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Research article

Some identities involving Gauss sums

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Abstract: We calculate several identities involving some Gauss sums of the 2^k-order character modulo an odd prime p by using the elementary and analytic methods, and finally give several exact and interesting formulae for them. The properties of the classical Gauss sums play an important role in the proof of this paper.

Keywords: 2^k-order character; the classical Gauss sums; analytic methods; identity; calculating formula
Mathematics Subject Classification: 11L10, 11L40

1. Introduction

Let $q > 1$ be an integer and let χ be Dirichlet character modulo q. The Gauss sums $G(m, \chi; q)$ is defined as

$$G(m, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma}{q}\right),$$

where m is integer and $e(y) = e^{2\pi i y}$.

Gauss sums is very important in the analytic number theory and related research filed. Many scholars studied its properties and obtained a series of interesting results (see [1–4, 6, 7, 9–14, 17, 18]). For example, let χ be the primitive character modulo q, we have

$$G(m, \chi; q) = \overline{\chi}(m) G(1, \chi; q) \equiv \overline{\chi}(m) \tau(\chi) \quad \text{and} \quad |\tau(\chi)| = \sqrt{q},$$

where $\tau(\chi) = \sum_{b=1}^{q} \chi(b) e\left(\frac{b}{q}\right)$ and $\overline{\chi}$ is the complex conjugate of χ.

Berndt and Evans [3] studied the properties of cubic Gauss sums. Zhang and Hu [15] studied the number of the solutions of the diagonal cubic congruence equation mod p, they obtained the following results: Let p be a prime with $p \equiv 1 \bmod 3$. Then for any third-order character λ modulo p, one has the identity

$$\tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) = dp, \tag{1.1}$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

In addition, Chen and Zhang [8] studied the case of the fourth-order character modulo p, and obtained an identity (see the Lemma 1). Chen [5] studied the properties of the Gauss sums of the sixth-order character modulo p. It is not hard to see from [5, 8, 15], the number of these characters is 2. What about the number of the characters > 2? Motivated by that, Zhang et al. [16] studied the eight-order and twelve-order characters.

In this article, we shall further study the generalization. That is, the number of the characters is $\phi\left(2^k\right) = 2^{k-1} \geq 4$. We prove several identities involving the Gauss sums of the 2^k-order character modulo an odd prime p with $p \equiv 1 \bmod 2^k (\ k \geq 3)$. We give several exact and interesting formulae for them.

Theorem 1. *Let p be an odd prime with $p \equiv 1 \bmod 16$. If χ_{16} denotes a sixteen-order character modulo p, then we have the identity*

$$\left|\frac{\tau^4\left(\chi_{16}^7\right)}{\tau^4\left(\chi_{16}\right)} + \frac{\tau^4\left(\chi_{16}^5\right)}{\tau^4\left(\chi_{16}^3\right)}\right| = \left|\frac{\tau^4\left(\chi_{16}^9\right)}{\tau^4\left(\chi_{16}^{15}\right)} + \frac{\tau^4\left(\chi_{16}^{11}\right)}{\tau^4\left(\chi_{16}^{13}\right)}\right| = \frac{2\cdot|\alpha|}{\sqrt{p}},$$

where $\alpha = \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a+\bar{a}}{p}\right)$, $\bar{a}$ denotes the multiplicative inverse of a modulo p, and $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p.

Theorem 2. *Let p be an odd prime with $p \equiv 1 \bmod 32$. If χ_{32} denotes a 32-order character modulo p, then we have the identity*

$$\left|\frac{\tau^4\left(\chi_{32}^7\right)\cdot\tau^4\left(\chi_{32}^{15}\right)}{\tau^4\left(\chi_{32}\right)\cdot\tau^4\left(\chi_{32}^9\right)} + \frac{\tau^4\left(\chi_{32}^5\right)\cdot\tau^4\left(\chi_{32}^{13}\right)}{\tau^4\left(\chi_{32}^3\right)\cdot\tau^4\left(\chi_{32}^{11}\right)}\right| = \frac{2\cdot|\alpha|}{\sqrt{p}}.$$

More generally, for any positive integer k, we have the following theorem.

