



Research article

A new approach to Bell and poly-Bell numbers and polynomials

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Abstract: Bell polynomials are widely applied in many problems arising from physics and engineering. The aim of this paper is to introduce new types of special polynomials and numbers, namely Bell polynomials and numbers of the second kind and poly-Bell polynomials and numbers of the second kind, and to derive their explicit expressions, recurrence relations and some identities involving those polynomials and numbers. We also consider degenerate versions of those polynomials and numbers, namely degenerate Bell polynomials and numbers of the second kind and degenerate poly-Bell polynomials and numbers of the second kind, and deduce their similar results.

Keywords: Bell polynomials of the second kind; degenerate Bell polynomials of the second kind; poly-Bell polynomials of the second kind; degenerate poly-Bell polynomials of the second kind

Mathematics Subject Classification: 11B73, 11B83

1. Introduction

Some special polynomials and numbers are diversely used in physics and engineering as well as in mathematics. For example, Bell polynomials play an important role in the studies of water waves which help energy development, mechanical engineering, marine/offshore engineering, hydraulic engineering, etc (see [9–12,22]). There are various ways of studying special numbers and polynomials, to mention a few, generating functions, combinatorial methods, p -adic analysis, umbral calculus, differential equations, probability theory, special functions and analytic number theory (see [1,2]).

The aim of this paper is to introduce several special polynomials and numbers, and to study their explicit expressions, recurrence relations and identities involving those polynomials and numbers by using generating functions.

Indeed, we introduce Bell polynomials and numbers of the second kind (see (2.3), (2.5)) and poly-Bell polynomials and numbers of the second kind (see (4.1)). The generating function of Bell numbers of the second kind is the compositional inverse of the generating function of Bell numbers minus the constant term. Then Bell polynomials of the second kind are natural extensions of those numbers (see [23]). The poly-Bell polynomials of the second kind, which are defined with the help of polylogarithm, become the Bell polynomials of the second kind up to sign when the index of the polylogarithm is $k = 1$.

We also consider degenerate versions of those numbers and polynomials, namely degenerate Bell numbers and polynomials of the second (see (3.3), (3.5)) and degenerate poly-Bell numbers and polynomials (see (5.1)), and derive similar results. It is worthwhile to note that degenerate versions of many special numbers and polynomials have been explored in recent years with aforementioned tools and many interesting arithmetical and combinatorial results have been obtained (see [14,15,18,19,26]). In fact, studying degenerate versions can be done not only for polynomials and numbers but also for transcendental functions like gamma functions. For the rest of this section, we recall the necessary facts that are needed throughout this paper.

The Stirling numbers of the first kind, $S_1(n, k)$, are given by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [7, 25]}), \quad (1.1)$$

As the inversion formula of (1.1), the Stirling numbers of the second kind, $S_2(n, k)$, are given by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [3, 13–20]}). \quad (1.2)$$

It is well known that the Bell polynomials are defined as

$$\text{Bel}_n(x) = \sum_{k=0}^n S_2(n, k) x^k, \quad (n \geq 0), \quad (\text{see [24, 25]}). \quad (1.3)$$

From (1.3), we note that

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!}, \quad (\text{see [4, 7, 8, 17, 27]}). \quad (1.4)$$

When $x = 1$, $\text{Bel}_n = \text{Bel}_n(1)$, $(n \geq 0)$ are called the Bell numbers.

For any $\lambda \in \mathbb{R}$, the degenerate exponential function is given by

$$e_{\lambda}^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad (\text{see [5, 6, 7, 26, 27]}), \quad (1.5)$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

When $x = 1$, we write $e_\lambda(t) = e_\lambda^1(t)$.

The degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [15]}), \quad (1.6)$$

where

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,1/\lambda} \frac{t^n}{n!}, \quad (\text{see [15]}). \quad (1.7)$$

In view of (1.2), the degenerate Stirling numbers of the second kind are defined by

$$\frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, \quad (\text{see [15]}). \quad (1.8)$$

In [17], the degenerate Bell polynomials are defined by

$$e_\lambda^x(e_\lambda(t) - 1) = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!}. \quad (1.9)$$

When $x = 1$, $\text{Bel}_{n,\lambda} = \text{Bel}_{n,\lambda}(1)$, ($n \geq 0$), are called the Bell numbers.

