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Research article

Regularity results of solutions to elliptic equations involving mixed local and nonlocal operators

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Abstract: In this paper, we study the summability of solutions to the following semilinear elliptic equations involving mixed local and nonlocal operators

$$\begin{cases} -\Delta u(x) + (-\Delta)^s u(x) = f(x), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $(-\Delta)^s$ is the fractional Laplace operator, f is a measurable function.

Keywords: fractional elliptic equations; mixed local and nonlocal operators; summability
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1. Introduction

The main aim of this paper is to investigate summability of the solutions to the following semilinear elliptic equations

$$\begin{cases} -\Delta u(x) + (-\Delta)^s u(x) = f(x), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary and $N > 2s, 0 < s < 1$, the data f is a nonnegative function that belongs to a suitable Lebesgue space. $(-\Delta)^s$ is defined by the following formula

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad u \in \mathcal{S}(\mathbb{R}^N),$$

where

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+2s}} d\xi \right)^{-1} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}.$$

During the last years, a lot of mathematical efforts have been devoted to the study of the fractional Laplacian, which can be used to describe many phenomena in life, such as financial mathematics, signal control processing, image processing, seismic analysis [2, 9, 10, 13, 22] and so on. Leonori et al. [21] established an L^p-theory to a family of integro-differential operators related to the fractional Laplacian

$$\begin{cases} \mathcal{L}u(x) = f(x), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $\mathcal{L}$ is integral operator with kernels functions $\mathcal{K}(x, y)$. It is worth pointing out that they established L^∞ estimates for solutions to problem (1.2) with $f \in L^m(\Omega)$, $m > \frac{N}{2s}$ by Moser and Stampacchia methods respectively, see Proposition 9 of [23] also. Dipierro et al. [18] obtained an L^∞ estimate for the solutions to some general kind of subcritical and critical problems in $\mathbb{R}^N$. Barrios et al. [3] extended the result of [21] to the following fractional p-Laplacian Dirichlet problem

$$\begin{cases} (-\Delta)_p^s u(x) = f(x), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

Abdellaoui et al. [1] obtained existence and summability of solutions to the following nonlocal nonlinear problem with Hardy potential

$$\begin{cases} (-\Delta)^s u(x) - \lambda \frac{u(x)}{|x|^{2s}} = f(x), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

Recently, an increasing attention has been focused on the study of the elliptic operators involving mixed local and nonlocal operators, which arise naturally in plasma physics [8] and population dynamics [17]. Biagi et al. [7] proved a radial symmetry result for the following elliptic equation by the moving planes method,

$$\begin{cases} -\Delta u(x) + (-\Delta)^s u(x) = f(u(x)), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function, $\Omega \subset \mathbb{R}^N$ is an open and bounded set with C^1 boundary, symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$. Biagi et al. [4] investigated the existence, maximum principles, interior Sobolev regularity and boundary regularity of solutions to problem (1.1). Dipierro et al. [16] discussed the spectral properties of mixed local and nonlocal equation under suitable Neumann conditions. For some other related results of mixed local and nonlocal equation, see [5,6,11,12,14,17,19,20] and the references therein.

The purpose of this paper is to study the summability of solutions to problem (1.1). The main results of this paper are the following theorems.

Theorem 1.1. *Suppose that $f \in L^m(\Omega)$ with $m > \frac{N}{s+1}$. Then there exists a constant $K > 0$, depending on N, Ω, s, $\|u\|_{H_0^1(\Omega)}$, $\|f\|_{L^m(\Omega)}$, such that any solutions to problem* (1.1) *satisfy*

$$\|u\|_{L^\infty(\Omega)} \leq K. \tag{1.5}$$

Remark 1.2. According to [24], we know that solutions to the following equations belong to $L^\infty(\Omega)$ if $f \in L^m(\Omega)$ with $m > \frac{N}{2}$,

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

While it is well known that for fractional elliptic equation [3,23],

$$\begin{cases} (-\Delta)^s u(x) = f(x), & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