Theorem 3. *Let $k \geq 5$ be an integer. For any prime p with $p \equiv 1 \bmod 2^k$, if $\chi_{2^k} = \psi$ denotes a 2^k-order character modulo p, then we have the identity*

$$\left|\frac{\prod\limits_{\substack{j=1\\ j\equiv -1 \bmod 8}}^{2^{k-1}-1}\tau^4\left(\psi^j\right)}{\prod\limits_{\substack{h=1\\ h\equiv 1 \bmod 8}}^{2^{k-1}-1}\tau^4\left(\psi^h\right)} + \frac{\prod\limits_{\substack{j=1\\ j\equiv -3 \bmod 8}}^{2^{k-1}-1}\tau^4\left(\psi^j\right)}{\prod\limits_{\substack{h=1\\ h\equiv 3 \bmod 8}}^{2^{k-1}-1}\tau^4\left(\psi^h\right)}\right| = \frac{2\cdot|\alpha|}{\sqrt{p}}.$$

2. Several lemmas

In this section, we will give several lemmas by using the relevant properties of character sums.

Lemma 1. *Let p be a prime with $p \equiv 1 \bmod 4$, then for any fourth-order character χ_4 modulo p, we have*

$$\tau^2(\chi_4) + \tau^2(\overline{\chi}_4) = 2\sqrt{p} \cdot \alpha,$$

where $\alpha = \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a+\bar{a}}{p}\right)$, *and* $\left(\frac{*}{p}\right) = \chi_2$ *denotes the Legendre's symbol modulo p.*

Proof. See Lemma 3 in [7]. □

Lemma 2. *Let p be a prime with $p \equiv 1 \bmod 4$. Then for any non-principal character ψ mod p, we have the identity*

$$\tau\left(\overline{\psi}^2\right) = \frac{\overline{\psi}(-4) \cdot \sqrt{p} \cdot \tau\left(\overline{\psi}\chi_2\right)}{\tau(\psi)}.$$

Proof. From the properties of the classical Gauss sums we have

$$\begin{aligned}
&\sum_{a=0}^{p-1}\psi\left(a^2-1\right) = \sum_{a=0}^{p-1}\psi\left((a+1)^2-1\right) = \sum_{a=1}^{p-1}\psi\left(a^2+2a\right) = \sum_{a=1}^{p-1}\psi(a)\psi(a+2)\\
=&\frac{1}{\tau\left(\overline{\psi}\right)}\sum_{a=1}^{p-1}\psi(a)\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{b(a+2)}{p}\right) = \frac{1}{\tau\left(\overline{\psi}\right)}\sum_{b=1}^{p-1}\overline{\psi}(b)\sum_{a=1}^{p-1}\psi(a)e\left(\frac{b(a+2)}{p}\right)\\
=&\frac{\tau(\psi)}{\tau\left(\overline{\psi}\right)}\sum_{b=1}^{p-1}\overline{\psi}(b)\overline{\psi}(b)e\left(\frac{2b}{p}\right) = \frac{\tau(\psi)}{\tau\left(\overline{\psi}\right)}\sum_{b=1}^{p-1}\overline{\psi}^2(b)e\left(\frac{2b}{p}\right) = \frac{\psi(4)\tau\left(\overline{\psi}^2\right)\tau(\psi)}{\tau\left(\overline{\psi}\right)}.
\end{aligned} \tag{2.1}$$

