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Research article

A new algorithm based on compressed Legendre polynomials for solving boundary value problems

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Abstract: In this paper, we discuss a novel numerical algorithm for solving boundary value problems. We introduce an orthonormal basis generated from compressed Legendre polynomials. This basis can avoid Runge phenomenon caused by high-order polynomial approximation. Based on the new basis, a numerical algorithm of two-point boundary value problems is established. The convergence and stability of the method are proved. The whole analysis is also applicable to higher order equations or equations with more complex boundary conditions. Four numerical examples are tested to illustrate the accuracy and efficiency of the algorithm. The results show that our algorithm have higher accuracy for solving linear and nonlinear problems.

Keywords: compressed Legendre polynomials; boundary value problems; convergence analysis; stability analysis; error estimation
Mathematics Subject Classification: 65L05, 65L10

1. Introduction

Boundary value problems(BVPs) of differential equations originate from many scientific fields such as applied mathematics, physics, chemistry, biology, medical science, economics, engineering science, etc. [1–4]. BVPs have always been the frontier of scientific research due to its important applications and scientific implications. It is difficult to obtain analytic solutions of these equations, which promotes the research of approximate methods to get numerical solutions. For instance, the authors in [5] develop a numerical method for second-order three-point BVPs by using the idea of piecewise approximation. A computational approach for multi-point BVPs is discussed based on the least squares approximation method and the Lagrange-multiplier method in [6]. The authors in [7] investigate the collocation method with linear/linear rational splines for the numerical solution of two-point boundary value problems. Numerical approaches based on B-spline method [8–11] and reproducing kernel techniques [12–14] are proposed respectively for solving singular linear and nonlinear equations. In [15] and [16], the authors applied Legendre polynomials to handle numerical solutions. Recently, multiscale algorithms with different multiscale orthogonal basis are introduced in [17, 18].

Numerical methods for solving linear equations have been studied extensively. This paper is concerned with the the following BVPs:

$$\begin{cases} y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = g(x), \quad x \in (0,1), \\ \mathcal{B}_i y = \alpha_i, \quad i = 1, \cdots, n. \end{cases} \tag{1.1}$$

where $a_i(x), i = 1, \cdots, n, g(x)$ are sufficiently smooth functions and $\mathcal{B}_i y = \alpha_i, i = 1, \cdots, n$ are linear boundary conditions. In this work, we will give a new approach to Eq (1.1) based on compressed Legendre polynomials. The method can reduce computational cost and provide highly accurate approximate solutions. What's more, this method can avoid Runge phenomenon caused by high-order polynomial approximation. Without loss of generality, we will discuss a class of equations with boundary conditions as in Eq (1.2) to show our method conveniently. The whole analysis is also applicable to higher order equations or equations with more complex boundary conditions by proper modifications.

$$\begin{cases} y'' + a_1(x)y' + a_2(x)y = g(x), \quad x \in (0,1), \\ y(0) = \alpha, \quad y(1) = \beta. \end{cases} \tag{1.2}$$

This paper is organized as follows. In section 2, we introduce an orthonormal basis generated from compressed Legendre polynomials. The numerical algorithm is established in section 3, and the convergence analysis is given in section 4. In section 5, we give some numerical examples to testify the effectiveness of our method.

2. A new basis based on compressed Legendre polynomials

For convenience, we homogenize the boundary conditions in Eq (1.2) so to get Eq (2.1).

$$\begin{cases} u'' + a_1(x)u' + a_2(u)u = f(x), \quad x \in (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{2.1}$$

We define
$W_2^2[0,1] = \{u(x) | u' \text{ is absolutely continuous on } [0,1], u'' \in L^2[0,1], u(0) = u(1) = 0\}$.
It is a reproducing kernel space as introduced in [19].
The inner product in $W_2^2[0,1]$ is given by

$$\langle u, v \rangle = \int_0^1 u'' v'' dx.$$

As we all know that Legendre polynomials $L_n(x)$ is an orthogonal basis in $L^2[-1,1]$. Therefore,

$$P_n(x) = \sqrt{2n+1} L_n(2x-1). \tag{2.2}$$

is an orthonormal basis of $L^2[0,1]$.

By using Eq (2.2), we will construct an orthonormal basis of $W_2^2[0,1]$. First, we compress $P_n(x)$ to obtain a new basis in $L^2[0,1]$.

