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Research article

On the general strong fuzzy solutions of general fuzzy matrix equation involving the Core-EP inverse

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Abstract: The inconsistent or consistent general fuzzy matrix equation are studied in this paper. The aim of this paper is threefold. Firstly, general strong fuzzy matrix solutions of consistent general fuzzy matrix equation are derived, and an algorithm for obtaining general strong fuzzy solutions of general fuzzy matrix equation by Core-EP inverse is also established. Secondly, if inconsistent or consistent general fuzzy matrix equation satisfies $X \in R(S^k)$, the unique solution or unique least squares solution of consistent or inconsistent general fuzzy matrix equation are given by Core-EP inverse. Thirdly, we present an algorithm for obtaining Core-EP inverse. Finally, we present some examples to illustrate the main results.

Keywords: Core-EP inverse; general strong fuzzy solution; unique least squares solution; fuzzy linear systems
Mathematics Subject Classification: 08A72,15A09

1. Introduction

The fuzzy arithmetic operations and concept of fuzzy numbers were firstly discussed and introduced by Dubois and Prade [8], Nahmias [17], and Zadeh [6, 23]. In [14], Mazandarani et al.introduced the concepts of granular difference, granular metric, continuous fuzzy functions, granular derivative and four basic operations. Based on the result of [14], Abbasi and Jalali in [1] introduced a novel approach for solving fully fuzzy linear systems and their duality. Fuzzy systems are used to a variety of problems ranging from fuzzy tracking control to fuzzy linear dynamical systems [2], fuzzy linear systems [9], fuzzy matrix discrete dynamical systems [19] and so on. In [7], Dong et al. introduced a novel approach of solving fuzzy matrix games through a ranking value function.

Fuzzy linear system plays an important role in various fields, such as optimization, physics, statistics, engineering, economics, information acquisition, and even social science. In [9], Friedman et al.

introduced a general model for solving fuzzy linear system (FLS), whose right-hand side column is an arbitrary fuzzy number vector and the coefficient matrix is crisp, by the embedded method. In [3], Asady et al. considered the full row rank system, using its similarity method to solve the $m \times n$ order fuzzy linear system for $m \leqslant n$. Later, Wang and Zheng [22, 24] studied the $m \times n$ order consistent and inconsistent fuzzy linear system by using generalized inverses of the coefficient matrix. In [13], Gong and Guo proposed a general model for solving inconsistent general fuzzy matrix equation (GFME), whose right-hand is an arbitrary fuzzy matrix and the coefficient matrix is crisp. However, by using aforementioned methods, the general strong solutions of the FLS can not be obtained. Mihailović, et al. in [15, 16] proposed two similarity methods for obtaining all solutions of the FLS by using the Moore-Penrose inverse and Group inverse. Based on the method of [9, 15], Jiang ang wang [12] proposed an algorithm for obtaining all solutions of the FLS by using Core inverse, and showed the importance of the Core-EP inverse of the coefficient matrix in solving GFME.

A matrix X satisfying the only equality $PXP = P$ is called an inner inverse of P; and a matrix X satisfying the only equality $XPX = X$ is called an outer inverse of P. As we all know, when the coefficient matrix belong to inner inverse, we can give the general strong solution of FLS, see [15, 16]. However, we know that the Core-EP inverse does not belong to inner inverse but it belong to outer inverse, see [10, 11]. The natural question arose: how can we give the general strong solutions to GFME through Core-EP inverse? For further investigations, there are more generalized inverses for different purposes [21]. Baksalary et al. [5] introduced the Core inverse and studied the properties of Core inverse and one special partial order. In [18], Prasad and Mohana proposed the Core-EP inverse, where the Core-EP inverse is a generalization of the Core inverse. Next, Wang H. [20] gave the Core-EP decomposition for studying the Core-EP inverse and its applications. In addition, if inconsistent or consistent matrix equation satisfies $X \in R(S^k)$, the unique solution or unique least squares solution of consistent or inconsistent matrix equation are given by Core-EP inverse. Therefore, our current purpose is to carefully study the square GFME and the unique Core-EP inverse block structure.
Inspired by the discussion above, in this paper, a numerical method is given for finding the general strong solution of GFME based on the Core-EP inverse calculation. Firstly, the effect of Core-EP inverse is extended and in solving singular consistent or inconsistent model matrix equation is studied. Secondly, we study the relationship between the Core-EP inverse of the coefficient matrix in GFME and the Core-EP inverse of the coefficient matrix in model GFME. Moreover, we discuss the nonnegativity of the Core-EP inverse of the coefficient matrix in model GFME. Finally, this paper presents a practical algorithm for solving consistent GFME and some examples are presented to illustrate the algorithm.

This paper is divided into five parts. In Section 2, we introduce some characteristics of generalized inverses and fuzzy numbers. In Section 3, a method for finding a strong fuzzy solution of the GFME based on Core-EP inverse calculation, is given when the coefficient matrix of model GFME is real $2n \times 2n$ matrix. In Section 4, another method for finding the general strong fuzzy matrix solutions of the GFME based on Core-EP inverse calculation, is given when the coefficient matrix of the GFME is real matrix. Next, the algorithm for solving the consistent GFME is derived, and we use some examples to explain the new algorithm. In Section 5, we give a summary of this work.

2. Preliminary

This section mainly contains two aspects. On one hand, we introduce generalized inverses and some common symbols. On the other hand, we review the definition of fuzzy numbers, fuzzy sets, and the symbols commonly used in GFME.