From (1.8) and (1.9), we note that

$$\text{Bel}_{n,\lambda}(x) = \sum_{k=0}^n S_{2,\lambda}(n,k) (x)_{k,\lambda}, \quad (n \geq 0), \quad (\text{see [17]}). \quad (1.10)$$

The polylogarithm of index k is given by

$$\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (k \in \mathbb{Z}, |x| < 1), \quad (\text{see [3, 13, 14, 16, 21]}). \quad (1.11)$$

Note that $\text{Li}_1(x) = -\log(1-x)$.

Recently, the degenerate polylogarithm is defined as

$$\text{Li}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1} (1)_{n,1/\lambda}}{(n-1)! n^k} x^n, \quad (|x| < 1, k \in \mathbb{Z}), \quad (\text{see [15]}). \quad (1.12)$$

Note that $\text{Li}_{1,\lambda}(x) = -\log_\lambda(1-x)$.

Here we mention that, to our best knowledge, the results of this paper are new.

2. Bell polynomials of the second kind

From (1.4), we note that

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!}$$

Let $x = 1$. Then we have

$$e^{e^t-1} - 1 = \sum_{n=1}^{\infty} \text{Bel}_n \frac{t^n}{n!}. \quad (2.1)$$

Let $f(t) = e^{e^t-1} - 1$. Then the compositional inverse of $f(t)$ is given by

$$f^{-1}(t) = \log(1 + \log(1 + t)). \quad (2.2)$$

We consider the new type Bell numbers, called *Bell numbers of the second kind*, defined by

$$\log(1 + \log(1 + t)) = \sum_{n=1}^{\infty} \text{bel}_n \frac{t^n}{n!}. \quad (2.3)$$

Now, we observe that

$$\begin{aligned} \log(1 + \log(1 + t)) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\log(1 + t))^k \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} (k-1)! \frac{1}{k!} (\log(1 + t))^k = \sum_{k=1}^{\infty} (-1)^{k-1} (k-1)! \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{k-1} (k-1)! S_1(n, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.4)$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 1. For $n \geq 1$, we have

$$(-1)^{n-1} \text{bel}_n = \sum_{k=1}^n (k-1)! \left[\begin{matrix} n \\ k \end{matrix} \right],$$

where $\left[\begin{matrix} n \\ k \end{matrix} \right]$ are the unsigned Stirling numbers of the first kind.

Also, we consider the new type Bell polynomials, called *Bell polynomials of the second kind*, defined by

$$\text{bel}_n(x) = \sum_{k=1}^n (-1)^{k-1} (k-1)! S_1(n, k) x^k, \quad (n \geq 1). \quad (2.5)$$

From (2.5), we can derive the following equation.

$$\begin{aligned} \sum_{n=1}^{\infty} \text{bel}_n(x) \frac{t^n}{n!} &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{k-1} (k-1)! S_1(n, k) x^k \right) \frac{t^n}{n!} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} (k-1)! x^k \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k!}{k} x^k \frac{1}{k!} (\log(1 + t))^k \end{aligned} \quad (2.6)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k (\log(1+t))^k = \log(1+x\log(1+t)).$$

Thus the generating function of Bell polynomials of the second kind is given by

$$\log(1+x\log(1+t)) = \sum_{n=1}^{\infty} \text{bel}_n(x) \frac{t^n}{n!}. \quad (2.7)$$

Note here that $\text{bel}_n = \text{bel}_n(1)$. From (2.7), we note that

$$\frac{x}{(1+x\log(1+t))(1+t)} = \frac{d}{dt} \log(1+x\log(1+t)) = \sum_{n=0}^{\infty} \text{bel}_{n+1}(x) \frac{t^n}{n!}. \quad (2.8)$$

Replacing t by $e^t - 1$ in (2.8), we get

$$\begin{aligned} \frac{x}{1+xt} e^{-t} &= \sum_{k=0}^{\infty} \text{bel}_{k+1}(x) \frac{1}{k!} (e^t - 1)^k \\ &= \sum_{k=0}^{\infty} \text{bel}_{k+1}(x) \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \text{bel}_{k+1}(x) S_2(n, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.9)$$

Taking $x = -1$ in (2.9), we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \text{bel}_{k+1}(-1) S_2(n, k) \right) \frac{t^n}{n!} = -\frac{1}{1-t} e^{-t} = -\sum_{n=0}^{\infty} d_n \frac{t^n}{n!}, \quad (2.10)$$

where d_n is the derangement number (see [19]).