Lemma 2.1. Given $k, p \in \mathbb{Z}^+$, with $1 \le k \le p$, define

$$\varphi_{nk}(x) = \begin{cases} \sqrt{p}P_n(px-k+1), & x \in [\frac{k-1}{p}, \frac{k}{p}], \\ 0, & else. \end{cases}$$

where p is the compression coefficient, then $\{\varphi_{nk}\}_{n=0,k=1}^{\infty \quad p}$ is an orthonormal basis of $L^2[0,1]$.

Proof. $\forall \varphi_{nk}, \varphi_{mj}, n \neq m$,

i) $k \neq j$, $\varphi_{nk}, \varphi_{mj}$ are orthogonal according to the definition;

ii) k=j, $\int_0^1 \varphi_{nk}\varphi_{mj}dx = \int_0^1 \varphi_{nk}\varphi_{mk}dx = \int_{\frac{k-1}{p}}^{\frac{k}{p}} pP_n(px-k+1)P_m(px-k+1)dx$.

Let $s = px - k + 1$,

$$\int_0^1 \varphi_{nk}\varphi_{mj}dx = \int_0^1 P_n(s)P_m(s)ds = 0.$$

$\varphi_{nk}(x)$ maintain the orthogonality.

And

$$\int_0^1 {\varphi_{nk}}^2 dx = \int_{\frac{k-1}{p}}^{\frac{k}{p}} pP_n^2(px-k+1)dx = \int_0^1 P_n^2(s)ds = 1.$$

It follows that $\varphi_{nk}(x)$ are orthonormal.

In addition,

$\forall u \in L^2[0,1]$, if $\int_0^1 u\varphi_{nk}dx = 0$, that is

$$\int_0^1 u\varphi_{nk}dx = \int_{\frac{k-1}{p}}^{\frac{k}{p}} u\sqrt{p}P_n(px-k+1)dx = \frac{\sqrt{p}}{p}\int_0^1 uP_n(s)ds = 0,$$

then $u = 0$.

So, $\varphi_{nk}(x)$ are orthonormal and complete. Therefore $\{\varphi_{nk}\}_{n=0,k=1}^{\infty \quad p}$ is an orthonormal basis of $L^2[0,1]$.

Next, we construct an orthonormal basis of $W_2^2[0,1]$.

Define $J^2\varphi_{nk}$ as

$$J^2\varphi_{nk}(x) = \int_0^x ds \int_0^s \varphi_{nk}\, dt - x\int_0^1 ds \int_0^s \varphi_{nk}\, dt. \tag{2.3}$$

Note that $J^2\varphi_{nk}(0) = J^2\varphi_{nk}(1) = 0$. We will prove that $\{J^2\varphi_{nk}\}_{n=0,k=1}^{\infty \quad p}$ is a basis of $W_2^2[0,1]$.

Theorem 2.1. $\{J^2\varphi_{nk}(x)\}_{n=0,k=1}^{\infty \quad p}$ is an orthonormal basis of $W_2^2[0,1]$.

Proof. $J^2\varphi_{nk}(x)$ are orthonormal. In fact,

$$\int_0^1 (J^2\varphi_{nk})''(J^2\varphi_{mj})''dt = \int_0^1 \varphi_{nk}\varphi_{mj}dx = \begin{cases} 1, & n = m,\ k = j, \\ 0, & otherwise. \end{cases}$$

$\forall u \in W_2^2[0,1]$, if $\int_0^1 u''(J^2\varphi_{nk})''dx = 0$, then $\int_0^1 u''\varphi_{nk}dx = 0$, which implies that $u'' = 0$.

Because u' is absolutely continuous on $[0,1]$, we have $u' = C$, C is a constant.

Notice that $u(0) = u(1) = 0$, so we have $u = 0$ on $[0,1]$ which shows the completeness.

Therefore, $\left\{J^2\varphi_{nk}(x)\right\}$ is an orthonormal basis of $W_2^2[0,1]$.

3. Numerical method

In this section, we introduce a numerical algorithm for solving Eq (2.1).