On the other hand, for any integer b with $(b, p) = 1$, we have the identity

$$\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right) = 1 + \sum_{a=1}^{p-1}(1+\chi_2(a))\,e\left(\frac{ba}{p}\right) = \sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ba}{p}\right) = \chi_2(b)\cdot\sqrt{p},$$

so note that $\chi_2(-1) = 1$ we also have the identity

$$\begin{aligned}
&\sum_{a=0}^{p-1}\psi\left(a^2-1\right) = \frac{1}{\tau\left(\overline{\psi}\right)}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{b(a^2-1)}{p}\right)\\
=&\frac{1}{\tau\left(\overline{\psi}\right)}\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{-b}{p}\right)\sum_{a=0}^{p-1}e\left(\frac{ba^2}{p}\right) = \frac{\tau(\chi_2)}{\tau\left(\overline{\psi}\right)}\sum_{b=1}^{p-1}\overline{\psi}(b)\chi_2(b)e\left(\frac{-b}{p}\right)\\
=&\frac{\sqrt{p}}{\tau\left(\overline{\psi}\right)}\sum_{b=1}^{p-1}\overline{\psi}\chi_2(b)e\left(\frac{-b}{p}\right) = \frac{\overline{\psi}(-1)\sqrt{p}\cdot\tau\left(\overline{\psi}\chi_2\right)}{\tau\left(\overline{\psi}\right)}.
\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) we have the identity

$$\tau\left(\overline{\psi}^2\right) = \frac{\overline{\psi}(-4) \cdot \sqrt{p} \cdot \tau\left(\overline{\psi}\chi_2\right)}{\tau(\psi)}.$$

This proves Lemma 2. □

Lemma 3. *Let p be an odd prime with $p \equiv 1 \bmod 8$. If χ_8 is an 8-order character modulo p, then we have*

$$\left|\tau^4(\chi_8) + \tau^4\left(\chi_8^3\right)\right| = 2 \cdot p^{\frac{3}{2}} \cdot |\alpha|.$$

Proof. Taking $\psi = \chi_8$, then $\psi^2 = \chi_4$ is a four-order character modulo p. Note that $\psi^2(-4) = 1$, from Lemmas 1 and 2 we have

$$2 \cdot \sqrt{p} \cdot \alpha = \tau^2\left(\overline{\psi}^2\right) + \tau^2\left(\psi^2\right) = \frac{p \cdot \tau^2\left(\overline{\psi}\chi_2\right)}{\tau^2(\psi)} + \frac{p \cdot \tau^2\left(\psi\chi_2\right)}{\tau^2\left(\overline{\psi}\right)}$$

or

$$\frac{\tau^2\left(\overline{\psi}\chi_2\right)}{\tau^2(\psi)} + \frac{\tau^2\left(\psi\chi_2\right)}{\tau^2\left(\overline{\psi}\right)} = \frac{2 \cdot \alpha}{\sqrt{p}}. \tag{2.3}$$

Note that $\overline{\psi}\chi_2 = \psi^3$, $\overline{\psi} = \psi^7$ and $\frac{\tau^2(\chi_8^5)}{\tau^2(\chi_8^7)} = \frac{\tau^2(\chi_8)}{\tau^2(\chi_8^3)}$, from (2.3) we have the identity

$$\tau^4(\chi_8) + \tau^4\left(\chi_8^3\right) = \frac{2 \cdot \alpha}{\sqrt{p}} \cdot \tau^2(\chi_8)\,\tau^2\left(\chi_8^3\right). \tag{2.4}$$

Now Lemma 3 follows from (2.4). □

3. Proofs of the theorems

In this section, we shall give the proof of all results by using the properties of the classical Gauss sums, the elementary and analytic methods. For Theorem 1, Let p be an odd prime with $p \equiv 1 \bmod 16$, and χ_{16} is a sixteen-order character modulo p, then from Lemma 2 we have

$$\tau^4\left(\overline{\chi}_8\right) = \frac{\overline{\chi}_{16}^4(-4) \cdot p^2 \cdot \tau^4\left(\chi_{16}^7\right)}{\tau^4\left(\chi_{16}\right)} = \frac{p^2 \cdot \tau^4\left(\chi_{16}^7\right)}{\tau^4\left(\chi_{16}\right)} \tag{3.1}$$

and

$$\tau^4\left(\overline{\chi}_8^3\right) = \frac{\overline{\chi}_{16}^{12}(-4) \cdot p^2 \cdot \tau^4\left(\chi_{16}^{21}\right)}{\tau^4\left(\chi_{16}^3\right)} = \frac{p^2 \cdot \tau^4\left(\chi_{16}^5\right)}{\tau^4\left(\chi_{16}^3\right)}. \tag{3.2}$$

