate various types of Hyers-Ulam stability for the class H, where H is the class of all continuously differentiable functions $\tau : [0, \infty) \to \infty$ of exponential order.

For any $a \in K$, we denote the real part of a by $R(a)$.

Theorem 3.1. *Assume that $a_{n-1} + \cdots + a_2 + a_1 + a_0$ is a constant with $R(a_{n-1} + \cdots + a_2 + a_1 + a_0) > 0$. Then the homogeneous linear differential equation* (1.1) *is Hyers-Ulam stable in the class H.*

Proof. Assume that $w \in H$ satisfies the inequality (2.2) for all $t \ge 0$. Let us define a function $i : [0, \infty) \to \mathbb{K}$ by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2(t)w^{''}(t) + a_1(t)w^{'}(t) + a_0 w(t) \tag{3.1}$$

for all $t \ge 0$.

In view of (2.2), the inequality $|i(t)| \le \epsilon$ holds for all $t \ge 0$. Mahgoub transform of $i(t)$ gives the following:

$$\begin{aligned} I(a) = \psi(i(t)) &= \psi[w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)] \\ &= \psi[w^{(n)}(t)] + a_{n-1}\psi[w^{(n-1)}(t)] + \cdots + a_2\psi[w^{''}(t)] + a_1\psi[w^{'}(t)] + a_0\psi[w(t)], \end{aligned}$$

where $\Omega(a) = \psi[w(t)]$, since

$$\begin{aligned}
\psi[w^{'}(t)] &= a\psi[w(t)] - aw(0) = a\Omega(a) - aw(0),\\
\psi[w^{''}(t)] &= a^2\Omega(a) - a^2w(0) - aw'(0),\\
\psi[w^{'''}(t)] &= a^3\Omega(a) - a^3w(0) - a^2w'(0) - aw^{''}(0),\\
&\vdots\\
\psi[w^{(n-1)}(t)] &= a^{n-1}\Omega(a) - \sum_{K=0}^{n-2} a^{n-K-1}w^K(0),\\
\psi[w^{(n)}(t)] &= a^n\Omega(a) - \sum_{K=0}^{n-1} a^{n-K}w^K(0). \qquad (3.2)
\end{aligned}$$

Now,

$$\begin{aligned}
I(a) =& a^n\Omega(a) - \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\right)\\
&+ \cdots + a_2(a^2\Omega(a) - a^2w(0) - aw'(0)) + a_1(a\Omega(a) - aw(0)) + a_0\Omega(a),\\
\psi[w(t)] =& \Omega(a) = I(a) + \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\\
&+ \cdots + \frac{a_2a^2w(0) + a_2aw'(0) + a_1aw(0)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}. \qquad (3.3)
\end{aligned}$$

If we put $\tau(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}w(0)$, then $\tau(0) = w(0)$ and $\tau \in H$. Mahgoub transform of $\tau(t)$ gives the following:

$$\begin{aligned}
\psi[\tau(t)] = T(a) = &\frac{1}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0} \qquad (3.4)\\
&\left\{\sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0) + \cdots + a_2a^2w(0) + a_2aw'(0) + a_1aw(0)\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t)]\\
&= a^nT(a) - \sum_{k=0}^{n-1} a^{n-k}\tau^{(k)}(0) + a_{n-1}\left(a^{n-1}T(a) - \sum_{k=0}^{n-2} a^{n-k-1}\tau^{(k)}(0)\right)\\
&\quad + \cdots + a_2(a^2T(a) - a^2\tau(0) - a\tau'(0)) + a_1(aT(a) - a\tau(0)) + a_0T(a)\\
&= a^nT(a) - \sum_{k=0}^{n-1} a^{n-k}\tau^{(k)}(0) + a_{n-1}a^{n-1}T(a) - a_{n-1}\sum_{k=0}^{n-2} a^{n-k-1}\tau^{(k)}(0)\\
&\quad + \cdots + a_2a^2T(a) - a_2a^2\tau(0) - a_2a\tau^1(0) + a_1aT(a) - a_1a\tau(0) + a_0T(a),
\end{aligned}$$