Define a linear operator

$$\mathcal{L} : W_2^2[0,1] \to L^2[0,1],$$

by

$$\mathcal{L}u = u'' + a_1(x)u' + a_2(x)u.$$

Eq (2.1) can be transformed into the following operator form

$$\mathcal{L}u = f. \tag{3.1}$$

It can be proved that the operator is bounded. Next, we demonstrate the implementation steps of our numerical algorithm for solving Eq (3.1). The theoretical background and the error analysis will be discussed in the upcoming section.

Definition 3.1. For any $\varepsilon > 0$, u is called an ε-approximate solution if $\|\mathcal{L}u - f\|^2_{L^2[0,1]} < \varepsilon$.

Let $F(c_{01}, \cdots\cdots, c_{np}) = \left\| \sum\limits_{j=0}^{n} \sum\limits_{k=1}^{p} c_{jk}\mathcal{L}J^2\varphi_{jk} - f \right\|^2_{L^2[0,1]}$, and $(c^*_{01}, \cdots\cdots, c^*_{np})$ is the minimum value point of $F(c_{01}, \cdots\cdots, c_{np})$.

Theorem 3.1. For any $\varepsilon > 0$, $\exists N$, when $n > N$, $u_{np} = \sum\limits_{j=0}^{n} \sum\limits_{k=1}^{p} c^*_{jk}J^2\varphi_{jk}$ is an ε-approximate solution of Eq (3.1).

Proof. Assume that u is the exact solution of Eq (3.1), $u = \sum\limits_{j=0}^{\infty} \sum\limits_{k=1}^{p} a_{jk}J^2\varphi_{jk}$.

$\forall \varepsilon > 0$, $\exists N$, when $n > N$, we have $w_{np} = \sum\limits_{j=0}^{n} \sum\limits_{k=1}^{p} a_{jk}J^2\varphi_{jk}$ satisfies

$$\left\|u - w_{np}\right\|_{W_2^2[0,1]} < \sqrt{\frac{\varepsilon}{\|\mathcal{L}\|^2}}.$$

Because

$$\left\|\mathcal{L}w_{np} - f\right\|^2_{L^2[0,1]} = \left\|\mathcal{L}w_{np} - \mathcal{L}u\right\|^2_{L^2[0,1]} \leq \|\mathcal{L}\|^2 \left\|w_{np} - u\right\|^2_{W_2^2[0,1]} < \varepsilon,$$

we get

$$\left\|\mathcal{L}u_{np} - f\right\|^2_{L^2[0,1]} = \left\|\mathcal{L}\sum_{j=0}^{n}\sum_{k=1}^{p} c^*_{jk}J^2\varphi_{jk} - f\right\|^2_{L^2[0,1]} = \min_{c_{jk}} \left\|\mathcal{L}\sum_{j=0}^{n}\sum_{k=1}^{p} c_{jk}J^2\varphi_{jk} - f\right\|^2_{L^2[0,1]}$$

$$\leq \left\|\mathcal{L}\sum_{j=0}^{n}\sum_{k=1}^{p} a_{jk}J^2\varphi_{jk} - f\right\|^2_{L^2[0,1]} = \left\|\mathcal{L}w_{np} - f\right\|^2_{L^2[0,1]} < \varepsilon.$$

According to Theorem 3.1, we obtain an ε-approximate solution u_{np} of Eq (3.1) by using the new basis. The coefficient is the minimum value point of F. Now, we will prove the minimum value point of F is unique.

Take partial derivatives of the function F with respect to $c_{ml}, m = 0 \cdots n, l = 1 \cdots p$.

$$\frac{\partial F}{\partial c_{ml}} = 2\sum_{j=0}^{n}\sum_{k=1}^{p} c_{jk}\langle \mathcal{L}J^2\varphi_{jk}, \mathcal{L}J^2\varphi_{ml}\rangle - 2\langle \mathcal{L}J^2\varphi_{ml}, f\rangle.$$

Let $\frac{\partial F}{\partial c_{ml}} = 0$, then

$$\sum_{j=0}^{n}\sum_{k=1}^{p} c_{jk}\langle \mathcal{L}J^2\varphi_{jk}, \mathcal{L}J^2\varphi_{ml}\rangle = \langle \mathcal{L}J^2\varphi_{ml}, f\rangle. \tag{3.2}$$

Define a matrix $\mathbf{D} = \left(\langle \mathcal{L}J^2\varphi_{jk}, \mathcal{L}J^2\varphi_{ml}\rangle\right)_{(n+1)p\times(n+1)p}$, and a vector $\mathbf{b} = (\langle \mathcal{L}J^2\varphi_{ml}, f\rangle)^T_{(n+1)p}$. Then $(c^*_{01}, \cdots\cdots c^*_{np})$ is the solution of $\mathbf{Dc} = \mathbf{b}$, where $\mathbf{c} = (c_{01}, \cdots\cdots c_{np})^T$.