$u \in L^\infty(\Omega)$ if $f \in L^m(\Omega)$ with $m > \frac{N}{2s}$. Theorem 1.1 shows that solutions to problem (1.1) are bounded if $f \in L^m(\Omega)$ with $m > \frac{N}{s+1}$. Note that for $0 < s < 1$,

$$\frac{N}{2} < \frac{N}{s+1} < \frac{N}{2s}. \tag{1.6}$$

Furthermore, according to Proposition 4.4 of [15], we know that

$$\lim_{s \to 0^+} (-\Delta)^s u = u.$$

Unfortunately, at least formally, (1.6) shows that the limit of $\Delta + (-\Delta)^s$ is not the operator $\Delta + I$ as $s \to 0^+$. In a forthcoming work, we consider the limiting behavior of solutions to boundary value nonlinear problem (1.1) when the parameter s tends to zero.

Theorem 1.3. *Suppose that $f \in L^m(\Omega)$ with*

$$1 < m < \frac{N}{s+1}. \tag{1.7}$$

Then, there exists a constant $c = c(N, m, s) > 0$, such that any solutions to problem (1.1) *satisfy*

$$\|u\|_{L^{m^{**}}(\Omega)} \leq c\|f\|_{L^m(\Omega)}, \tag{1.8}$$

where

$$m^{**} = \frac{mN(N-2s)}{(N-2)(N-2ms)}. \tag{1.9}$$

Remark 1.4. Obviously, m^{**} is monotone increasing in s and

$$\lim_{s\to 1^-} m^{**} = \frac{mN}{N-2m}.$$

It is interesting to note that

$$\frac{mN}{N-2ms} < m^{**} < \frac{mN}{N-2m}, \tag{1.10}$$

which shows that the exponent in (1.9) better than the one coming from the fractional Laplace $(-\Delta)^s$ only, while which worse that the one coming from the Laplace operator $-\Delta$ only.

The surprising character of Theorem 1.3 lies mainly in the fact that, the mixed local and nonlocal operators has its own features, one can not consider the fractional Laplacian as a lower order perturbation only of the classical elliptic problem.

The paper is organized as follows. In Section 2 we present the relevant definitions and lemmas. Section 3 is devoted to the proof of Theorem 1.1 and Section 4 contains the proof of Theorem 1.3.

2. Preliminaries

The definition of solution in this paper is defined as

Definition 2.1. A function $u \in H_0^1(\mathbb{R}^N)$ is a weak solution to (1.1), if for every test function $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla \phi dx + \iint_{\mathcal{D}(\Omega)} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dxdy = \int_\Omega f\phi dx,$$

where

$$\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega).$$

Here, we also need the Sobolev embedding theorem. Suppose that for $s \in (0,1)$ and $N > 2s$, there exists a constant $\mathcal{S} = \mathcal{S}(N,s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$,

$$||u||^2_{L^{2_s^*}} \le \mathcal{S} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy,$$

where $2_s^* = \frac{2N}{N-2s}$.

In the proof of main theorem, we need some base results of [21]. For any $k \ge 0$, define

$$T_k(u) = \max\{-k, \min\{k,u\}\}, \quad G_k(u) = u(x) - T_k(u).$$

Lemma 2.2 (Lemma 4 of [21])**.** *Let $u(x)$ be a positive measurable function in $\mathbb{R}^N$. Then for any $k \ge 0$,*

$$[T_k(u(x)) - T_k(u(y))][G_k(u(x)) - G_k(u(y))] \ge 0 \ \ a.e \ in \ \mathcal{D}(\Omega).$$

Lemma 2.3 (Proposition 3 of [21])**.** *Let v be a function in $H_0^s(\Omega)$. For any $k \geq 0$, we have*

$$\lambda \|G_k(v)\|^2_{H_0^s(\Omega)} \leq \int_\Omega G_k(v)(-\Delta)^s v dx,$$

and

$$\lambda \|T_k(v)\|^2_{H_0^s(\Omega)} \leq \int_\Omega T_k(v)(-\Delta)^s v dx.$$