Therefore, by comparing the coefficients on both sides of (2.10), we obtain the following theorem.

Theorem 2. For $n \geq 0$, we have

$$\sum_{k=0}^n \text{bel}_{k+1}(-1) S_2(n, k) = -d_n.$$

Replacing t by $e^{e^t-1} - 1$ in (2.3), we get

$$\begin{aligned} t &= \sum_{k=1}^{\infty} \text{bel}_k \frac{1}{k!} (e^{e^t-1} - 1)^k = \sum_{k=1}^{\infty} \text{bel}_k \sum_{j=k}^{\infty} S_2(j, k) \frac{1}{j!} (e^t - 1)^j \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^j \text{bel}_k S_2(j, k) \sum_{n=j}^{\infty} S_2(n, j) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{j=1}^n \sum_{k=1}^j \text{bel}_k S_2(j, k) S_2(n, j) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.11)$$

Thus we obtain following theorem.

Theorem 3. For $n \geq 2$, we have

$$\sum_{j=1}^n \sum_{k=1}^j \text{bel}_k S_2(j, k) S_2(n, j) = 0, \text{ and } \text{bel}_1 = 1.$$

Replacing t by $e^t - 1$ in (2.7), we get

$$\begin{aligned} \log(1 + xt) &= \sum_{k=1}^{\infty} \text{bel}_k(x) \frac{1}{k!} (e^t - 1)^k \\ &= \sum_{k=1}^{\infty} \text{bel}_k(x) \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \text{bel}_k(x) S_2(n, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.12)$$

On the other hand,

$$\log(1 + xt) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n t^n. \quad (2.13)$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 4. For $n \geq 1$, we have

$$x^n = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^n \text{bel}_k(x) S_2(n, k).$$

In particular,

$$1 = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^n \text{bel}_k S_2(n, k).$$

3. Degenerate Bell polynomials of the second kind

From (1.3), we note that

$$e_{\lambda}(e_{\lambda}(t) - 1) - 1 = \sum_{n=1}^{\infty} \text{Bel}_{n,\lambda} \frac{t^n}{n!}. \quad (3.1)$$

Let $f_{\lambda}(t) = e_{\lambda}(e_{\lambda}(t) - 1) - 1$. Then the compositional inverse of $f_{\lambda}(t)$ is given by

$$f_{\lambda}^{-1}(t) = \log_{\lambda}(1 + \log_{\lambda}(1 + t)). \quad (3.2)$$

We consider the new type degenerate Bell numbers, called *degenerate Bell numbers of the second kind*, defined by

$$\log_{\lambda}(1 + \log_{\lambda}(1 + t)) = \sum_{n=1}^{\infty} \text{bel}_{n,\lambda} \frac{t^n}{n!}. \quad (3.3)$$

Now, we observe that

$$\begin{aligned}\log_{\lambda}(1 + \log_{\lambda}(1 + t)) &= \sum_{k=1}^{\infty} \lambda^{k-1}(1)_{k,1/\lambda} \frac{1}{k!} (\log_{\lambda}(1 + t))^k \\ &= \sum_{k=1}^{\infty} \lambda^{k-1}(1)_{k,1/\lambda} \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \lambda^{k-1}(1)_{k,1/\lambda} S_{1,\lambda}(n, k) \right) \frac{t^n}{n!}.\end{aligned}\tag{3.4}$$

Therefore, by (3.3) and (3.4), we obtain the following theorem.

Theorem 5. For $n \geq 1$, we have

$$\text{bel}_{n,\lambda} = \sum_{k=1}^n \lambda^{k-1}(1)_{k,1/\lambda} S_{1,\lambda}(n, k).$$

Also, we define the *degenerate Bell polynomials of second kind* by

$$\text{bel}_{n,\lambda}(x) = \sum_{k=1}^n \lambda^{k-1}(1)_{k,1/\lambda} S_{1,\lambda}(n, k) x^k.\tag{3.5}$$

Note that $\text{bel}_{n,\lambda} = \text{bel}_{n,\lambda}(1)$.