Theorem 3.2. Suppose that $\mathcal{L}$ is invertible, then the solution of (3.2) is unique.

Proof. The homogeneous equation corresponding to (3.2) is

$$\sum_{j=0}^{n}\sum_{k=1}^{p} c_{jk}\langle \mathcal{L}J^2\varphi_{jk}, \mathcal{L}J^2\varphi_{ml}\rangle = 0, m = 0\cdots n, l = 1\cdots p.$$

Let's multiply both sides of the above equations by $c_{ml}, m = 0\cdots n, l = 1\cdots p$, and add them all. We have

$$\langle\sum_{j=0}^{n}\sum_{k=1}^{p} c_{jk}\mathcal{L}J^2\varphi_{jk}, \sum_{m=0}^{n}\sum_{l=1}^{p} c_{ml}\mathcal{L}J^2\varphi_{ml}\rangle = 0,$$

which is just $\left\|\sum\limits_{j=0}^{n}\sum\limits_{k=1}^{p} c_{jk}\mathcal{L}J^2\varphi_{jk}\right\|^2_{L^2[0,1]} = 0$. Then $\sum\limits_{j=0}^{n}\sum\limits_{k=1}^{p} c_{jk}\mathcal{L}J^2\varphi_{jk} = 0$.

$\mathcal{L}$ is invertible and $\{J^2\varphi_{nk}(x)\}_{n=0,k=1}^{\infty\ \ p}$ is an orthonormal basis , so

$$c_{jk} = 0, j = 0, \cdots, n, k = 0, \cdots, p.$$

Hence, Eq (3.2) has a unique solution.

4. Convergence and stability analysis

In this section, we will prove the convergence and stability of our method.

Let $u(x) = \sum\limits_{j=0}^{\infty}\sum\limits_{k=1}^{p} a_{jk}J^2\varphi_{jk}$ is the exact solution of Eq (3.1), $u_{np}(x) = \sum\limits_{j=0}^{n}\sum\limits_{k=1}^{p} c^*_{jk}J^2\varphi_{jk}$ is the ε-approximate solution as constructed in the previous section.

Theorem 4.1. $u_{np}(x)$ uniformly converges to $u(x)$.

Proof. Theorem 3.1 implies that $\left\|\mathcal{L}u_{np} - f\right\|^2_{L^2[0,1]} \to 0$. We obtain that

$$\begin{aligned}\left\|u_{np} - u\right\|_{W_2^2[0,1]} &= \left\|\mathcal{L}^{-1}\mathcal{L}u_{np} - \mathcal{L}^{-1}\mathcal{L}u\right\|_{W_2^2[0,1]} \\ &\le \left\|\mathcal{L}^{-1}\right\|\left\|\mathcal{L}u_{np} - \mathcal{L}u\right\|_{L^2[0,1]} = \left\|\mathcal{L}^{-1}\right\|\left\|\mathcal{L}u_{np} - f\right\|_{L^2[0,1]}.\end{aligned} \tag{4.1}$$

So,

$$\left\|u_{np} - u\right\|_{W_2^2[0,1]} \to 0.$$

Furthermore,

$$\left|u_{np}(x)-u(x)\right| = \left|\langle u_{np}(\cdot)-u(\cdot), R_x(\cdot)\rangle\right| \leq \left\|u_{np}-u\right\|_{W_2^2[0,1]}\|R_x\|_{W_2^2[0,1]},$$

where $R_x(t)$ is the reproducing kernel of $W_2^2[0,1]$.

Because $R_x(t)$ is bounded on $[0,1]$, we get

$$\left|u_{np}(x)-u(x)\right| \to 0.$$

We now present the error analysis of our method.