Applying (3.1), (3.2) and Lemma 3 we have

$$2 \cdot p^{\frac{3}{2}} \cdot |\alpha| = \left|\tau^4(\chi_8) + \tau^4\left(\chi_8^3\right)\right| = \left|\tau^4\left(\overline{\chi}_8\right) + \tau^4\left(\overline{\chi}_8^3\right)\right| = p^2 \cdot \left|\frac{\tau^4\left(\chi_{16}^7\right)}{\tau^4\left(\chi_{16}\right)} + \frac{\tau^4\left(\chi_{16}^5\right)}{\tau^4\left(\chi_{16}^3\right)}\right|$$

or identity

$$\left|\frac{\tau^4\left(\chi_{16}^7\right)}{\tau^4\left(\chi_{16}\right)}+\frac{\tau^4\left(\chi_{16}^5\right)}{\tau^4\left(\chi_{16}^3\right)}\right|=\frac{2\cdot|\alpha|}{\sqrt{p}}.$$

This proves Theorem 1.

Now we prove the Theorem 2. Let p be an odd prime with $p \equiv 1 \bmod 32$, and χ_{32} is a 32-order character modulo p, then from Lemma 2 we have

$$\tau^4\left(\overline{\chi}_{16}\right)=\frac{\overline{\chi}_{32}^4(-4)\cdot p^2\cdot\tau^4\left(\chi_{32}^{15}\right)}{\tau^4\left(\chi_{32}\right)}=\frac{\overline{\chi}_4(2)\cdot p^2\cdot\tau^4\left(\chi_{32}^{15}\right)}{\tau^4\left(\chi_{32}\right)} \tag{3.3}$$

and

$$\tau^4\left(\overline{\chi}_{16}^7\right)=\frac{\overline{\chi}_{32}^{28}(-4)\cdot p^2\cdot\tau^4\left(\chi_{32}^{105}\right)}{\tau^4\left(\chi_{32}^7\right)}=\frac{\chi_4(2)\cdot p^2\cdot\tau^4\left(\chi_{32}^{9}\right)}{\tau^4\left(\chi_{32}^7\right)}. \tag{3.4}$$

Since $\chi_4^2(2)=\chi_2(2)=1$, from (3.3) and (3.4) we get

$$\frac{\tau^4\left(\overline{\chi}_{16}^7\right)}{\tau^4\left(\overline{\chi}_{16}\right)}=\frac{\tau^4\left(\chi_{32}\right)\cdot\tau^4\left(\chi_{32}^9\right)}{\tau^4\left(\chi_{32}^7\right)\cdot\tau^4\left(\chi_{32}^{15}\right)} \tag{3.5}$$

and

$$\frac{\tau^4\left(\overline{\chi}_{16}^5\right)}{\tau^4\left(\overline{\chi}_{16}^3\right)}=\frac{\tau^4\left(\chi_{32}^3\right)\cdot\tau^4\left(\chi_{32}^{11}\right)}{\tau^4\left(\chi_{32}^5\right)\cdot\tau^4\left(\chi_{32}^{13}\right)}. \tag{3.6}$$