Lemma 2.4 (Theorem 16 of [21])**.** *Let f be a positive function that belong to $L^m(\Omega)$ with $\frac{2N}{N+2S} \leq m < \frac{N}{2s}$. Then, there exists a constant $c = c(N, m, s) > 0$ such that the unique energy solution to* (1.2) *satisfies*

$$\|u\|_{L^{m_s^{**}}(\Omega)} \leq c \|f\|_{L^m(\Omega)},$$

where

$$m_s^{**} = \frac{mN}{N - 2ms}.$$

The following numerical iteration result is important in proving the boundedness results.

Lemma 2.5 (Lemma 4.1 in [24])**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that*

$$\psi(h) \leq \frac{M\psi(k)^\delta}{(h-k)^\gamma}, \qquad \forall h > k > 0,$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where $d^\gamma = M\psi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}$.

3. Proof of main results

The main tool for the proof of Theorem 1.1 is Stampacchia method.

Proof. For any $k > 0$, taking $G_k(u)$ as text function in the definition of weak solution, we have

$$\begin{aligned}&\int_\Omega \nabla u(x) \cdot \nabla G_k u(x) dx + \iint_{\mathcal{D}(\Omega)} \frac{[u(x) - u(y)][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+2s}} dxdy \\ &= \int_\Omega f(x) G_k u(x) dx,\end{aligned} \tag{3.1}$$

where $\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega)$.

Obviously, by $u(x) = T_k(u(x)) + G_k(u(x))$, we get

$$\begin{aligned}&\iint_{\mathcal{D}(\Omega)} \frac{[u(x) - u(y)][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+2s}} dxdy \\ &= \iint_{\mathcal{D}(\Omega)} \frac{[T_k(u(x)) + G_k(u(x)) - T_k(u(y)) - G_k(u(y))][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+2s}} dxdy\end{aligned}$$

$$= \iint_{\mathcal{D}(\Omega)} \frac{[T_k(u(x)) - T_k(u(y))][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+2s}} dxdy$$
$$+ \iint_{\mathcal{D}(\Omega)} \frac{|G_k(u(x)) - G_k(u(y))|^2}{|x-y|^{N+2s}} dxdy. \tag{3.2}$$

According to Lemma 2.2, we have

$$[T_k(u(x)) - T_k(u(y))][G_k(u(x)) - G_k(u(y))] \geq 0, \text{a.e}, (x,y) \in \mathcal{D}(\Omega), \tag{3.3}$$

which, together with (3.2), implies that

$$\iint_{\mathcal{D}(\Omega)} \frac{|G_k(u(x)) - G_k(u(y))|^2}{|x-y|^{N+2s}} dxdy$$
$$\leq \iint_{\mathcal{D}(\Omega)} \frac{[u(x) - u(y)][G_k(u(x)) - G_k(u(y))]}{|x-y|^{N+2s}} dxdy. \tag{3.4}$$

Note that $\int_\Omega \nabla u(x) \cdot \nabla G_k u(x) dx \geq 0$, this fact, combine Sobolev's embedding theorem, (3.1) with (3.4), leads to

$$\begin{aligned} &\|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \\ \leq &\mathcal{S} \iint_{\mathcal{D}(\Omega)} \frac{|G_k(u(x)) - G_k(u(y))|^2}{|x-y|^{N+2s}} dxdy \\ \leq &\mathcal{S} \int_{A_k} f(x) G_k(u(x)) dx \\ \leq &\mathcal{S} \|f\|_{L^m(A_k)} \|G_k(u)\|_{L^{2^*_s}(A_k)} |A_k|^{1-\frac{1}{m}-\frac{1}{2^*_s}}, \end{aligned} \tag{3.5}$$

where $A_k = \{x \in \Omega : u(x) \geq k\}$ and $2^*_s = \frac{2N}{N-2s}$. Here we have used the Hölder inequality and the fact that $G_k(u(x)) = 0$, $x \in \Omega \setminus A_k$. Therefore,

$$\|G_k(u)\|_{L^{2^*_s}(\Omega)} \leq \mathcal{S} \|f\|_{L^m(A_k)} |A_k|^{1-\frac{1}{m}-\frac{1}{2^*_s}}, \tag{3.6}$$