Let

$$w_{np} = \sum_{j=0}^{n}\sum_{k=1}^{p} a_{jk}J^2\varphi_{jk}.$$

By (4.1) , we get

$$\left\|u_{np}-u\right\|_{W_2^2[0,1]} \leq \left\|\mathcal{L}^{-1}\right\|\left\|\mathcal{L}u_{np}-f\right\|_{L^2[0,1]}.$$

By Theorem 3.1, it follows that

$$\left\|\mathcal{L}u_{np}-f\right\|_{L^2[0,1]} \leq \left\|\mathcal{L}w_{np}-f\right\|_{L^2[0,1]} = \left\|\mathcal{L}w_{np}-\mathcal{L}u\right\|_{L^2[0,1]} \leq \|\mathcal{L}\|\left\|w_{np}-u\right\|_{W_2^2[0,1]}.$$

So,

$$\left\|u_{np}-u\right\|_{W_2^2[0,1]} \leq \left\|\mathcal{L}^{-1}\right\|\|\mathcal{L}\|\left\|w_{np}-u\right\|_{W_2^2[0,1]}. \tag{4.2}$$

In the above formula,

$$\left\|u-w_{np}\right\|^2_{W_2^2[0,1]} = \int_0^1 (u''-w''_{np})^2dx = \left\|u''-w''_{np}\right\|^2_{L^2[0,1]}. \tag{4.3}$$

Since $u'' = \sum\limits_{j=0}^{\infty}\sum\limits_{k=1}^{p} a_{jk}\varphi_{jk}, w''_{np} = \sum\limits_{j=0}^{n}\sum\limits_{k=1}^{p} a_{jk}\varphi_{jk}$, we get

$$\begin{aligned}a_{jk} = \langle u'', \varphi_{jk}\rangle &= \int_0^1 u''\varphi_{jk}dx = \int_{\frac{k-1}{p}}^{\frac{k}{p}} u''\sqrt{p}P_j(px-k+1)dx\\ &= \frac{1}{p}\int_0^1 u''(\frac{s+k-1}{p})\sqrt{p}P_j(s)ds.\end{aligned}$$

Assume $v(s,k) = u''(\frac{s+k-1}{p})$,$v_n(s,k) = w''_{np}(\frac{s+k-1}{p})$, then

$$a_{jk} = \frac{\sqrt{p}}{p}\int_0^1 v(s,k)P_j(s)ds = \frac{\sqrt{p}}{p}\langle v(s,k), P_j(s)\rangle = \frac{\sqrt{p}}{p}b_{jk},$$

where b_{jk} are the coefficients of the generalized Fourier expansion with respect to the basis $\{P_j(x)\}_{j=1}^{\infty}$. So $v(s,k) = \sum\limits_{j=0}^{\infty} b_{jk}P_j(s), v_n(s,k) = \sum\limits_{j=0}^{n} b_{jk}P_j(s)$.

Now Eq (4.3) becomes

$$\left\|u''-w''_{np}\right\|^2_{L^2[0,1]} = \left\|u''-\sum_{j=0}^{n}\sum_{k=1}^{p} a_{jk}\varphi_{nk}\right\|^2_{L^2[0,1]} = \left\|u''-\frac{\sqrt{p}}{p}\sum_{j=0}^{n}\sum_{k=1}^{p} b_{jk}\varphi_{nk}\right\|^2_{L^2[0,1]}$$

$$= \int_0^1 (u'' - \frac{\sqrt{p}}{p}\sum_{j=0}^{n}\sum_{k=1}^{p} b_{jk}\varphi_{jk})^2 dx = \sum_{k=1}^{p}\int_{\frac{k-1}{p}}^{\frac{k}{p}} (u'' - \frac{\sqrt{p}}{p}\sum_{j=0}^{n} b_{jk}\sqrt{p}P_j(px-k+1))^2 dx$$
$$= \frac{1}{p}\sum_{k=1}^{p}\int_0^1 (v(s,k) - \sum_{j=0}^{n} b_{jk}P_j(s))^2 ds = \frac{1}{p}\sum_{k=1}^{p}\int_0^1 (v(s,k) - v_n(s,k))^2 ds.$$

According to [20], we can get

$$\left\|u'' - w''_{np}\right\|^2_{L^2[0,1]} = \frac{1}{p}\sum_{k=1}^{p}\int_0^1 (v(s,k)-v_n(s,k))^2 ds = \frac{1}{p}\sum_{k=1}^{p}\|v(s,k)-v_n(s,k)\|^2_{L^2[0,1]}$$
$$\leq \frac{1}{p}\sum_{k=1}^{p}(C\frac{1}{n^{2m}}\sum_{l=\min(m,n+1)}^{m}\left\|\partial_s^l v(s,k)\right\|^2_{L^2[0,1]}).$$

where C is a constant and m makes $u^{(m+2)} \in L^2[0,1]$.