From (3.5), we note that

$$\begin{aligned}\sum_{n=1}^{\infty} \text{bel}_{n,\lambda}(x) \frac{t^n}{n!} &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \lambda^{k-1}(1)_{k,1/\lambda} S_{1,\lambda}(n, k) x^k \right) \frac{t^n}{n!} \\ &= \sum_{k=1}^{\infty} \lambda^{k-1}(1)_{k,1/\lambda} x^k \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \\ &= \sum_{k=1}^{\infty} \lambda^{k-1}(1)_{k,1/\lambda} x^k \frac{1}{k!} (\log_{\lambda}(1 + t))^k \\ &= \log_{\lambda}(1 + x \log_{\lambda}(1 + t)).\end{aligned}\tag{3.6}$$

Thus the generating function of $\text{bel}_{n,\lambda}(x)$ is given by

$$\log_{\lambda}(1 + x \log_{\lambda}(1 + t)) = \sum_{n=1}^{\infty} \text{bel}_{n,\lambda}(x) \frac{t^n}{n!}.\tag{3.7}$$

Replacing t by $e_{\lambda}(t) - 1$ in (3.7), we get

$$\begin{aligned}\log_{\lambda}(1 + xt) &= \sum_{k=1}^{\infty} \text{bel}_{k,\lambda}(x) \frac{1}{k!} (e_{\lambda}(t) - 1)^k \\ &= \sum_{k=1}^{\infty} \text{bel}_{k,\lambda}(x) \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!}\end{aligned}\tag{3.8}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \text{bel}_{k,\lambda}(x) S_{2,\lambda}(n,k) \right) \frac{t^n}{n!}.$$

On the other hand,

$$\log_{\lambda}(1+xt) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n,1/\lambda} x^n \frac{t^n}{n!}. \quad (3.9)$$

Therefore, by (3.8) and (3.9), we obtain the following theorem.

Theorem 6. For $n \geq 1$, we have

$$x^n = \frac{\lambda^{1-n}}{(1)_{n,1/\lambda}} \sum_{k=1}^n \text{bel}_{k,\lambda}(x) S_{2,\lambda}(n,k).$$

In particular,

$$\lambda^{n-1} (1)_{n,1/\lambda} = \sum_{k=1}^n \text{bel}_{k,\lambda} S_{2,\lambda}(n,k).$$

Replacing t by $e_{\lambda}(e_{\lambda}(t) - 1) - 1$ in (3.3), we have

$$\begin{aligned} t &= \sum_{k=1}^{\infty} \text{bel}_{k,\lambda} \frac{1}{k!} (e_{\lambda}(e_{\lambda}(t) - 1) - 1)^k \\ &= \sum_{k=1}^{\infty} \text{bel}_{k,\lambda} \sum_{j=k}^{\infty} S_{2,\lambda}(j,k) \frac{1}{j!} (e_{\lambda}(t) - 1)^j \\ &= \sum_{j=1}^{\infty} \left(\sum_{k=1}^j \text{bel}_{k,\lambda} S_{2,\lambda}(j,k) \right) \sum_{n=j}^{\infty} S_{2,\lambda}(n,j) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{j=1}^n \sum_{k=1}^j \text{bel}_{k,\lambda} S_{2,\lambda}(j,k) S_{2,\lambda}(n,j) \right) \frac{t^n}{n!}. \end{aligned} \quad (3.10)$$

Therefore, by comparing the coefficients on both sides of (3.10), we obtain the following theorem.

Theorem 7. For $n \geq 2$, we have

$$\sum_{j=1}^n \sum_{k=1}^j \text{bel}_{k,\lambda} S_{2,\lambda}(j,k) S_{2,\lambda}(n,j) = 0, \text{ and } \text{bel}_{1,\lambda} = 1.$$

4. Poly-Bell polynomials of the second kind

Now, we consider the *poly-Bell polynomials of the second kind* which are defined as

$$\text{Li}_k(-x \log(1-t)) = \sum_{n=1}^{\infty} \text{bel}_n^{(k)}(x) \frac{t^n}{n!}. \quad (4.1)$$

When $x = 1$, $\text{bel}_n^{(k)} = \text{bel}_n^{(k)}(1)$ are called the *poly-Bell numbers of the second kind*.