2.1. The block representation of the Core-EP inverse

In this part, we review the characteristics of the Core-EP inverse and the Schur decomposition. The symbols $\mathbb{R}_{n\times n}$, I_n, P^*, $\mathcal{R}(A)$, and $\mathrm{rk}(P)$ denote the set of $m\times n$ real matrices, the identity matrix of rank n, the conjugate transpose, range, and rank, respectively of $P\in\mathbb{R}_{n\times n}$. For a $n\times n$ matrix P, the index of A is the smallest nonnegative integer k, denoted $\mathrm{Ind}(P)$ as follow:

$$\mathbb{R}_n^{CM} = \{P\in\mathbb{R}_{n,n} : rk(P^{k+1}) = rk(P^k)\}.$$

Let $P = (p_1, p_2\cdots, p_n)$, where $p_j = (p_{1j}, p_{2j}, \cdots, p_{mj})^*, 1\le j\le n.$ According to [21], we know that $||.||_F$ is the F-norm of P as follow:

$$||P||_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}\right)^{\frac{1}{2}} = [\sum_{j=1}^{n}(p_j)^* p_j]^{\frac{1}{2}},\ i = 1, 2\cdots m,$$

such as

$$||P||_F^2 = \left\|\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\right\|_F^2 = 25. \tag{2.1}$$

According to [20], each matrix has the following form of decomposition (called the Schur decomposition): For any real $n\times n$ matrix P of index k, there exists an $n\times n$ unitary matrix U such that

$$P = U\begin{bmatrix}T & G\\ 0 & N\end{bmatrix}U^*, \tag{2.2}$$

and

$$P^{\circledS} = U\begin{bmatrix}T^{-1} & 0\\ 0 & 0\end{bmatrix}U^*, \tag{2.3}$$

where $T\in\mathbb{R}_{k\times k}$ is invertible, $G\in\mathbb{R}_{k\times(n-k)}$, $N\in\mathbb{R}_{(n-k)\times(n-k)}$ is nilpotent, and $N^k = 0$.
Some matrices equations for a matrix $P\in\mathbb{R}_{m\times n}$ will be reviewed as follows:

$$PXP = P\ \ (1),\ XPX = X\ \ (2),\ (PX)^* = PX\ \ (3),$$
$$(XP)^* = XP\ \ (4),\ PX^2 = X\ \ (2)',\ XP^{k+1} = P^k\ \ (5).$$

Definition 2.1 ([10, 21]). *For any $P\in\mathbb{R}_{m\times n}$, let $\mathbb{T}\{i, j, \ldots h\}$ be the set of $X\in\mathbb{R}_{m\times n}$ fulfilling equations $(i), (j), \ldots, (h)$ in the equations (1)-(5) and (2)′. A matrix $X\in\mathbb{T}\{i, j, \ldots h\}$ is said to be an $\{i, j, \ldots h\}$-inverse of P and we denote it by $P^{\{i,j,\ldots h\}}$.*
(i) If $X\in\mathbb{R}_{m\times n}$ satisfies (1)–(4), then it is said to be the Moore-Penrose inverse of $P\in\mathbb{R}_{m\times n}$. It is

denoted by $P^{\dagger}$ *or* $P^{\{1,2,3,4\}}$.
(*ii*) *If* $X \in \mathbb{R}_{n\times n}$ *satisfies* (3), (2)′ *and* (5), *then it is said to be Core-EP inverse of* $P \in \mathbb{R}_n^{CM}$. *It is denoted by* $P^{\textcircled{\dagger}}$ *or* $P^{\{3,2',5\}}$.
(*iii*) *If* $X \in \mathbb{R}_{n\times n}$ *satisfies* (1), (2)′ *and* (3), *then it is said to be Core inverse of* $Ind(P) = 1$. *It is denoted by* $P^{\textcircled{\#}}$ *or* $P^{\{1,2',3\}}$.
For the given $P = [p_{ij}]$, $P \in \mathbb{R}_{m\times n}$, *we denote* $|P| = [|p_{ij}|]$, $|P| \in \mathbb{R}_{m\times n}$. *P is said to be non-negative if* $p_{ij} \geq 0$, *for each i and j.*

Table 1. Common mathematical symbols.

Notation	Symbolic meaning
$\mathbb{R}_{m\times n}$	$m \times n$ real matrices.
$\mathbb{R}_n^{CM}$	The set of matrices of the index are one or zero.
$P^{\{i,j,\ldots h\}}$	An $\{i, j, \ldots h\}$-inverse of P
P^*	Transposition of P
$P^{\dagger}$	Moore-Penrose inverse of P
$P^{\textcircled{\dagger}}$	Core-EP inverse of P
P^{-1}	Inverse of P
$P^{\sharp}$	Group inverse of P
$R(P)$	Range space of P
$\|.\|_F$	F-norm of P

Theorem 2.1 ([20]). *Let* $P \in \mathbb{R}_{n\times n}$ *with Ind(P)=k. Then*

$$P^{\textcircled{\dagger}} = \{PP^k(P^k)^{\dagger}\}^{\dagger}. \tag{2.4}$$

Algorithm 1. Computing Core − EP inverse of matrix P $\in \mathbb{R}_{n\times n}$ **using MATLAB**

1. $Input\ P\ is\ the\ n-by-n\ matrix.$
2. $Input\ k\ is\ the\ index\ of\ matrix\ P.$
3. $J = pinv(mpower(P, k)).$
4. $L = mtimes(mpower(P, k+1), J).$
5. $P^{\textcircled{\dagger}} = pinv(L).$

The MATLAB software incorporates built in functions pinv and mpower for computing the Moore-Penrose inverse and the matrix power respectively.