On the other hand, by $\nabla u(x) = \nabla G_k(u(x))$ for $x \in A_k$, we find

$$\int_\Omega \nabla u \cdot \nabla G_k(u) dx = \int_{A_k} |\nabla G_k(u)|^2 dx. \tag{3.7}$$

This fact, combine with the Sobolev embedding theorem, (3.1), (3.7) and Lemma 2.3, leads to

$$\begin{aligned} \|G_k(u)\|^2_{L^{2^*}(\Omega)} &\leq \int_{A_k} |\nabla G_k(u)|^2 dx \\ &\leq \int_{A_k} f(u) G_k(u) dx \\ &\leq \|f\|_{L^m(A_k)} \|G_k(u)\|_{L^{2^s_*}(A_k)} |A_k|^{1-\frac{1}{m}-\frac{1}{2^s_*}}. \end{aligned} \tag{3.8}$$

where $2^* = \frac{2N}{N-2}$. Now combine (3.6) with (3.8), we have

$$\|G_k(u)\|^2_{L^{2^*}(\Omega)}$$

$$\le \|f\|_{L^m(A_k)} \|G_k(u)\|_{L^{2_*^s}(A_k)} |A_k|^{1-\frac{1}{m}-\frac{1}{2_*^s}}$$

$$\le \|f\|^2_{L^m(\Omega)} |A_k|^{2(1-\frac{1}{m}-\frac{1}{2_s^*})} \tag{3.9}$$

For every $h > k$ we know that $A_h \subset A_k$ and $|G_k(u(x))|\chi_{A_h(x)} \ge (h-k)$ in Ω, we have that

$$\begin{aligned}(h-k)|A_h|^{\frac{1}{2^*}} &\le \left(\int_{A_h} |G_k(u)|^{2^*}\right)^{\frac{1}{2^*}} \\ &\le \|G_k(u)\|_{L^{2^*}(\Omega)} \\ &\le \|f\|_{L^m(\Omega)} |A_k|^{(1-\frac{1}{m}-\frac{1}{2_s^*})}.\end{aligned} \tag{3.10}$$

Therefore

$$|A_h| \le \frac{\|f\|^{2^*}_{L^m(\Omega)} |A_k|^{2^*(1-\frac{1}{m}-\frac{1}{2_s^*})}}{(h-k)^{2^*}}. \tag{3.11}$$

Note that

$$2^*(1-\frac{1}{m}-\frac{1}{2_s^*}) > 1 \tag{3.12}$$

if $m > \frac{N}{s+1}$. Finally, we apply the Lemma 2.5 with the choice $\psi(u) = |A_u|$, hence there exists k_0 such that $\psi(k) \equiv 0$ for any $k \ge k_0$ and thus ess $\sup_\Omega u \le k_0$. □

4. Proof of Theorem 1.3

The main tools for the proof of Theorem 1.3 are Calderón-Zygmund theory and Sobolev embedding theorem. The proof is divided into two parts.

Proof. Define

$$\Phi(\sigma) = \begin{cases} \sigma^\beta, & 0 \le \sigma \le T, \\ \beta T^{\beta-1}(\sigma - T) + T^\beta, & \sigma > T, \end{cases} \tag{4.1}$$

where $\beta = \frac{N(m-1)}{N-2ms} > 1$. Taking $\Phi(u)$ as text function in the definition of weak solution to (1.1), we have

$$\begin{aligned}&\int_\Omega \nabla u \cdot \nabla \Phi(u)dx + \iint_{\mathbb{R}^{2N}} \frac{[u(x)-u(y)][\Phi(u(x))-\Phi(u(y))]}{|x-y|^{N+2s}} dxdy \\ &= \int_\Omega f\Phi(u)dx.\end{aligned} \tag{4.2}$$