Since $v(s,k) = u''(\frac{s+k-1}{p})$, $\partial_s^l v(s,k) = \frac{1}{p^l}u^{(l+2)}(x)$.

Hence,

$$\left\|u'' - w''_{np}\right\|^2_{L^2[0,1]} \leq \frac{1}{p}\sum_{k=1}^{p}(C\frac{1}{n^{2m}}\sum_{l=\min(m,n+1)}^{m}\frac{1}{p^{2l}}\left\|u^{(l+2)}\right\|^2_{L^2[0,1]}) \leq C_0\frac{1}{n^{2m}}\sum_{l=\min(m,n+1)}^{m}\frac{1}{p^{2l}}. \tag{4.4}$$

where C_0 is a constant.

By (4.3) and (4.4), we have that

$$\left\|u - w_{np}\right\|^2_{W_2^2[0,1]} \leq C_0\frac{1}{n^{2m}}\sum_{l=\min(m,n+1)}^{m}\frac{1}{p^{2l}}.$$

Plug it into (4.2), we conclude that

$$\left\|u_{np} - u\right\|^2_{W_2^2[0,1]} \leq M_0\frac{1}{n^{2m}}\sum_{l=\min(m,n+1)}^{m}\frac{1}{p^{2l}}, \tag{4.5}$$

where M_0 is a constant.

Inequality (4.5) gives the error bound of our new approach. We summarize our result in the following theorem.

Theorem 4.2. If $u^{(m+2)} \in L^2[0,1]$ with $m \geq 0$, and $n \geq m-1$, then there exists a positive constant M_0 so that $\left\|u_{np} - u\right\|_{W_2^2[0,1]} \leq \sqrt{M_0}\frac{1}{n^m}\frac{1}{p^m}$.

Theorem 4.2 indicates the convergence of our algorithm is influenced by both of n and p. The rate of convergence gets faster with the increasing of the two parameters.

In the end of this section, the stability of our algorithm is discussed by using the condition number of the matrix $\mathbf{D}$.

Let $\{\psi_i\}_{i=1}^{\infty} = \{J^2\varphi_{01}, \cdots, J^2\varphi_{0p}, \cdots, J^2\varphi_{n1}, \cdots, J^2\varphi_{np}, \cdots\}$, then $\mathbf{D} = \left(\langle \mathcal{L}J^2\varphi_{jk}, \mathcal{L}J^2\varphi_{ml}\rangle\right)_{M\times M}$ could be rewritten as $\mathbf{D} = \left(\langle \mathcal{L}\psi_i, \mathcal{L}\psi_j\rangle\right)_{M\times M}$, $M = (n+1)p$.

Lemma 4.1 ([17]) If $u \in W_2^2[0,1]$ with $\|u\|_{W_2^2[0,1]} = 1$, then $\|\mathcal{L}u\|_{L^2[0,1]} \geq \frac{1}{\|\mathcal{L}^{-1}\|}$.

Theorem 4.3. The condition number of $\mathbf{D}$ satisfies $Cond(\mathbf{D})_2 \leq \|\mathcal{L}\|^2\left\|\mathcal{L}^{-1}\right\|^2$.

Proof. Assume $\mathbf{x} = (x_1, \cdots x_M)^T$ is a unit eigenvector belonging to λ, i.e., $\lambda\mathbf{x} = \mathbf{D}\mathbf{x}$, $\|\mathbf{x}\| = 1$.