$$a^nT(a) + a_{n-1}a^{n-1}T(a) + \cdots + a_2a^2T(a) + a_1aT(a) + a_0T(a)$$
$$= \sum_{k=0}^{n-1} a^{n-k}\tau^{(k)}(0) + a_{n-1}\sum_{k=0}^{n-2} a^{n-k-1}\tau^{(k)}(0) + \cdots + a_2a^2\tau(0) + a_2a\tau'(0) + a_1a\tau(0),$$
$$T(a)[a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0]$$
$$= \sum_{k=0}^{n-1} a^{n-k}\tau^{(k)}(0) + a_{n-1}\sum_{k=0}^{n-2} a^{n-k-1}\tau^{(k)}(0) + \cdots + a_2a^2\tau(0) + a_2a\tau'(0) + a_1a\tau(0),$$
$$T(a) = \frac{1}{[a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0]}$$
$$\left\{\sum_{k=0}^{n-1} a^{n-k}\tau^{(k)}(0) + a_{n-1}\sum_{k=0}^{n-2} a^{n-k-1}\tau^{(k)}(0) + \cdots + a_2a^2\tau(0) + a_2a\tau'(0) + a_1a\tau(0)\right\}$$

By (3.4), we have

$$\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau''(t) + a_1\tau'(t) + a_0\tau(t)] = 0.$$

Since ψ is a one-to-one operator,

$$\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau''(t) + a_1\tau'(t) + a_0\tau(t) = 0.$$

Hence $\tau(t)$ is a solution of the differential equation (1.1).

By (3.3) and (3.4), we obtain

$$\psi[w(t)] - \psi[\tau(t)] = \Omega(a) - T(a) = \frac{I(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}$$
$$= I(a)\frac{1}{a}\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}$$
$$= \frac{1}{a}I(a)L(a) = \psi[i(t) * l(t)], \quad (by\ using\ Theorem\ 2.3)$$

where $L(a) = \frac{a}{(a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0)t}$.

Here, we use the inverse Mahgoub transform method.

We know that, $\psi^{-1}\{L(a)\} = l(t)$. This implies

$$l(t) = \psi^{-1}\left\{\frac{a}{(a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0)t}\right\}.$$

So, $l(t) = \left[(a^n + a^{n-1} + \cdots + a^2 + a^1 + a^0) + (a_{n-1} + \cdots + a_2 + a_1 + a_0)\right]t$.

For this, $e^{-Vt} = \psi^{-1}\left(\frac{a}{V+a}\right)$. Thus we have

$$e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t} = \psi^{-1}\left\{\frac{a}{(a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0)t}\right\}.$$

Then

$$l(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}.$$

Consequently,

$$\psi[w(t) - \tau(t)] = \psi[i(t) * l(t)]$$

and thus

$$w(t) - \tau(t) = i(t) * l(t).$$

Taking modulus on both sides, we have

$$|w(t) - \tau(t)| = |i(t) * l(t)| = \left| \int_0^t i(s) l(t-s) ds \right| \leq \int_0^t |i(s)||l(t-s)| ds \leq \epsilon \int_0^t |l(t-s)| ds.$$

Since $l(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}$ (or) $l(t) = e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}$,

$$\begin{aligned}
|w(t) - \tau(t)| &\leq \epsilon \int_0^t e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)} ds \\
&\leq \epsilon \int_0^t e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)} ds \\
&\leq \epsilon \int_0^t e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t} e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s} ds \\
&\leq \epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t} \int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s} ds \\
&\leq \epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t} \left[\frac{e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)} \right]_0^t \\
&\leq \epsilon \left[\frac{e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s} \cdot e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)} - \frac{e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)} \right] \\
&\leq \epsilon \left[\frac{1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)} \right] \\
&\leq \frac{\epsilon}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)} \left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t} \right) \\
&\leq K\epsilon
\end{aligned}$$

for all $t \geq 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$.