Noting that

$$\begin{aligned}&\left|\frac{\tau^4\left(\overline{\chi}_{32}\right)\cdot\tau^4\left(\overline{\chi}_{32}^9\right)}{\tau^4\left(\overline{\chi}_{32}^7\right)\cdot\tau^4\left(\overline{\chi}_{32}^{15}\right)}+\frac{\tau^4\left(\overline{\chi}_{32}^3\right)\cdot\tau^4\left(\overline{\chi}_{32}^{11}\right)}{\tau^4\left(\overline{\chi}_{32}^5\right)\cdot\tau^4\left(\overline{\chi}_{32}^{13}\right)}\right|\\=&\left|\frac{\tau^4\left(\chi_{32}^7\right)\cdot\tau^4\left(\chi_{32}^{15}\right)}{\tau^4\left(\chi_{32}\right)\cdot\tau^4\left(\chi_{32}^9\right)}+\frac{\tau^4\left(\chi_{32}^5\right)\cdot\tau^4\left(\chi_{32}^{13}\right)}{\tau^4\left(\chi_{32}^3\right)\cdot\tau^4\left(\chi_{32}^{11}\right)}\right|.\end{aligned} \tag{3.7}$$

Combining (3.5)–(3.7) and Theorem 1 we immediately have the identity

$$\begin{aligned}&\left|\frac{\tau^4\left(\chi_{32}^7\right)\cdot\tau^4\left(\chi_{32}^{15}\right)}{\tau^4\left(\chi_{32}\right)\cdot\tau^4\left(\chi_{32}^9\right)}+\frac{\tau^4\left(\chi_{32}^5\right)\cdot\tau^4\left(\chi_{32}^{13}\right)}{\tau^4\left(\chi_{32}^3\right)\cdot\tau^4\left(\chi_{32}^{11}\right)}\right|\\=&\left|\frac{\tau^4\left(\overline{\chi}_{32}\right)\cdot\tau^4\left(\overline{\chi}_{32}^9\right)}{\tau^4\left(\overline{\chi}_{32}^7\right)\cdot\tau^4\left(\overline{\chi}_{32}^{15}\right)}+\frac{\tau^4\left(\overline{\chi}_{32}^3\right)\cdot\tau^4\left(\overline{\chi}_{32}^{11}\right)}{\tau^4\left(\overline{\chi}_{32}^5\right)\cdot\tau^4\left(\overline{\chi}_{32}^{13}\right)}\right|\\=&\left|\frac{\tau^4\left(\chi_{32}\right)\cdot\tau^4\left(\chi_{32}^9\right)}{\tau^4\left(\chi_{32}^7\right)\cdot\tau^4\left(\chi_{32}^{15}\right)}+\frac{\tau^4\left(\chi_{32}^3\right)\cdot\tau^4\left(\chi_{32}^{11}\right)}{\tau^4\left(\chi_{32}^5\right)\cdot\tau^4\left(\chi_{32}^{13}\right)}\right|\\=&\left|\frac{\tau^4\left(\overline{\chi}_{16}^7\right)}{\tau^4\left(\overline{\chi}_{16}\right)}+\frac{\tau^4\left(\overline{\chi}_{16}^5\right)}{\tau^4\left(\overline{\chi}_{16}^3\right)}\right|=\left|\frac{\tau^4\left(\chi_{16}^7\right)}{\tau^4\left(\chi_{16}\right)}+\frac{\tau^4\left(\chi_{16}^5\right)}{\tau^4\left(\chi_{16}^3\right)}\right|=\frac{2\cdot|\alpha|}{\sqrt{p}}.\end{aligned}$$

This proves Theorem 2.

From Theorems 1 and 2 we know that Theorem 3 is true for $k = 4$ and 5. Assuming that Theorem 3 holds true for $k = n \geq 5$. That is,

$$\left| \frac{\prod\limits_{\substack{j=1 \\ j\equiv -1 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\psi^j\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 1 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\psi^h\right)} + \frac{\prod\limits_{\substack{j=1 \\ j\equiv -3 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\psi^j\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 3 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\psi^h\right)} \right| = \frac{2\cdot|\alpha|}{\sqrt{p}}, \tag{3.8}$$

where $\psi = \chi_{2^n}$ is a 2^n-order character modulo p.