From (1.11), we note that

$$\begin{aligned} \text{Li}_k(-x \log(1-t)) &= \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} x^l l! \frac{1}{l!} (\log(1-t))^l \\ &= \sum_{l=1}^{\infty} \frac{(-1)^l}{l^{k-1}} (l-1)! x^l \sum_{n=l}^{\infty} (-1)^n S_1(n, l) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{l=1}^n \frac{(-1)^{n-l}}{l^{k-1}} (l-1)! x^l S_1(n, l) \right) \frac{t^n}{n!}. \end{aligned} \quad (4.2)$$

Therefore, by (4.1) and (4.2), we obtain the following theorem.

Theorem 8. For $n \geq 1$, we have

$$\text{bel}_n^{(k)}(x) = \sum_{l=1}^n \frac{x^l}{l^{k-1}} (l-1)! \left[\begin{matrix} n \\ l \end{matrix} \right].$$

In particular,

$$\text{bel}_n^{(k)} = \sum_{l=1}^n \frac{1}{l^{k-1}} (l-1)! \left[\begin{matrix} n \\ l \end{matrix} \right].$$

Note that

$$\text{bel}_n^{(1)}(x) = \sum_{l=1}^n x^l (l-1)! \left[\begin{matrix} n \\ l \end{matrix} \right] = (-1)^{n-1} \text{bel}_n(x).$$

Indeed,

$$\begin{aligned} \text{Li}_1(-x \log(1-t)) &= -\log(1+x \log(1-t)) \\ &= \sum_{n=1}^{\infty} \text{bel}_n(x) (-1)^{n-1} \frac{t^n}{n!}. \end{aligned}$$

Replacing t by $1 - e^{-t}$ in (4.1), we get

$$\begin{aligned} \text{Li}_k(xt) &= \sum_{l=1}^{\infty} \text{bel}_l^{(k)}(x) \frac{1}{l!} (1 - e^{-t})^l \\ &= \sum_{l=1}^{\infty} \text{bel}_l^{(k)}(x) (-1)^l \sum_{n=l}^{\infty} S_2(n, l) (-1)^n \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{l=1}^n (-1)^{n-l} \text{bel}_l^{(k)}(x) S_2(n, l) \right) \frac{t^n}{n!}. \end{aligned} \quad (4.3)$$

From (1.11) and (4.3), we note that

$$\frac{x^n}{n^k} = \frac{1}{n!} \sum_{l=1}^n (-1)^{n-l} \text{bel}_l^{(k)}(x) S_2(n, l). \quad (4.4)$$

Therefore, by (4.4), we obtain the following theorem.

Theorem 9. For $n \geq 1$, we have

$$x^n = \frac{n^{k-1}}{(n-1)!} \sum_{l=1}^n (-1)^{n-l} \text{bel}_l^{(k)}(x) S_2(n, l).$$

5. Degenerate poly-Bell polynomials of the second kind

We define the *degenerate poly-Bell polynomials of the second kind* by

$$\text{Li}_{k,\lambda}(-x \log_\lambda(1-t)) = \sum_{n=1}^{\infty} \text{bel}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \quad (5.1)$$

When $x = 1$, $\text{bel}_{n,\lambda}^{(k)} = \text{bel}_{n,\lambda}^{(k)}(1)$ are called the *degenerate poly-Bell numbers of the second*.

From (2.1), we note that

$$\begin{aligned} \text{Li}_{k,\lambda}(-x \log_\lambda(1-t)) &= \sum_{l=1}^{\infty} \frac{(-\lambda)^{l-1} (1)_{l,1/\lambda}}{(l-1)! l^k} (-x \log_\lambda(1-t))^l \\ &= - \sum_{l=1}^{\infty} \frac{(1)_{l,1/\lambda}}{l^{k-1}} \lambda^{l-1} x^l \frac{1}{l!} (\log_\lambda(1-t))^l. \\ &= - \sum_{l=1}^{\infty} \frac{(1)_{l,1/\lambda}}{l^{k-1}} \lambda^{l-1} x^l \sum_{n=l}^{\infty} S_{1,\lambda}(n, l) (-1)^n \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left((-1)^{n-1} \sum_{l=1}^n \frac{1}{l^{k-1}} (1)_{l,1/\lambda} \lambda^{l-1} x^l S_{1,\lambda}(n, l) \right) \frac{t^n}{n!}. \end{aligned} \quad (5.2)$$

Therefore, by (5.1) and (5.2), we obtain the following theorem.