This implies that the homogeneous linear differential equation (1.1) has Hyers-Ulam stability for the class H. □

We note that if $-R(a_{n-1} + \cdots + a_2 + a_1 + a_0) < 0$, then

$$\frac{\epsilon}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)} \left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t} \right)$$

diverges to infinity as t tends to infinity.

Hence, in the case of $-R(a_{n-1}+\cdots+a_2+a_1+a_0) < 0$, we notice that we cannot prove the Hyers-Ulam stability by applying the Mahgoub transform method.

Similar to Theorem 3.1, we will prove the Hyers-Ulam σ-stability for the differential equation (1.1). For the sake of the completeness of this paper, the proof is introduced here in detail.

Theorem 3.2. *Assume that* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_{n-1} + \cdots + a_2 + a_1 + a_0$ *is a constant with* $R(a_{n-1} + \cdots + a_2 + a_1 + a_0) > 0$. *Then the differential equation* (1.1) *has the Hyers-Ulam* σ*-stability for the class* H.

Proof. Assume that $w \in H$ and $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function the inequality (2.4) for all $t \geq 0$. If we define a function $i : [0,\infty) \to \mathbb{K}$ by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)$$

for all $t \geq 0$, then $|i(t)| \leq \sigma(t)\epsilon$ for all $t \geq 0$.

As we did in the first part of theorem 3.1, we can prove that $\tau(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}w(0)$ is a solution of the differential equation (1.1). Of course, $\tau \in H$. On the other hand,

$$L(a) = \frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0},$$

which gives

$$l(t) = \psi^{-1}\left[\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\right] = e^{-(a_{n-1}+\cdots+a_2+a_1+[a_0-x])t}.$$

Moreover, it follows from (3.3) and (3.4) that

$$\begin{aligned}\psi[w(t)] - \psi[\tau(t)] = \Omega(a) - T(a) &= \frac{I(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\\ &= I(a)\frac{1}{a}\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\\ &= \frac{1}{a}I(a)L(a) = \psi[i(t) * l(t)],\end{aligned}$$

which yields that

$$\psi[w(t) - \tau(t)] = \psi[i(t) * l(t)]. \tag{3.5}$$

Therefore,

$$w(t) - \tau(t) = i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}.$$

By a similar method to the proof of Theorem 3.1, we can show that

$$\begin{aligned}|w(t) - \tau(t)| = \left|i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right| &= \left|\int_0^t i(s)e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}ds\right|\\ &\leq \int_0^t |i(s)||e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}|ds\\ &\leq \sigma(t)\epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)}\int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds\\ &\leq \frac{\sigma(t)\epsilon}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)}\left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right)\\ &\leq K\sigma(t)\epsilon\end{aligned}$$

for all $t \geq 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$. □

Now, we are going to establish the Mittag-Liffler-Hyers-Ulam stability of the differential equation (1.1) by using Mahgoub transform.

Theorem 3.3. *Let $a_{n-1} + \cdots + a_2 + a_1 + a_0$ and β be constants satisfying $R(a_{n-1} + ... + a_2 + a_1 + a_0) > 0$ and $\beta > 0$. Then the homogeneous linear differential equation* (1.1) *has Mittag-Liffler-Hyers-Ulam stability for the class H.*

Proof. Assume that $w(t) \in H$ and it satisfies the inequality (2.6) for all $t \geq 0$. Let $i : [0, \infty) \to \mathbb{K}$ be a function defined by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)$$

for all $t \geq 0$.