2.2. *The concept of the GFME*

An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $[\tilde{z}]_{\alpha} = [\underline{z}(\alpha), \overline{z}(\alpha)]$, which satisfy the following requirements (see [15]). Meanwhile, A fuzzy set of $\tilde{z}$ with a membership function $\tilde{z} \to [0, 1]$ satisfying the following three conditions is called a fuzzy

number.
1. $\tilde{z}(x) = 0$ outside of interval $[a, b]$.
2. $\tilde{z}$ is the upper semi-consistent continuous function.
3. There exists constants c and d such that $a \leq c \leq d \leq b$.
3.1. $\tilde{z}(x)$ is monotonic increasing on $[a, c]$,
3.2. $\tilde{z}(x)$ is monotonic decreasing on $[d, b]$,
3.3. $\tilde{z}(x) = 1$, $c \leq x \leq d$.

Denote ξ by the sets of all fuzzy numbers. The $\alpha - cut$ of a fuzzy number is the crisp set, a bounded closed interval for each $\alpha \in [0, 1]$, denoted with $[\tilde{z}]_\alpha$, such that $[\tilde{z}]_\alpha = [\underline{z}(\alpha), \bar{z}(\alpha)]$, where $\bar{z}(\alpha) = sup\{x \in \mathbb{R} : \tilde{z}(x) \geq \alpha\}$ and $\underline{z}(\alpha) = inf\{x \in \mathbb{R} : \tilde{z}(x) \geq \alpha\}$. Using the lower and upper branches, $\underline{z}$ and $\bar{z}$, a fuzzy number $\tilde{z}$ can be equivalently defined as a pair of function $(\underline{z}, \bar{z})$ where $\underline{z} : [0, 1] \to \mathbb{R}$ is a non-increasing left-continuous function, $\bar{z} : [0, 1] \to \mathbb{R}$ is a non-decreasing left-continuous function and $\underline{z}(\alpha) \leq \bar{z}(\alpha)$ for each $\alpha \in (0, 1]$.

Definition 2.2 ([16, Definition3]). *Let $\tilde{z} = (\underline{z}(\alpha), \bar{z}(\alpha))$, $\tilde{u} = (\underline{u}(\alpha), \bar{u}(\alpha))$ be two arbitrary fuzzy numbers and t be a real number, we define the scalar multiplication and the addition of fuzzy numbers.*
1. $[\tilde{u} + \tilde{z}]_\alpha = [\underline{z}(\alpha) + \underline{u}(\alpha), \bar{z}(\alpha) + \bar{u}(\alpha)]$,
2. $[k\tilde{z}]_\alpha = \begin{cases} [k\underline{z}(\alpha), k\bar{z}(\alpha)], & k \geq 0, \\ [k\bar{z}(\alpha), k\underline{z}(\alpha)], & k < 0, \end{cases}$
3. $\tilde{z} = \tilde{u} \Leftrightarrow \underline{z}(\alpha) = \underline{u}(\alpha)$ *and* $\bar{z}(\alpha) = \bar{u}(\alpha)$.

Definition 2.3 ([13, Definition2.4]). *The fuzzy matrix system $A\tilde{X} = \tilde{Y}$ is as follow:*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1m} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nm} \end{pmatrix} = \begin{pmatrix} \tilde{y}_{11} & \tilde{y}_{12} & \cdots & \tilde{y}_{1m} \\ \tilde{y}_{21} & \tilde{y}_{22} & \cdots & \tilde{y}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{y}_{n1} & \tilde{y}_{n2} & \cdots & \tilde{y}_{nm} \end{pmatrix},$$

where the matrix $A = [a_{ij}]$ is a square matrix ($\tilde{y}_{ij} \in \xi, \tilde{x}_{ij} \in \xi$). Satisfying the above equations and conditions is said to be general fuzzy matrix equation (GFME). Using matrix notation, we have

$$A\tilde{X} = \tilde{Y} \tag{2.5}$$

A fuzzy number matrix, given by

$$\tilde{X} = (x_1, x_2, \cdots, x_m),$$

where $x_j = \left((\underline{x}_{1j}(\alpha), \tilde{x}_{1j}(\alpha)), (\underline{x}_{2j}(\alpha), \tilde{x}_{2j}(\alpha)), \ldots, (\underline{x}_{nj}(\alpha), \tilde{x}_{nj}(\alpha))\right)^*$, $1 \leq j \leq m$, $0 \leq \alpha \leq 1$, is called a solution of the GFME (2.5). We have

$$\left[\sum_{j=1}^{n} a_{ij}\tilde{x}_{ij}\right]_\alpha = [\tilde{y}_{ij}]_\alpha, \; i = 1, 2 \cdots n.$$

Then,

$$\sum_{j=1}^{n} a_{ij}^{+}\underline{x}_{ij}(\alpha) - \sum_{j=1}^{n} a_{ij}^{-}\bar{x}_{ij}(\alpha) = \underline{y}_{ij}(\alpha),$$

$$\sum_{j=1}^{n} a_{ij}^{+}\bar{x}_{ij}(\alpha) - \sum_{j=1}^{n} a_{ij}^{-}\underline{x}_{ij}(\alpha) = \bar{y}_{ij}(\alpha),$$

where $a_{ij}^{+} = a_{ij} \vee 0$ *and* $a_{ij}^{-} = -a_{ij} \vee 0$. *Then, the form of model GFME is as follow:*

$$\begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{2n1} & s_{2n2} & \cdots & s_{2n2n} \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} & \cdots & \underline{x}_{1m} \\ \underline{x}_{21} & \underline{x}_{22} & \cdots & \underline{x}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{x}_{n1} & \underline{x}_{n2} & \cdots & \underline{x}_{nm} \\ -\bar{x}_{11} & -\bar{x}_{12} & \cdots & -\bar{x}_{1m} \\ -\bar{x}_{21} & -\bar{x}_{22} & \cdots & -\bar{x}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{x}_{n1} & -\bar{x}_{n2} & \cdots & -\bar{x}_{nm} \end{pmatrix} = \begin{pmatrix} \underline{y}_{11} & \underline{y}_{12} & \cdots & \underline{y}_{1m} \\ \underline{y}_{21} & \underline{y}_{22} & \cdots & \underline{y}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{y}_{n1} & \underline{y}_{n2} & \cdots & \underline{y}_{nm} \\ -\bar{y}_{11} & -\bar{y}_{12} & \cdots & -\bar{y}_{1m} \\ -\bar{y}_{21} & -\bar{y}_{22} & \cdots & -\bar{y}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{y}_{n1} & -\bar{y}_{n2} & \cdots & -\bar{y}_{nm} \end{pmatrix}.$$