Firstly, we consider $\int_\Omega \nabla u \cdot \nabla \Phi(u)dx$. It is esaily to see that

$$\int_\Omega \nabla u \cdot \nabla \Phi(u)dx = \int_{\Omega\cup\{u>T\}} \nabla u \cdot \nabla \Phi(u)dx + \int_{\Omega\cup\{0\le u\le T\}} \nabla u \cdot \nabla \Phi(u)dx$$

$$=\beta T^{\beta-1}\int_{\Omega\cup\{u>T\}}|\nabla u|^2dx+\int_{\Omega\cup\{0\le u\le T\}}\nabla u\cdot\nabla u^\beta dx$$
$$\ge 0. \tag{4.3}$$

Using (4.2) and (4.3), we get

$$\iint_{\mathbb{R}^{2N}}\frac{[u(x)-u(y)][\Phi(u(x))-\Phi(u(y))]}{|x-y|^{N+2s}}dxdy\le\int_\Omega f\Phi(u)dx. \tag{4.4}$$

Similar to the proof of Lemma 2.4, we know that

$$\|u\|_{L^{m_s^{**}}(\Omega)}\le c\,\|f\|_{L^m(\Omega)}, \tag{4.5}$$

where c depends, on λ, S, s, N, m and Ω, $m_s^{**}=\frac{mN}{N-2ms}$.

Secondly, we show that

$$\iint_{\mathbb{R}^{2N}}\frac{[u(x)-u(y)][\Phi(u(x))-\Phi(u(y))]}{|x-y|^{N+2s}}dxdy\ge 0. \tag{4.6}$$

In fact, decompose $\mathbb{R}^N$ as

$$\mathbb{R}^N=\{x\in\mathbb{R}^N:u(x)>T\}\cup\{x\in\mathbb{R}^N:0\le u(x)\le T\}.$$

Denote

$$\begin{aligned}\Omega_1&=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N:u(x)>T,u(y)>T\},\\ \Omega_2&=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N:u(x)>T,0\le u(y)\le T\},\\ \Omega_3&=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N:0\le u(x)\le T,u(y)>T\},\\ \Omega_4&=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N:0\le u(x)\le T,0\le u(x)\le T\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{[u(x)-u(y)][\Phi(u(x))-\Phi(u(y))]}{|x-y|^{N+2s}}dxdy\\ =&\left(\iint_{\Omega_1}+\iint_{\Omega_2}+\iint_{\Omega_3}+\iint_{\Omega_4}\right)\frac{[u(x)-u(y)][\Phi(u(x))-\Phi(u(y))]}{|x-y|^{N+2s}}dxdy\\ :=&I_1+I_2+I_3+I_4.\end{aligned} \tag{4.7}$$

Firstly, we consider I_1. By the definition of Φ, which given by (4.1), we find, for $(x,y)\in\Omega_1$,

$$\Phi(u(x))=\beta T^{\beta-1}(u(x)-T)+T^\beta,\ \ \Phi(u(y))=\beta T^{\beta-1}(u(y)-T)+T^\beta,$$

which implies that $\Phi(u(x))-\Phi(u(y))=\beta T^{\beta-1}[u(x)-u(y)]$. Therefore

$$\begin{aligned}I_1=&\iint_{\Omega_1}\frac{[u(x)-u(y)][\Phi(u(x))-\Phi(u(y))]}{|x-y|^{N+2s}}dxdy\\ =&\beta T^{\beta-1}\iint_{\Omega_1}\frac{[u(x)-u(y)]^2}{|x-y|^{N+2s}}dxdy\\ \ge&0.\end{aligned} \tag{4.8}$$

For I_2, it is obvious that, for $(x, y) \in \Omega_2$,

$$\Phi(u(x)) = \beta T^{\beta-1}(u(x) - T) + T^\beta, \ \ \Phi(u(y)) = u(y)^\beta \le T^\beta.$$