In view of (2.6), we have $|i(t)| \leq \epsilon$ for all $t \geq 0$. Mahgoub transform of $i(t)$ gives the following:

$$\begin{aligned} I(a) = \psi(i(t)) &= \psi[w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t)] \\ &= a^n\Omega(a) - \sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ &+ \cdots + a_2(a^2\Omega(a) - a^2 w(0) - a w'(0)) + a_1(a\Omega(a) - a w(0)) + a_0\Omega(a). \end{aligned}$$

Thus we get

$$\begin{aligned} \psi[w(t)] = \Omega(t) = I(a) &+ \sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ &+ \cdots + \frac{a_2 a^2 w(0) + a_2 a w'(0) + a_1 a w(0)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0}. \end{aligned} \tag{3.6}$$

If we put $\tau(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t} w(0)$, then $\tau(0) = w(0)$ and $\tau \in H$. Mahgoub transform of $\tau(t)$ gives

$$\begin{aligned} \psi[\tau(t)] = T(t) = &\sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ &+ \cdots + \frac{a_2 a^2 w(0) + a_2 a w'(0) + a_1 a w(0)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0}. \end{aligned} \tag{3.7}$$

Thus, it follows from (3.7) that

$$\begin{aligned} &\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t)] \\ &= a^n T(a) - \sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ &+ \cdots + a_2(a^2 T(a) - a^2\tau(0) - a\tau'(0)) + a_1(aT(a) - a\tau(0)) + a_0 T(a) = 0. \end{aligned}$$

Since ψ is a one-to-one operator,

$$\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t) = 0.$$

If we set $L(a) = \frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}$, then we get

$$l(t) = \psi^{-1}\left[\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\right] = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}. \tag{3.8}$$

By (3.6) and (3.7), we obtain

$$\begin{aligned}\psi[w(t)] - \psi[\tau(t)] = \Omega(a) - T(a) &= \frac{I(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0} \\ &= I(a)\frac{1}{a}\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0} \\ &= \frac{1}{a}I(a)L(a) = \psi[i(t) * l(t)]. \end{aligned} \tag{3.9}$$

This gives $w(t) - \tau(t) = i(t) * l(t) = i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}$.

Taking modulus on both sides and using the fact that $|i(t)| \leq \epsilon E_\beta(t)$ for $t \geq 0$ and $E_\beta(t)$ is increasing for $t \geq 0$, we have

$$\begin{aligned}|w(t) - \tau(t)| &= \left|i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right| \\ &= \left|\int_0^t i(s)e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}ds\right| \\ &\leq \int_0^t |i(s)||e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}|ds \\ &\leq E_\beta(t)\epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)}\int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds \\ &\leq \frac{E_\beta(t)\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}\left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right) \\ &\leq KE_\beta(t)\epsilon\end{aligned}$$

for all $t \geq 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$.

Then, by Definition 2.11, we can confirm that the homogeneous linear differential equation (1.1) has Mittag-Liffler-Hyers-Ulam stability for the class H. □

Similar to the case of Theorem 3.3, the Mittag-Liffler-Hyers-Ulam σ-stability for the linear differential equation (1.1) can be proved.

Theorem 3.4. *Assume that* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_{n-1} + \cdots + a_2 + a_1 + a_0$ *and* β *are constants which satisfy* $R(a_{n-1} + \cdots + a_2 + a_1 + a_0) > 0$. *Then the differential equation* (1.1) *has the Mittag-Liffler-Hyers-Ulam* σ*-stability for the class* H.

Proof. Assume that $w \in H$ and $\sigma : [0,\infty) \to (0,\infty)$ is a function and that $w(t)$ and $\tau(t)$ satisfy the inequality (2.8) for all $t \geq 0$. We will prove that there exist a positive integer $K > 0$ (independent of ϵ) and a solution $\tau : [0,\infty) \to \mathbb{K}$ of the differential equation (1.1) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \leq K\sigma(t)\epsilon E_\beta(t)$$

for all $t \geq 0$.

If we define a function $i : [0, \infty) \to \mathbb{K}$ by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2w^{''}(t) + a_1w^{'}(t) + a_0w(t) \tag{3.10}$$

for all $t \geq 0$, then we have $|i(t)| \leq \sigma(t)\epsilon E_\beta(t)$ for all $t \geq 0$.