Using the matrix notation, we obtain

$$SX(\alpha) = Y(\alpha),\ \alpha \in [0,1]. \tag{2.6}$$

where

$$s_{kp} = \begin{cases} a_{ij}^{+} & k = i,\ p = j+n \text{ or } k = i+n,\ p = j, \\ a_{ij}^{-} & k = i,\ p = j+n \text{ or } k = i+n,\ p = j, \end{cases}$$

The matrix S is as follows:

$$S = \begin{bmatrix} D & E \\ E & D \end{bmatrix}, \tag{2.7}$$

where D and E are $n \times n$ matrices, $D = [a_{ij}^{+}]$ and $E = [a_{ij}^{-}]$. The coefficient matrix of (2.6) is S. According to [9], if S is non-negative and defines $X^0 = S^{-1}Y(\alpha)$ as a solution of (2.6), then $\tilde{X}^0 \in \xi$ is a strong fuzzy solution of GFME (2.5).

Lemma 2.2. *Let $S \in \mathbb{R}_{2n\times 2n}$ be the coefficient matrix of consistent (2.6). A matrix X is a solution of the consistent (2.6) if and only if*

$$SX = Y,\ Y \in R(S^k).$$

Thus, a matrix solution is

$$X = S^{\textcircled{\tiny T}}Y.$$

Proof. The proof can be found in the proof of [21, Theorem 3.2.2]. □

Lemma 2.3. *Let $S \in \mathbb{R}_{2n\times 2n}$ be the coefficient matrix of inconsistent (2.6). The matrix X is unique least square solution of the inconsistent (2.6) if and only if*

$$SX = Y,\ X \in R(S^k).$$

Thus, the unique least squares matrix solution is

$$X = S^{\textcircled{\tiny T}}Y.$$

Proof. From $X \in R(S^k)$, it follows that there exists $b \in \mathbb{R}_{2n\times 2n}$ for which $X = S^k b$. Let the decomposition of S be as in (2.2). We have

$$U^*Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, U^*b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, and\ S^{\textcircled{\dagger}}Y = U\begin{pmatrix} T^{-1}y_1 \\ 0 \end{pmatrix}. \tag{2.8}$$

It follows that

$$\begin{aligned} \|SX - Y\|_F^2 = \|SS^k b - Y\|_F^2 &= \left\| U\begin{pmatrix} T & G \\ 0 & N \end{pmatrix} U^*U \begin{pmatrix} T^k & V \\ 0 & 0 \end{pmatrix} U^*b - UU^*Y \right\|_F^2 \\ &= \left\| U\begin{pmatrix} T^{k+1} & TV \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - U\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_F^2 \\ &= \left\| \begin{pmatrix} T^{k+1}b_1 + TVb_2 - y_1 \\ -y_2 \end{pmatrix} \right\|_F^2, \\ &= \|T^{k+1}b_1 + TVb_2 - y_1\|_F^2 + \|y_2\|_F^2. \end{aligned}$$

where $V = T^{k-1}G + T^{k-2}GN + T^{k-3}GN^2 \cdots + GN^{k-1}$. Since T is invertible, we have $\min \|T^{k+1}b_1 + TVb_2 - y_1\|_F^2 = 0$, when

$$b_1 = T^{-(k+1)}y_1 - T^{-k}Vb_2.$$

Therefore,

$$\begin{aligned} X = S^k b = U\begin{pmatrix} T^k & V \\ 0 & 0 \end{pmatrix} U^*b = U\begin{pmatrix} T^k b_1 + Vb_2 \\ 0 \end{pmatrix} \\ = U\begin{pmatrix} T^{-1}y_1 \\ 0 \end{pmatrix} = S^{\textcircled{\dagger}}Y. \end{aligned}$$

□

Corollary 2.4. *Let $S \in \mathbb{R}_{2n\times 2n}$ be the coefficient matrix of consistent (2.6). The matrix X is unique solution of the consistent (2.6) if and only if*

$$SX = Y,\ X \in R(S^k),\ Y \in R(S^k).$$

Thus, the unique solution is

$$X = S^{\textcircled{\dagger}}Y.$$

Through the above Lemma and Corollary, we know that if the coefficient matrix S of consistent or inconsistent (2.6) has a unique Core-EP inverse, we will systematically study this kind of GFME (2.5).

3. Properties of matrix S

In this part, a matrix block structure of Core-EP inverse of matrix S is extremely important for our further study.