So the conjugate of (3.8) is

$$\left| \frac{\prod\limits_{\substack{j=1 \\ j\equiv -1 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\overline{\psi}^j\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 1 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\overline{\psi}^h\right)} + \frac{\prod\limits_{\substack{j=1 \\ j\equiv -3 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\overline{\psi}^j\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 3 \bmod 8}}^{2^{n-1}-1} \tau^4\left(\overline{\psi}^h\right)} \right| = \frac{2\cdot|\alpha|}{\sqrt{p}}. \tag{3.9}$$

Then for $k = n + 1$ and j with $j \equiv 1 \bmod 8$, from Lemma 2 we have

$$\frac{\tau^4\left(\overline{\chi}^{7j}_{2^n}\right)}{\tau^4\left(\overline{\chi}^{j}_{2^n}\right)} = \frac{\overline{\chi}^{3j}_{2^{n-1}}\left(2^4\right)\cdot\tau^4\left(\chi^{j}_{2^{n+1}}\right)\cdot\tau^4\left(\chi^{j(2^n-2^3+1)}_{2^{n+1}}\right)}{\tau^4\left(\chi^{7j}_{2^{n+1}}\right)\cdot\tau^4\left(\chi^{j(2^n-1)}_{2^{n+1}}\right)}. \tag{3.10}$$

Note that the identity

$$\prod_{s=0}^{2^{n-3}-1}\overline{\chi}^{3(8s+1)}_{2^n}\left(2^4\right) = \overline{\chi}_{2^n}^{2^{n-3}+2^{n-1}\left(2^{n-1}-1\right)}\left(2^{12}\right) = 1.$$

From (3.9) and (3.10) we may immediately deduce the identity

$$\left| \frac{\prod\limits_{\substack{j=1 \\ j\equiv -1 \bmod 8}}^{2^{n}-1} \tau^4\left(\overline{\chi}^j_{2^{n+1}}\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 1 \bmod 8}}^{2^{n}-1} \tau^4\left(\overline{\chi}^h_{2^{n+1}}\right)} + \frac{\prod\limits_{\substack{j=1 \\ j\equiv -3 \bmod 8}}^{2^{n}-1} \tau^4\left(\overline{\chi}^j_{2^{n+1}}\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 3 \bmod 8}}^{2^{n}-1} \tau^4\left(\overline{\chi}^h_{2^{n+1}}\right)} \right| = \frac{2\cdot|\alpha|}{\sqrt{p}}. \tag{3.11}$$

Taking the conjugate for (3.11), this implies that Theorem 3 is complete for $k = n + 1$.

This proves Theorem 3 by mathematical induction.

4. Conclusions

The main result of this paper is to prove a new identity for the classical Gauss sums. That is, if p is a prime with $p \equiv 1 \bmod 2^k$, $k \geq 4$, then for any 2^k-order character ψ modulo p, we have the identity

$$\left| \frac{\prod\limits_{\substack{j=1 \\ j\equiv -1 \bmod 8}}^{2^{k-1}-1} \tau^4\left(\psi^j\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 1 \bmod 8}}^{2^{k-1}-1} \tau^4\left(\psi^h\right)} + \frac{\prod\limits_{\substack{j=1 \\ j\equiv -3 \bmod 8}}^{2^{k-1}-1} \tau^4\left(\psi^j\right)}{\prod\limits_{\substack{h=1 \\ h\equiv 3 \bmod 8}}^{2^{k-1}-1} \tau^4\left(\psi^h\right)} \right| = \frac{2 \cdot |\alpha|}{\sqrt{p}}.$$

These results give the exact values of some special Gauss sums and reveal the values distribution of classical Gauss sums. This not only promotes the research of some well-known sums, such as Kloosterman sums, which plays an important role in the study of the Diophantine equation, but also makes new contributions to the research of other related fields.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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