Then, by applying the same method as presented in the proof of Theorem 3.3, we can easily get

$$\begin{aligned}
|w(t) - \tau(t)| = \left|i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right| &= \left|\int_0^t i(s)e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}ds\right| \\
&\leq \int_0^t |i(s)||e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}|ds \\
&\leq \sigma(t)E_\beta(t)\epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)} \int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds \\
&\leq \frac{\sigma(t)E_\beta(t)\epsilon}{R(a_{n-1} + \cdots + a_2 + a_1 + a_0)}\left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right) \\
&\leq K\sigma(t)E_\beta(t)\epsilon
\end{aligned}$$

for all $t \geq 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$.

Then, by Definition 2.12, we can confirm that the homogeneous linear differential equation (1.1) has Mittag-Liffler-Hyers-Ulam σ-stability for the class H. □

4. Hyers-Ulam stability of (1.2)

In this section, we prove several types of Hyers-Ulam stability of the nonhomogeneous n^{th} order linear differential equation (1.2) by using Mahgoub transform.

Theorem 4.1. *Assume that $m : [0, \infty) \to \infty$ is a continuous function of exponential order and $a_{n-1} + \cdots + a_2 + a_1 + a_0$ is a constant with $R(a_{n-1} + \cdots + a_2 + a_1 + a_0) > 0$. The homogeneous linear differential equation* (1.1) *has the Hyers-Ulam stability for the class H.*

Proof. Suppose that $w \in H$ satisfies the inequality (2.3) for all $t \geq 0$. Consider the function $i : [0, \infty) \to \mathbb{K}$ by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2w^{''}(t) + a_1w^{'}(t) + a_0(t)w(t) - m(t) \tag{4.1}$$

for all $t \geq 0$.

Then it holds that $|i(t)| \leq \epsilon$ holds for all $t \geq 0$. Mahgoub transform of $i(t)$ gives the following:

$$I(a) = \psi(i(t)) = \psi[w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2w^{''}(t) + a_1w^{'}(t) + a_0w(t) - m(t)]. \tag{4.2}$$

This implies that

$$\begin{aligned}
\psi[w(t)] = \Omega(t) = I(a) + \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\right) \\
+ \cdots + \frac{a_2a^2w(0) + a_2aw'(0) + a_1aw(0) - M(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}.
\end{aligned} \tag{4.3}$$

If we set

$$\tau(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}w(0) + \left(m(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right),$$

then $\tau(0) = w(0)$ and $\tau \in H$. Mahgoub transform of $\tau(t)$ gives the following:

$$\begin{aligned}\psi[\tau(t)] = T(a) = \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\right)\\ + \cdots + \frac{a_2a^2w(0) + a_2aw'(0) + a_1aw(0) + M(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}.\end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned}&\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau''(t) + a_1\tau'(t) + a_0\tau(t)]\\ &= a^nT(a) - \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}T(a) - \sum_{k=0}^{n-2} a^{n-k-1}\tau^{(k)}(0)\right)\\ &+ \cdots + a_2(a^2T(a) - a^2\tau(0) - a\tau'(0)) + a_1(aT(a) - a\tau(0)) + a_0T(a).\end{aligned}$$

By (4.4), we have

$$\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau''(t) + a_1\tau'(t) + a_0\tau(t)] = M(a) = \psi[m(t)]$$

and thus

$$\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau''(t) + a_1\tau'(t) + a_0\tau(t) = m(t). \tag{4.5}$$

Hence $\tau(t)$ is a solution of the differential equation (1.1).