Theorem 3.1. *Let $S \in \mathbb{R}_{2n\times 2n}$ be the coefficient matrix of (2.7). The Core-EP inverse of matrix*

$$S = \begin{bmatrix} C & D \\ D & C \end{bmatrix}, \tag{3.1}$$

is

$$S^{\textcircled{\dagger}} = \begin{bmatrix} H & I \\ I & H \end{bmatrix}, \tag{3.2}$$

where

$$H = \frac{1}{2}[(C+D)^{\textcircled{\dagger}} + (C-D)^{\textcircled{\dagger}}],\ I = \frac{1}{2}[(C+D)^{\textcircled{\dagger}} - (C-D)^{\textcircled{\dagger}}].$$

Proof. Let A be the matrix in (2.5) and S its associated matrix from (2.6). We have $A = A^{+} - A^{-} = C - D$ and $|A| = A^{+} + A^{-} = C + D$.
Proof of necessity: we know $SS^{\textcircled{\dagger}}S^{\textcircled{\dagger}} = S^{\textcircled{\dagger}}$, therefore

$$\begin{bmatrix} C & D \\ D & C \end{bmatrix}\begin{bmatrix} H & I \\ I & H \end{bmatrix}\begin{bmatrix} H & I \\ I & H \end{bmatrix} = \begin{bmatrix} H & I \\ I & H \end{bmatrix},$$

and get

$$(HC + ID)H + (IC + HD)I = H,\ (HC + ID)I + (IC + HD)H = I.$$

We have

$$(C + D)(H + I)(H + I) = (H + I),\ (C - D)(H - I)(H - I) = (H - I).$$

We know $(SS^{\textcircled{\dagger}})^{*} = SS^{\textcircled{\dagger}}$, hence

$$\begin{bmatrix} H & I \\ I & H \end{bmatrix}^{*}\begin{bmatrix} C & D \\ D & C \end{bmatrix}^{*} = \begin{bmatrix} C & D \\ D & C \end{bmatrix}\begin{bmatrix} H & I \\ I & H \end{bmatrix},$$

and get

$$(CH + DI)^{*} = CH + DI,\ (CI + DH)^{*} = CI + DH.$$

We have

$$[(C + D)(H + I)]^{*} = (C + D)(H + I),\ [(C - D)(H - I)]^{*} = (C - D)(H - I).$$

We know $S^{\textcircled{\dagger}}S^{k+1} = S^{k}$, therefore

$$\begin{bmatrix} H & I \\ I & H \end{bmatrix}\begin{bmatrix} C & D \\ D & C \end{bmatrix}^{k+1} = \begin{bmatrix} C & D \\ D & C \end{bmatrix}^{k}.$$

According to [16], we have

$$(H + I)(C + D)^{k_1+1} = (C + D)^{k_1},\ (H - I)(C - D)^{k_2+1} = (C - D)^{k_2},$$

where, $k_1 = ind(A)$ and $k_2 = ind(|A|)$. Therefore, $S^{\textcircled{\dagger}}$ have the structure given by (3.2). In order to calculate H and I, we know

$$H + I = (C + D)^{\textcircled{\dagger}},\ H - I = (C - D)^{\textcircled{\dagger}},$$

and consequently we get,

$$H = \frac{1}{2}[(C + D)^{\textcircled{\dagger}} + (C - D)^{\textcircled{\dagger}}],\ I = \frac{1}{2}[(C + D)^{\textcircled{\dagger}} - (C - D)^{\textcircled{\dagger}}].$$

□

Theorem 3.2. *Let S be the coefficient matrix of consistent (2.6) with $X \in S^k$. If $S^{\textcircled{\dagger}}$ is a non-negative matrix satisfying (3.2), then one of the consistent (2.6) represents the unique solution matrix $X^0 = S^{\textcircled{\dagger}}Y$, then the correlated fuzzy linear matrix $\tilde{X}^0$ is a strong solution of consistent GFME (2.5).*

Proof. We know $X^0 = S^{\textcircled{\dagger}}Y$, then

$$\underline{X}^0 = \begin{bmatrix} H & I \end{bmatrix} Y, \tag{3.3}$$

$$\bar{X}^0 = \begin{bmatrix} -I & -H \end{bmatrix} Y. \tag{3.4}$$

Subtract the above two formulas, we get

$$\begin{aligned} \bar{X}^0 - \underline{X}^0 &= \begin{bmatrix} -H & -I \end{bmatrix} Y - \begin{bmatrix} H & I \end{bmatrix} Y \\ &= \begin{bmatrix} -(I + H) & -(Z + I) \end{bmatrix} Y \\ &= (H + Z)(\bar{Y} - \underline{Y}). \end{aligned}$$

Then

$$\bar{X}^0 - \underline{X}^0 = (H + I)(\bar{Y} - \underline{Y}). \tag{3.5}$$

Because $H + I$ is nonnegative and $\bar{Y} \geq \underline{Y}$. Since $\bar{Y}$ is non-decreasing and $\underline{Y}$ is non-increasing, then $X^0 = S^{\textcircled{\dagger}}Y$ holds if $\bar{X}$ and $\underline{X}$ are non-decreasing and non-increasing, respectively. The bounded left continuity of $\bar{X}$ and $\underline{X}$ are obvious since they are the linear combinations of $\bar{Y}$ and $\underline{Y}$, respectively. □

Corollary 3.3. *Let S be the coefficient matrix of inconsistent (2.6) and $X \in S^k$. If $S^{\textcircled{\dagger}}$ is a non-negative matrix satisfying (3.2), then one of the inconsistent (2.6) represents the unique least squares solution matrix $X^0 = S^{\textcircled{\dagger}}Y$, then the correlated fuzzy linear matrix $\tilde{X}^0$ is a least squares solution of inconsistent GFME (2.5).*

Theorem 3.4. $S^{\textcircled{\dagger}} \geq 0$ *if and only if*

$$S^{\textcircled{\dagger}} = \begin{bmatrix} BC^* & BD^* \\ BD^* & BC^* \end{bmatrix} \tag{3.6}$$

for some positive diagonal matrix B. Meanwhile, $(C + D)^{\textcircled{\dagger}} = B(C + D)^*$, $(C - D)^{\textcircled{\dagger}} = B(C - D)^*$.