We have

$$\lambda x_i = \sum_{j=1}^{M} \langle \mathcal{L}\psi_i, \mathcal{L}\psi_j \rangle x_j = \sum_{j=1}^{M} \langle \mathcal{L}\psi_i, x_j \mathcal{L}\psi_j \rangle = \langle \mathcal{L}\psi_i, \sum_{j=1}^{M} x_j \mathcal{L}\psi_j \rangle. \tag{4.6}$$

Multiplying Eq (4.6) by $x_i, i = 1 \cdots M$, and add all the equations.

$$\lambda = \lambda \sum_{i=1}^{M} x_i^{\ 2} = \langle \sum_{i=1}^{M} x_i \mathcal{L}\psi_i, \sum_{j=1}^{M} x_j \mathcal{L}\psi_j \rangle = \left\| \sum_{i=1}^{M} x_i \mathcal{L}\psi_i \right\|_{L^2[0,1]}^2 \leq \|\mathcal{L}\|^2 \sum_{i=1}^{M} x_i^{\ 2} = \|\mathcal{L}\|^2.$$

On the other hand,

$$\lambda = \left\| \sum_{i=1}^{M} x_i \mathcal{L}\psi_i \right\|_{L^2[0,1]}^2 = \left\| \mathcal{L} \sum_{i=1}^{M} x_i \psi_i \right\|_{L^2[0,1]}^2 = \|\mathcal{L}u\|_{L^2[0,1]}^2,$$

where $u = \sum_{i=1}^{M} x_i \psi_i$.

From Lemma 4.1, $\lambda \geq \frac{1}{\|\mathcal{L}^{-1}\|^2}$.

So

$$Cond(\mathbf{D})_2 = \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right| \leq \|\mathcal{L}\|^2 \left\| \mathcal{L}^{-1} \right\|^2.$$

The stability is proved.

5. Numerical examples

In this part, we will test four examples of BVPs which have been discussed by different algorithms in [17, 21–24]. In the following examples, comparison of numerical results demonstrate the efficiency and stability of our method. The absolute error function is defined as $E = \left| y_{np}(x) - y(x) \right|, 0 \leq x \leq 1$, where $y(x)$ is the exact solution and $y_{np}(x)$ is an ε-approximate solution.

Example 1. ([17, 21]) Consider an equation with Robin boundary condition:

$$\begin{cases} y''(x) + xy(x) = x^3 + 2, & x \in (0, 1), \\ y'(0) - y(0) = 0, & y'(1) - y(1) = 1. \end{cases} \tag{5.1}$$

$y = x^2$ is the exact solution of Eq (5.1). The numerical results using our method are compared with [17, 21] in Table 1. It shows that our method converges rapidly with higher accuracy.

Table 1. Absolute error for Example 1.

x	Method in [21]	Method in [17]	Present method	
			$n=4, p=3$	$n=7, p=3$
0.0	1.91E-5	1.12E-10	3.52E-11	3.82E-12
0.1	2.10E-5	1.39E-10	3.69E-11	3.96E-12
0.2	2.29E-5	1.66E-10	3.37E-11	3.56E-12
0.3	2.46E-5	1.96E-10	2.49E-11	2.57E-12
0.4	2.60E-5	2.28E-10	5.40E-12	6.78E-13
0.5	2.71E-5	2.63E-10	2.03E-11	2.04E-12
0.6	2.78E-5	2.96E-10	5.11E-11	5.51E-12
0.7	2.79E-5	3.26E-10	8.62E-11	9.91E-12
0.8	2.73E-5	3.54E-10	1.22E-10	1.48E-11
0.9	2.61E-5	3.82E-10	1.53E-10	1.89E-11
1.0	2.41E-5	4.09E-10	1.72E-10	2.19E-11

Example 2. Consider the following two-point BVP:

$$\begin{cases} y''(x) + \sin xy'(x) + y(x) = f(x), & x \in (0,1), \\ y'(0) = 0, \quad y(1) = 0. \end{cases} \tag{5.2}$$

where $f(x) = \begin{cases} x\cos x, & 0 \le x < \frac{1}{2}, \\ (-x/2+1)\sinh x, & \frac{1}{2} \le x \le 1. \end{cases}$. Since the exact solution of Eq (5.2) is not known, we discuss the absolute residual error function defined as $R = |\mathcal{L}u_{np} - f|$. The results are listed in Table 2. It shows that the numerical error is rather smaller by our method.