Thus, for $(x, y) \in \Omega_2$, $u(x) \ge u(y)$ and

$$\Phi(u(x)) - \Phi(u(y)) = \beta T^{\beta-1}(u(x) - T) + (T^\beta - u^\beta(y)) \ge 0.$$

This fact gives that

$$\begin{aligned} I_2 =& \iint_{\Omega_2} \frac{[u(x) - u(y)][\Phi(u(x)) - \Phi(u(y))]}{|x - y|^{N+2s}} dxdy \\ =& \iint_{\Omega_2} \frac{[u(x) - u(y)][\beta T^{\beta-1}(u(x) - T) + (T^\beta - u^\beta(y))]}{|x - y|^{N+2s}} dxdy \\ \ge& 0. \end{aligned} \tag{4.9}$$

For I_3, it is easy to check that, for $(x, y) \in \Omega_3$,

$$\Phi(u(x)) = u(x)^\beta, \ \ \Phi(u(y)) = \beta T^{\beta-1}(u(y) - T) + T^\beta,$$

and

$$\Phi(u(x)) - \Phi(u(y)) = u(x)^\beta - T^\beta - \beta T^{\beta-1}(u(y) - T) \le 0.$$

Consequently

$$\begin{aligned} I_3 =& \iint_{\Omega_3} \frac{[u(x) - u(y)][\Phi(u(x)) - \Phi(u(y))]}{|x - y|^{N+2s}} dxdy \\ =& \iint_{\Omega_3} \frac{[u(x) - u(y)][(u^\beta(x) - T^\beta) - (\beta T^{\beta-1}(u(x) - T))]}{|x - y|^{N+2s}} dxdy \\ \ge& 0, \end{aligned} \tag{4.10}$$

here we use that fact that $u(x) - u(y) \le 0$ for $(x, y) \in \Omega_3$.

For I_4, obviously, for $(x, y) \in \Omega_4$,

$$\Phi(u(x)) - \Phi(u(y)) = u(x)^\beta - u(y)^\beta.$$

This fact, together with the monotonicity of t^β, leads to

$$\begin{aligned} I_4 =& \iint_{\Omega_4} \frac{[u(x) - u(y)][\Phi(u(x)) - \Phi(u(y))]}{|x - y|^{N+2s}} dxdy \\ =& \iint_{\Omega_4} \frac{[u(x) - u(y)][u^\beta(x) - u^\beta(y)]}{|x - y|^{N+2s}} dxdy \\ \ge& 0. \end{aligned} \tag{4.11}$$

Using (4.8)–(4.11), we derive that (4.6) holds.

According to (4.2) and (4.6), we have

$$\int_\Omega \nabla u \cdot \nabla \Phi(u) dx \leq \int_\Omega f \Phi(u) dx. \tag{4.12}$$

For large $T > 0$, by (4.1) we know that $\Phi(u) = u^\beta$ if $0 \leq u(x) \leq T$. Thus (4.12), together with the Hölder inequalities, yields

$$\frac{4}{(\beta+1)^2} \int_\Omega |\nabla u^{\frac{\beta+1}{2}}|^2 dx \leq \int_\Omega f u^\beta dx \leq \|f\|_{L^m(\Omega)} \|u\|^\beta_{L^{m_s^{**}}(\Omega)},$$

where

$$\frac{1}{m} + \frac{\beta}{m_s^{**}} = 1, \quad \frac{2^*}{2}(\beta+1) > 1, \quad m_s^{**} = \frac{mN}{N-2ms}.$$

Therefore, using (4.5), we get

$$\|u^{\frac{\beta+1}{2}}\|^2_{L^{2^*}(\Omega)} \leq c\|f\|_{L^m(\Omega)} \|u\|^\beta_{L^{m_s^{**}}(\Omega)} \leq c\|f\|^{\beta+1}_{L^m(\Omega)}.$$

This fact implies that

$$\|u\|_{L^{m^{**}}(\Omega)} \leq c\|f\|_{L^m(\Omega)}, \tag{4.13}$$

where

$$m^{**} = \frac{Nm(N-2s)}{(N-2)(N-2ms)}.$$

□

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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