In addition, by applying (4.3) and (4.4), we obtain

$$\begin{aligned}\psi[w(t)] - \psi[\tau(t)] = \Omega(a) - T(a) &= \frac{I(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\\ &= I(a)\frac{1}{a}\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\\ &= \frac{1}{a}I(a)L(a) = \psi[i(t) * l(t)],\end{aligned}$$

where $L(a) = \frac{a}{a^n+a_{n-1}a^{n-1}+\cdots+a_2a^2+a_1a+a_0}$. This gives

$$l(t) = \psi^{-1}\left[\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\right] = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}.$$

Therefore, we have

$$\psi[w(t) - \tau(t)] = \psi[i(t) * l(t)] = \psi[i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}],$$

which yields

$$w(t) - \tau(t) = i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}. \tag{4.6}$$

Furthermore,

$$\begin{aligned}|w(t)-\tau(t)| &= \left|\int_0^t i(s)l(t-s)ds\right| \le \int_0^t |i(s)||l(t-s)|ds \le \epsilon\int_0^t |l(t-s)|ds\\ &\le \epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+[a_0-x])(t)}\int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds\\ &\le \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}\left(1-e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right)\\ &\le K\epsilon\end{aligned}$$

for all $t \ge 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$. □

For the Hyers-Ulam σ-stability of the nonhomogeneous linear differential equation (1.2), we obtain the following theorem.

Theorem 4.2. *Assume that $m : [0,\infty) \to (0,\infty)$ is an continuous function of exponential order and $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function and that $a_{n-1}+\cdots+a_2+a_1+a_0$ is a constant with $R(a_{n-1}+\cdots+a_2+a_1+a_0) > 0$. Then the differential equation* (1.2) *has the Hyers-Ulam σ-stability for the class H.*

Proof. We consider an arbitrary function $w \in H$ that satisfies the inequality (2.5) for all $t \ge 0$. Now, we define a function $i : [0,\infty) \to \mathbb{K}$ by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) - m(t)$$

for all $t \ge 0$. Then $|i(t)| \le \sigma(t)\epsilon$ for all $t \ge 0$.

It is not difficult to check that

$$\begin{aligned}\psi[w(t)] = \Omega(t) = I(a) + \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\right)\\ + \cdots + \frac{a_2a^2w(0) + a_2aw'(0) + a_1aw(0) + M(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}.\end{aligned} \tag{4.7}$$

If we set

$$\tau(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}w(0) + \left(m(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right),$$

then $\tau(0) = w(0)$ and $\tau \in H$. Further, we apply the Mahgoub transform on both sides to get

$$\begin{aligned}\psi[\tau(t)] = T(a) = \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\right)\\ + \cdots + \frac{a_2a^2w(0) + a_2aw'(0) + a_1aw(0) = M(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}.\end{aligned} \tag{4.8}$$

On the other hand,

$$\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t)]$$

$$= a^nT(a) - \sum_{k=0}^{n-1} a^{n-k}w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1}w^{(k)}(0)\right)$$
$$+ \cdots + a_2(a^2T(a) - a^2\tau(0) - a\tau'(0)) + a_1(aT(a) - a\tau(0)) + a_0T(a).$$

The relation (4.8) implies that

$$\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t)] = M(a) = \psi[m(t)]$$

and thus

$$\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t) = m(t).$$

That is, $\tau(t)$ is a solution of the differential equation (1.2). Using (4.7) and (4.8), we obtain

$$\psi[w(t)] - \psi[\tau(t)] = \Omega(a) - T(a) = \frac{I(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}$$
$$= I(a)\frac{1}{a}\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}$$
$$= \frac{1}{a}I(a)L(a) = \psi[i(t) * l(t)],$$

where $L(a) = \frac{a}{a^n+a_{n-1}a^{n-1}+\cdots+a_2a^2+a_1a+a_0}$. This gives

$$l(t) = \psi^{-1}\left[\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\right] = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}.$$

Therefore, we have

$$\psi[w(t) - \tau(t)] = \psi[i(t) * l(t)],$$

which gives

$$w(t) - \tau(t) = i(t) * l(t).$$

Similar to the proof of Theorem 3.2, we have

$$|w(t) - \tau(t)| = |i(t) * l(t)| = \left|\int_0^t i(s)l(t-s)ds\right| \le \int_0^t |i(s)||l(t-s)|ds \le \epsilon\int_0^t |l(t-s)|ds$$
$$\le \sigma(t)\epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)}\int_0^t e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds$$
$$\le K\sigma(t)\epsilon$$

for all $t \ge 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$. □

Now, we prove the Mittag-Liffler-Hyers-Ulam stability of the nonhomogeneous linear differential equation (1.1) by using Mahgoub transform method.