Proof. According to [4], it is clear that $S^{\textcircled{\dagger}} \geq 0$ if and only if $S^{\textcircled{\dagger}} = B^{\bullet} S^{*}$ for some positive diagonal matrix $B^{\bullet} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$. We have

$$\begin{bmatrix} \frac{1}{2}[(C+D)^{\textcircled{\dagger}} + (C-D)^{\textcircled{\dagger}}] & \frac{1}{2}[(C+D)^{\textcircled{\dagger}} - (C-D)^{\textcircled{\dagger}}] \\ \frac{1}{2}[(C+D)^{\textcircled{\dagger}} - (C-D)^{\textcircled{\dagger}}] & \frac{1}{2}[(C+D)^{\textcircled{\dagger}} + (C-D)^{\textcircled{\dagger}}] \end{bmatrix} = \begin{bmatrix} B_1 C^* & B_1 D^* \\ B_2 D^* & B_2 C^* \end{bmatrix}.$$

Therefore, $B_1 C^* = B_2 C^*$ and $B_1 D^* = B_2 D^*$. Let $B_1 = diag(b_{11}, b_{12}, \ldots, b_{1n})$, $B_2 = diag(b_{21}, b_{22}, \ldots, b_{2n})$,

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \ D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}.$$

We have

$$B_1 C^* = \begin{bmatrix} b_{11}c_{11} & b_{11}c_{21} & \cdots & b_{11}c_{n1} \\ b_{12}c_{12} & b_{12}c_{22} & \cdots & b_{12}c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n}c_{1n} & b_{1n}c_{2n} & \cdots & b_{1n}c_{nn} \end{bmatrix} = \begin{bmatrix} b_{21}c_{11} & b_{21}c_{21} & \cdots & b_{21}c_{n1} \\ b_{22}c_{12} & b_{22}c_{22} & \cdots & b_{22}c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{2n}c_{1n} & b_{2n}c_{2n} & \cdots & b_{2n}c_{nn} \end{bmatrix} = B_2 C^*,$$

$$B_1 D^* = \begin{bmatrix} b_{11}d_{11} & b_{11}d_{21} & \cdots & b_{11}d_{n1} \\ b_{12}d_{12} & b_{12}d_{22} & \cdots & b_{12}d_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n}d_{1n} & b_{1n}d_{2n} & \cdots & b_{1n}d_{nn} \end{bmatrix} = \begin{bmatrix} b_{21}d_{11} & b_{21}d_{21} & \cdots & b_{21}d_{n1} \\ b_{22}d_{12} & b_{22}d_{22} & \cdots & b_{22}d_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{2n}d_{1n} & b_{2n}d_{2n} & \cdots & b_{2n}d_{nn} \end{bmatrix} = B_2 D^*.$$

From the structure of the 2.7, of $c_{1i}, \cdots, c_{ni}$, $d_{1i}, \cdots, d_{ni}$ $(i = 1, ..., n)$, at least one is nonzero. Let $c_{ni} \neq 0$, we know $b_{1i}c_{ni} = b_{2i}c_{ni}$, then $c_{1i} = c_{2i}$ $(i = 1, ..., n)$, etc. We know $B_1 = B_2 = B$. Since

$$S^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{2}[(C+D)^{\textcircled{\dagger}} + (C-D)^{\textcircled{\dagger}}] & \frac{1}{2}[(C+D)^{\textcircled{\dagger}} - (C-D)^{\textcircled{\dagger}}] \\ \frac{1}{2}[(C+D)^{\textcircled{\dagger}} - (C-D)^{\textcircled{\dagger}}] & \frac{1}{2}[(C+D)^{\textcircled{\dagger}} + (C-D)^{\textcircled{\dagger}}] \end{bmatrix} = \begin{bmatrix} BC^* & BD^* \\ BD^* & BC^* \end{bmatrix},$$

it is easy to obtain $(C+D)^{\textcircled{\dagger}} = B(C+D)^*$, $(C-D)^{\textcircled{\dagger}} = B(C-D)^*$. □

4. The general strong fuzzy solutions of GFME

Now, we show the general solutions of the GFME (2.5). First, we seek a fuzzy number matrix $\tilde{X}^{\imath}$ which refers to general solutions set of GFME (2.5). Let $F \in \mathbb{C}_{n\times n}$, $F = A^{\textcircled{\dagger}}$ and $|F| = [|f_{ij}|]$. Let the form of $S_F \in \mathbb{C}_{2n\times 2n}$ be as follows

$$S_F = \begin{bmatrix} F^+ & F^- \\ F^- & F^+ \end{bmatrix}, \tag{4.1}$$

where $F^+ = [f_{ij}^+]$ and $F^- = [f_{ij}^-]$. For any representative matrix Y, let $X^{\imath} = S_F Y$. Since F^+, F^- and S_F are non-negative, similar to the proof of Theorem 3.2, we can deduce that $\tilde{X}^{\imath}$ is a fuzzy number matrix, even if (2.5) has no solution.

Theorem 4.1. *$A \in \mathbb{R}_n^{CM}$ is a singular coefficient matrix of the consistent GFME (2.5), where $\tilde{Y}$ is a column of fuzzy matrix as the GFME (2.5). If $X^\iota = S_F Y,\ F = A^{\textcircled{\dagger}},\ |F| = [|f_{ij}|]$, where S_F is in the form (4.1). The following statements hold:*
(i) $A(\bar{X}^\iota + \underline{X}^\iota) = \bar{Y} + \underline{Y}$.
(ii) If $|F|$ is one of Core-EP inverse of $|A|$, then $|A|(\bar{X}^\iota - \underline{X}^\iota) = \bar{Y} - \underline{Y}$, and $\tilde{X}^\iota$ is a solution of (2.5).