Table 2. Absolute residual error for Example 2.

x	$n=4, p=4$	$n=5, p=4$	$n=5, p=8$
0.0	1.54E-6	4.06E-8	7.05E-10
0.1	4.60E-7	3.45E-9	6.86E-10
0.2	2.37E-7	8.77E-9	6.83E-10
0.3	1.48E-8	1.22E-8	7.39E-10
0.4	6.07E-8	4.94E-9	7.24E-10
0.5	8.95E-7	7.00E-9	6.26E-10
0.6	2.81E-7	1.52E-9	6.30E-11
0.7	1.34E-7	1.50E-10	2.33E-10
0.8	1.11E-7	3.37E-9	4.54E-11
0.9	2.52E-7	1.76E-9	9.79E-11
1.0	7.12E-7	1.74E-8	9.74E-10

Example 3. ([22, 23] We consider the following sixth-order BVP:

$$\begin{cases} y^{(6)}(x) + 6e^x = y(x), & x \in (0,1), \\ y(0) = 1, y(1) = 0, y''(0) = -1, y''(1) = -2e, y^{(4)}(0) = -3, y^{(4)}(1) = -4e. \end{cases} \tag{5.3}$$

$y = (1 - x)e^x$ is the exact solution of Eq (5.3). The results are reported in the Table 3. The absolute error of our method is smaller than those obtained in [22] by homotopy perturbation method and in [23] by Legendre wavelet collocation method.

Table 3. Absolute error for Example 3.

x	Method in [22]	Method in [23]	E(Present method)	
			$n = 4, p = 2$	$n = 4, p = 8$
0.0	0	0	0	0
0.1	4.1E-4	1.3E-10	1.23E-12	1.29E-16
0.2	7.8E-4	2.4E-10	2.21E-12	3.73E-17
0.3	1.1E-3	3.3E-10	2.82E-12	5.72E-17
0.4	1.3E-3	3.7E-10	3.32E-12	3.64E-17
0.5	1.3E-3	3.9E-10	3.65E-12	1.73E-17
0.6	1.3E-3	3.6E-10	3.63E-12	1.14E-16
0.7	1.1E-3	3.0E-10	3.06E-12	9.02E-17
0.8	7.8E-4	2.8E-10	1.91E-12	3.82E-17
0.9	4.1E-4	1.1E-10	8.81E-13	0
1.0	0	3.2E-16	3.47E-17	3.47E-17

Example 4. ([24]) Consider the following nonlinear fourth-order BVP:

$$\begin{cases} y^{(4)}(x) - e^x y''(x) + y(x) + \sin(y(x)) = f(x), & x \in (0, 1), \\ y(0) = 1, y'(0) = 1, y(1) = 1 + \sinh(1), y'(1) = \cosh(1). \end{cases} \tag{5.4}$$

where $f(x) = 1+\sin(1+\sinh(x))-(e^x-2)\sinh(x)$. The exact solution of Eq (5.4) is $y(x) = 1+\sinh(x)$. Firstly, Eq (5.4) is linearized by Newton iterative method [13]. Then, we use our method to solve the linear equation. The numerical results obtained after three iterations are given in Table 4. It is seen from Table 4 that we have got a better approximation to the exact solution of the nonlinear problem.

Table 4. Absolute error for Example 4.

x	Method in [24]	Present method	
		$n = 3, p = 3$	$n = 3, p = 8$
0.0	0	0	0
0.1	2.78E-8	1.78E-12	1.52E-14
0.2	8.09E-8	3.90E-12	3.83E-14
0.3	1.20E-7	2.63E-12	3.40E-14
0.4	1.25E-7	3.87E-12	1.76E-14
0.5	9.56E-8	9.55E-12	1.07E-13
0.6	4.82E-8	2.09E-12	1.92E-13
0.7	7.38E-9	1.62E-13	2.20E-13
0.8	1.07E-8	1.31E-11	1.65E-13
0.9	7.08E-9	6.80E-12	6.01E-14
1.0	0	0	0

6. Conclusions

In this paper, a new basis $\{J^2\varphi_{nk}(x)\}_{n=0,k=1}^{\infty\ \ p}$ of $W_2^2[0,1]$ is constructed from compressed Legendre polynomials. This basis is simple, efficient, and practical. Our method can avoid Runge phenomenon caused by high-order polynomial approximation. Using the concept of ε-approximate solution, we describe a method with the new basis to solve boundary value problems. The convergence and stability of our algorithm are proved. Our algorithm is tested by some numerical examples with different boundary conditions. The numerical results show that our method have higher accuracy for solving linear and nonlinear problems.

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Conflict of interest

The authors declare that they have no competing interests.

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