Theorem 4.3. *Assume that $m : [0,\infty) \to (0,\infty)$ is an continuous function of exponential order and that $a_{n-1}+\cdots+a_2+a_1+a_0$ and β are constants satisfying $R(a_{n-1}+\cdots+a_2+a_1+a_0) > 0$ and $\beta > 0$. Then the nonhomogeneous linear differential equation* (1.2) *has Mittag-Liffler-Hyers-Ulam stability for the class H.*

Proof. Suppose that $w \in H$ and $w(t)$ satisfies the inequality (2.7) for alll $t \geq 0$. Consider a function $i : [0, \infty) \to \mathbb{K}$ defined by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) - m(t)$$

for all $t \geq 0$.

It follows from (2.7) that $|i(t)| \leq E_\beta(t)\epsilon$ for all $t \geq 0$. Mahgoub transform of $i(t)$ gives the following:

$$I(a) = \psi(i(t)) = \psi[w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2 w^{''}(t) + a_1 w^{'}(t) + a_0 w(t) - m(t)].$$

That is,

$$\begin{aligned}\psi[w(t)] = \Omega(t) = I(a) + \sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ + \cdots + \frac{a_2 a^2 w(0) + a_2 a w'(0) + a_1 a w(0) + M(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0}. \end{aligned} \tag{4.9}$$

If we set $\tau(t) = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t} w(0) + \left(m(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right)$, then $\tau(0) = w(0)$ and $\tau \in H$. We apply the Mahgoub transform on both sides of the last equality to get

$$\begin{aligned}\psi[\tau(t)] = T(a) = \sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}\Omega(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ + \cdots + \frac{a_2 a^2 w(0) + a_2 a w'(0) + a_1 a w(0) + M(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0}. \end{aligned} \tag{4.10}$$

On the other hand

$$\begin{aligned}&\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t)] \\ &= a^n T(a) - \sum_{k=0}^{n-1} a^{n-k} w^{(k)}(0) + a_{n-1}\left(a^{n-1}T(a) - \sum_{k=0}^{n-2} a^{n-k-1} w^{(k)}(0)\right) \\ &+ \cdots + a_2(a^2 T(a) - a^2\tau(0) - a\tau'(0)) + a_1(aT(a) - a\tau(0)) + a_0 T(a) = 0.\end{aligned}$$

Then by (4.10), we have

$$\psi[\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t)] = M(a) = \psi[m(t)]$$

and thus

$$\tau^{(n)}(t) + a_{n-1}\tau^{(n-1)}(t) + \cdots + a_2\tau^{''}(t) + a_1\tau^{'}(t) + a_0\tau(t) = m(t).$$

Hence $\tau(t)$ is a solution of the differential equation (1.2). In addition, by applying (3.6) and (4.10), we can obtain

$$\psi[w(t)] - \psi[\tau(t)] = \Omega(a) - T(a) = \frac{I(a)}{a^n + a_{n-1}a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0}$$

$$= I(a)\frac{1}{a}\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}$$
$$= \frac{1}{a}I(a)L(a) = \psi[i(t) * l(t)], \tag{4.11}$$

where $L(a) = \frac{a}{a^n+a_{n-1}a^{n-1}+\cdots+a_2a^2+a_1a+a_0}$. This gives

$$l(t) = \psi^{-1}\left[\frac{a}{a^n + a_{n-1}a^{n-1} + \cdots + a_2a^2 + a_1a + a_0}\right] = e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}. \tag{4.12}$$

Therefore, we have $\psi[w(t)] - \psi[\tau(t)] = \psi[i(t) * l(t)]$, which yields $w(t) - \tau(t) = i(t) * l(t)$ for all $t \geq 0$.