Proof. (i) Since $X^\iota = S_F Y$, we have

$$\underline{X}^\iota = \begin{bmatrix} F^+ & F^- \end{bmatrix} Y, \tag{4.2}$$

$$\bar{X}^\iota = \begin{bmatrix} -F^- & -F^+ \end{bmatrix} Y. \tag{4.3}$$

Add (4.2) and (4.3) together to get the following form

$$\begin{aligned} \bar{X}^\iota + \underline{X}^\iota &= \begin{bmatrix} -F^- & -F^+ \end{bmatrix} Y + \begin{bmatrix} F^+ & F^- \end{bmatrix} Y \\ &= \begin{bmatrix} (F^+ - F^-) & -(F^+ - F^-) \end{bmatrix} Y \\ &= \begin{bmatrix} F & -F \end{bmatrix} Y. \end{aligned}$$

Then

$$\bar{X}^\iota + \underline{X}^\iota = F(\bar{Y} + \underline{Y}). \tag{4.4}$$

Since the GFME (2.5) is consistent. Then, $A(\bar{X} + \underline{X}) = \bar{Y} + \underline{Y}$ has solution (for $\alpha \in [0, 1]$). Furthermore, $F = A^{\textcircled{\dagger}}$, so it follows from (4.4) that

$$\bar{Y} + \underline{Y} = A(\bar{X}^\iota + \underline{X}^\iota). \tag{4.5}$$

(ii) Subtracting (4.2) and (4.3) together to get the following form:

$$\bar{X}^\iota - \underline{X}^\iota = |F|(\bar{Y} - \underline{Y}). \tag{4.6}$$

Since $|F|$ is one of Core-EP inverse of $|A|$, we have $|A|^{\textcircled{\dagger}} = |A^{\textcircled{\dagger}}| = |F|$ then

$$|A|^{\textcircled{\dagger}} = |F| = F^+ + F^-,\ A^{\textcircled{\dagger}} = F = F^+ - F^-.$$

According to the Theorem 3.2, we have $H = F^+,\ Z = F^-$. Then

$$\begin{aligned} H &= \frac{1}{2}[|A|^{\textcircled{\dagger}} + A^{\textcircled{\dagger}}], \\ I &= \frac{1}{2}[|A|^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}]. \end{aligned}$$

Since $S^{\textcircled{\dagger}} = S_F$, therefore X^ι is a solution to GFME (2.5). Through (4.6) we have

$$\bar{Y} - \underline{Y} = |A|(\bar{X}^\iota - \underline{X}^\iota). \tag{4.7}$$

Any matrix $A \in \mathbb{R}_n^{CM}$ has $A^+ = \frac{1}{2}(|A| + A)$ and $A^- = \frac{1}{2}(|A| - A)$, and (4.5), (4.7) are added and subtracted. We obtain

$$\begin{aligned} \bar{Y} &= A^+ \bar{X}^\iota - A^- \underline{X}^\iota = \begin{bmatrix} -A^- & -A^+ \end{bmatrix} X^\iota, \\ \underline{Y} &= -A^- \bar{X}^\iota + A^+ \underline{X}^\iota = \begin{bmatrix} A^+ & A^- \end{bmatrix} X^\iota. \end{aligned}$$

Therefore, the conclusion is proved. □

We will study a form for correlated the general strong solutions to GFME (2.5) in the following theorem.

Theorem 4.2. *$A \in \mathbb{R}_n^{CM}$ is a singular coefficient matrix of the consistent GFME (2.5), an arbitrary fuzzy matrix $\tilde{Y}$, since $X^\iota = S_F Y$ it have $A(\bar{X}^\iota + \underline{X}^\iota) = \bar{Y} + \underline{Y}$. Let $W = \begin{pmatrix} w_{11}(\alpha) & \cdots & w_{1m}(\alpha) \\ \vdots & & \vdots \\ w_{n1}(\alpha) & \cdots & w_{nm}(\alpha) \end{pmatrix}$, define $W = \underline{Y} - \begin{bmatrix} A^+ & A^- \end{bmatrix} X^\iota$ where $\begin{bmatrix} A^+ & A^- \end{bmatrix}$ is $n \times 2n$ order matrix. Define $\Lambda = \begin{pmatrix} \lambda_{11}(\alpha) & \cdots & \lambda_{1m}(\alpha) \\ \vdots & & \vdots \\ \lambda_{n1}(\alpha) & \cdots & \lambda_{nm}(\alpha) \end{pmatrix}$ and $\Theta = \begin{pmatrix} \Theta_{11}(\alpha) & \cdots & \Theta_{1m}(\alpha) \\ \vdots & & \vdots \\ \Theta_{n1}(\alpha) & \cdots & \Theta_{nm}(\alpha) \end{pmatrix}$, where Λ and Θ are solutions of $A\Lambda = 0$ and $|A|\Theta = W$, respectively. We have*

$$\tilde{X} = \{\underline{X}^\iota + \frac{1}{2}\Lambda + \Theta,\ \bar{X}^\iota + \frac{1}{2}\Lambda - \Theta\}.$$

Proof. The proof can be found in the proof of [[15, Theorem 8]]. □

Next we will present an algorithm to solve the GFME (2.5). The coefficient matrix of GFME (2.5) is $A = [a_{ij}]$. The matrix S_F is given by the formula (4.1), and F is the Core-EP inverse of the matrix A.

Algorithm 2.