Theorem 10. For $n \geq 1$, we have

$$(-1)^{n-1} \text{bel}_{n,\lambda}^{(k)}(x) = \sum_{l=1}^n \frac{1}{l^{k-1}} (1)_{l,1/\lambda} \lambda^{l-1} x^l S_{1,\lambda}(n, l).$$

For $k = 1$, we have

$$(-1)^{n-1} \text{bel}_{n,\lambda}^{(1)}(x) = \sum_{l=1}^n (1)_{l,1/\lambda} \lambda^{l-1} x^l S_{1,\lambda}(n, l) = \text{bel}_{n,\lambda}(x), \quad (n \geq 0).$$

Indeed,

$$\text{Li}_{1,\lambda}(-x \log_\lambda(1-t)) = -\log_\lambda(1 + x \log_\lambda(1-t)) = \sum_{n=1}^{\infty} (-1)^{n-1} \text{bel}_{n,\lambda}(x) \frac{t^n}{n!}.$$

Replacing t by $1 - e_\lambda(-t)$ in (5.1), we get

$$\text{Li}_{k,\lambda}(xt) = \sum_{l=1}^{\infty} \text{bel}_{l,\lambda}^{(k)}(x) \frac{1}{l!} (1 - e_\lambda(-t))^l \quad (5.3)$$

$$\begin{aligned}
&= \sum_{l=1}^{\infty} \text{bel}_{l,\lambda}^{(k)}(x) (-1)^l \frac{1}{l!} (e_{\lambda}(-t) - 1)^l \\
&= \sum_{l=1}^{\infty} \text{bel}_{l,\lambda}^{(k)}(x) (-1)^l \sum_{n=l}^{\infty} S_{2,\lambda}(n, l) (-1)^n \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} \left(\sum_{l=1}^n (-1)^{n-l} \text{bel}_{l,\lambda}^{(k)}(x) S_{2,\lambda}(n, l) \right) \frac{t^n}{n!}.
\end{aligned}$$

On the other hand,

$$\text{Li}_{k,\lambda}(xt) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1} (1)_{n,1/\lambda}}{(n-1)! n^k} x^n t^n = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1} (1)_{n,1/\lambda}}{n^{k-1}} x^n \frac{t^n}{n!}. \quad (5.4)$$

From (5.3) and (5.4), we get the following result.

Theorem 11. For $n \geq 1$, we have

$$\frac{(-\lambda)^{n-1} (1)_{n,1/\lambda}}{n^{k-1}} x^n = \sum_{l=1}^n (-1)^{n-l} \text{bel}_{l,\lambda}^{(k)}(x) S_{2,\lambda}(n, l).$$

6. Conclusions

Many special polynomials and numbers are widely used in physics and engineering as well as in mathematics. In recent years, degenerate versions of some special polynomials and numbers have been studied by means of various different tools. Here we introduced Bell polynomials of the second kind, poly-Bell polynomials of the second kind and their degenerate versions, namely degenerate Bell polynomials of the second kind and degenerate poly-Bell polynomials of the second kind. By using generating functions, we explored their explicit expressions, recurrence relations and some identities involving those polynomials and numbers.

It is one of our future projects to continue this line of research, namely to explore many special numbers and polynomials and their degenerate versions with the help of various different tools.

Acknowledgments

This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, (NRF-2021R1F1A1050151).

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Boussayoud, A. Abderrezzak, Complete homogeneous symmetric functions and Hadamard product, *Ars Comb.*, **144** (2019), 81–90.