Furthermore,

$$|w(t) - \tau(t)| = \left|i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right| = \left|\int_0^t i(s)e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}ds\right|$$
$$\leq \int_0^t |i(s)||e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}|ds$$
$$\leq E_\beta(t)\epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)}\int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds$$
$$\leq \frac{E_\beta(t)\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}\left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right)$$
$$\leq KE_\beta(t)\epsilon$$

for all $t \geq 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$. This completes the proof. □

Similar to the case of Theorem 4.3, the Mittag-Liffler-Hyers-Ulam σ-stability for the linear differential equation (1.2) can be proved.

Theorem 4.4. *Assume that* $m : [0,\infty) \to (0,\infty)$ *is an continuous function of exponential order and that* $\sigma : [0,\infty) \to (0,\infty)$ *is an increasing function and* $a_{n-1}+\cdots+a_2+a_1+a_0$ *and* β *are constants which satisfy* $R(a_{n-1}+\cdots+a_2+a_1+a_0) > 0$. *Then the nonhomogeneous linear differential equation* (1.2) *has the Mittag-Liffler-Hyers-Ulam* σ*-stability for the class* H.

Proof. Assume that $w \in H$ satisfies the inequality (2.9) for all $t \geq 0$. It is easy to prove that there exist a constant $K > 0$ (independent of ϵ) and a solution $\tau : [0,\infty) \to \mathbb{K}$ of the differential equation (1.2) such that $\tau \in H$ and

$$|w(t) - \tau(t)| \leq K\sigma(t)\epsilon E_\beta(t),$$

for all $t \geq 0$.

If we define a function $i : [0,\infty) \to \mathbb{K}$ by

$$i(t) = w^{(n)}(t) + a_{n-1}w^{(n-1)}(t) + \cdots + a_2w^{''}(t) + a_1w^{'}(t) + a_0w(t) - m(t)$$

for all $t \geq 0$, then we have $|i(t)| \leq \sigma(t)\epsilon E_\beta(t)$ for all $t \geq 0$.

By applying a similar method as in the proof of Theorem 4.3, we can easily prove that there exists a solution $\tau : [0,\infty) \to \mathbb{K}$ of (1.2) satisfying $\tau \in H$ and

$$|w(t) - \tau(t)| = \left|i(t) * e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right| = \left|\int_0^t i(s)e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}ds\right|$$

$$\leq \int_0^t |i(s)||e^{-(a_{n-1}+\cdots+a_2+a_1+a_0)(t-s)}|ds$$
$$\leq \sigma(t)E_\beta(t)\epsilon e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)} \int_0^t e^{R(a_{n-1}+\cdots+a_2+a_1+a_0)s}ds$$
$$\leq \frac{\sigma(t)E_\beta(t)\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}\left(1 - e^{-R(a_{n-1}+\cdots+a_2+a_1+a_0)t}\right)$$
$$\leq K\sigma(t)E_\beta(t)\epsilon$$

for all $t \geq 0$, where $K = \frac{\epsilon}{R(a_{n-1}+\cdots+a_2+a_1+a_0)}$. This completes the proof. □

5. Conclusions

In this paper, we demonstrated the Hyers-Ulam stability, Hyers-Ulam σ-stability, Mittag-Leffler-Hyers-Ulam stability, and Mittag-Leffler-Hyers-Ulam σ-stability of the linear differential equations of n^{th}-order with constant coefficients using Mahgoub transform method. All in all, we set up adequate models for the Hyers-Ulam stability of n^{th}-order linear differential equations with steady coefficients utilizing the Mahgoub transform method. Additionally, this paper gives another technique to research the Hyers-Ulam stability of differential equations. This is the primary endeavor to utilize the Mahgoub transform to demonstrate the Hyers-Ulam stability for linear differential equations of the n^{th}-order. Besides, this paper shows that the Mahgoub transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients.

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Conflict of interest

The authors declare that they have no competing interests.

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