2. A. Boussayoud, M. Kerada, R. Sahali, W. Rouibah, Some applications on generating functions, *J. Concr. Appl. Math.*, **12** (2014), 321–330.
3. K. N. Boyadzhiev, Polyexponentials, *arXiv:0710.1332*.
4. R. A. Brualdi, *Introductory combinatorics*, 5 Eds., Upper Saddle River, NJ: Pearson Prentice Hall, 2010.
5. L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Util. Math.*, **15** (1979), 51–88.
6. L. Carlitz, Arithmetic properties of the Bell polynomials, *J. Math. Anal. Appl.*, **15** (1966), 33–52. doi: 10.1016/0022-247X(66)90135-1.
7. L. Comtet, *Advanced combinatorics: The art of finite and infinite expansions*, Revised and enlarged edition, Dordrecht: D. Reidel Publishing Co., 1974. doi: 10.1007/978-94-010-2196-8.
8. G. B. Djordjevic, G. V. Milovanovic, *Special classes of polynomials*, University of Nis, Faculty of Technology, 2014.
9. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Beholding the shallow water waves near an ocean beach or in a lake via a Boussinesq-Burgers system, *Chaos Soliton. Fract.*, **147** (2021), 110875. doi: 10.1016/j.chaos.2021.110875.
10. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Bilinear forms through the binary Bell polynomials, N solitons and Bäcklund transformations of the Boussinesq–Burgers system for the shallow water waves in a lake or near an ocean beach, *Commun. Theor. Phys.*, **72** (2020), 095002. doi: 10.1088/1572-9494/aba23d.
11. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Scaling transformation, hetero-Backlund transformation and similarity reduction on a (2+1)-dimensional generalized variable-coefficient Boiti-Leon-Pempinelli system for water waves, *Rom. Rep. Phys.*, **73** (2021), 111.
12. X.-Y. Gao, Y.-J. Guo, W.-R. Shan, Looking at an open sea via a generalized (2+1) -dimensional dispersive long-wave system for the shallow water: scaling transformations, hetero-Bäcklund transformations, bilinear forms and N solitons, *Eur. Phys. J. Plus*, **136** (2021), 893. doi: 10.1140/epjp/s13360-021-01773-6.
13. G. H. Hardy, On the zeroes certain classes of integral Taylor series, Part I, On the integral function formula, *Proc. Lond. Math. Soc.*, **2** (1905), 332—339. doi:10.1112/plms/s2-2.1.332.
14. W. A. Khan, A note on degenerate Hermite poly-Bernoulli numbers and polynomials, *J. Class. Anal.*, **8** (2016), 65–76. doi: 10.7153/jca-08-06.
15. D. S. Kim, T. Kim, A note on a new type of degenerate Bernoulli numbers, *Russ. J. Math. Phys.*, **27** (2020), 227–235. doi: 10.1134/S1061920820020090.
16. D. S. Kim, T. Kim, A note on polyexponential and unipoly functions, *Russ. J. Math. Phys.*, **26** (2019), 40–49. doi: 10.1134/S1061920819010047.
17. T. Kim, D. S. Kim, L.-C. Kim, H. Lee, Jindalrae and Gaenari numbers and polynomials in connection with Jindalrae-Stirling numbers, *Adv. Differ. Equ.*, **2020** (2020), 245. doi: 10.1186/s13662-020-02701-1.
18. T. Kim, D. S. Kim, L.-C. Jang, H. Lee, H. Kim, Generalized degenerate Bernoulli numbers and polynomials arising from Gauss hypergeometric function, *Adv. Differ. Equ.*, **2021** (2021), 175. doi: 10.1186/s13662-021-03337-5.

19. T. Kim, D. S. Kim, H. Lee, L.-C. Jang, A note on degenerate derangement polynomials and numbers, *AIMS Mathematics*, **6** (2021), 6469–6481. doi: 10.3934/math.2021380.
20. V. Kurt, On the generalized q -poly-Euler polynomials of the second kind, *Filomat*, **34** (2020), 475–482. doi: 10.2298/FIL2002475K.
21. L. Lewin, *Polylogarithms and associated functions*, 2 Eds., New York-Amsterdam: North-Holland Publishing Co., 1981.
22. J.-C. Li, B.-C. Nie, A few frontier issues in ocean engineering mechanics, *China Ocean Eng.*, **35** (2021), 1–11. doi: 10.1007/s13344-021-0001-8.
23. F. Qi, D.-W. Niu, D. Lim, Y. Yao, Special values of the Bell polynomials of the second kind for some sequences and functions, *J. Math. Anal. Appl.*, **491** (2020), 124382. doi: 10.1016/j.jmaa.2020.124382.
24. M. Rigo, *Advanced graph theory and combinatorics*, Hoboken, NJ: John Wiley & Sons, Inc., 2016. doi: 10.1002/9781119008989.
25. S. Roman, *The umbral calculus*, New York: Academic Press, Inc., 1984.
26. S. K. Sharma, W. A. Khan, S. Araci, S. S. Ahmed, New type of degenerate Daehee polynomials of the second kind, *Adv. Differ. Equ.*, **2020** (2020), 428. doi: 10.1186/s13662-020-02891-8.
27. Y. Simsek, Identities on the Changhee numbers and Apostol-type Daehee polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)*, **27** (2017), 199–212.



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