1. *Calculate $X^\iota = S_F Y$, if equation $A(\bar{X}^\iota + \underline{X}^\iota) = \bar{Y} + \underline{Y}$ is satisfied, proceed to the next step.*
2. *Let $\Lambda = \begin{pmatrix} \lambda_{11}(\alpha) & \cdots & \lambda_{1m}(\alpha) \\ \vdots & & \vdots \\ \lambda_{n1}(\alpha) & \cdots & \lambda_{nm}(\alpha) \end{pmatrix}$ $\alpha \in [0, 1]$ satisfy the homogeneous equation $A\Lambda = 0$.*
   Then, $\underline{X}^{\iota\Lambda} = \underline{X}^\iota + \frac{1}{2}\Lambda,\ \bar{X}^{\iota\Lambda} = \bar{X}^\iota + \frac{1}{2}\Lambda$.
3. *Calculate $W = \begin{pmatrix} w_{11}(\alpha) & \cdots & w_{1m}(\alpha) \\ \vdots & & \vdots \\ w_{n1}(\alpha) & \cdots & w_{nm}(\alpha) \end{pmatrix}$, $\alpha \in [0, 1]$, by using $W = \underline{Y} - \underline{S}X^\iota$,*
   where $\underline{S} = [A^+ \ \ A^-]$ is an $n \times 2n$ matrix.
4. *If the family of classical systems $|A|\Theta = W$, where $W = \begin{pmatrix} w_{11}(\alpha) & \cdots & w_{1m}(\alpha) \\ \vdots & & \vdots \\ w_{n1}(\alpha) & \cdots & w_{nm}(\alpha) \end{pmatrix}$, have a solution $\Theta = \begin{pmatrix} \Theta_{11}(\alpha) & \cdots & \Theta_{1m}(\alpha) \\ \vdots & & \vdots \\ \Theta_{n1}(\alpha) & \cdots & \Theta_{nm}(\alpha) \end{pmatrix}$, $\alpha \in [0, 1]$, then : $\underline{X} = \underline{X}^{\iota\Lambda} + \Theta,\ \bar{X} = \bar{X}^{\iota\Lambda} - \Theta$.*
5. *From all determined Θ, Λ and for each $\alpha \in [0, 1]$, we have $\theta_{ij}(\alpha) \leq \dfrac{\bar{x}^\iota_{ij}(\alpha) - \underline{x}^\iota_{ij}(\alpha)}{2}$, $i = 1, \ldots, n,\ j = 1, \ldots, m$, where $\underline{x}^\iota_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) + \theta_{ij}(\alpha)$ ($\underline{x}^\iota_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) - \theta_{ij}(\alpha)$) is monotonic bounded non − decreasing (monotonic bounded non − increasing) left continuous function.*

We will explain our previous Theorems, Definitions and validity of Algorithm through some examples. Exampe 4.1 is a 2×2 order inconsistent fuzzy matrix equation with $X \in R(S)$. In Example 4.1, A

and $|A|$ are singular, $ind(A) = ind(|A|) = ind(S) = 1$, and $S^{\textcircled{\dagger}}$ is nonnegative. We know that we can give a least squares strong fuzzy solution of Exampe 4.1 through the unique least squares solution matrix $X^0 = S^{\textcircled{\dagger}}Y$. Example 4.2 is a 4×4 order consistent fuzzy matrix equation. In Example 4.2, A and S are singular, and $|A|$ is reversible. Moreover, we know $ind(A) = ind(S) = 2$, and $ind(|A|) = 0$. Next, we will get the general strong fuzzy solution of Exampe 4.2 through above Algorithm. In Example 4.3, we constrain Example 4.2 to satisfy $X \in R(A^2)$. Next, we will get a strong fuzzy solution of Exampe 4.1 by the unique least squares solution matrix $X^0 = S^{\textcircled{\dagger}}Y$.

Example 4.1. *It is a 2×2 order inconsistent fuzzy matrix equation with $X \in R(A)$ as floown:*

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (-1+\alpha, 1-\alpha) & (-2+2\alpha, 2-2\alpha) \\ (-2+\alpha, 2-3\alpha) & (-2+3\alpha, 4-3\alpha) \end{pmatrix}.$$

The model fuzzy matrix equation is as follows:

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 4 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 4 & 0 \end{pmatrix}\begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ -\bar{x}_{11} & -\bar{x}_{12} \\ -\bar{x}_{21} & -\bar{x}_{22} \end{pmatrix} = \begin{pmatrix} -1+\alpha & -2+2\alpha \\ -2+\alpha & -2+3\alpha \\ -1+\alpha & -2+2\alpha \\ -2+3\alpha & -4+3\alpha \end{pmatrix}.$$

According to Algorithm 1, we have

$$S^{\textcircled{\dagger}} = \begin{pmatrix} 0 & 0.1000 & 0.05 & 0 \\ 0.1000 & 0 & 0 & 0.2000 \\ 0.0500 & 0 & 0 & 0.1000 \\ 0 & 0.2000 & 0.1000 & 0 \end{pmatrix} = \begin{bmatrix} BC^* & BD^* \\ BD^* & BC^* \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0.025 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

By formula $X(\alpha) = S^{\textcircled{\dagger}}Y$, we obtain the unique least squares matrix solution as follow:

$$X(\alpha) = \begin{pmatrix} -0.2500+0.1500\alpha & -0.3000+0.4000\alpha \\ -0.5000+0.7000\alpha & -1.0000+0.8000\alpha \\ -0.2500+0.3500\alpha & -0.5000+0.4000\alpha \\ -0.5000+0.3000\alpha & -0.6000+0.8000\alpha \end{pmatrix}.$$

Then, we obtain a least squares strong fuzzy matrix solution $\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix}$ as follow:

$$\begin{aligned}
\tilde{x}_{11} &= (-0.2500+0.1500\alpha,\ 0.2500-0.3500\alpha), \\
\tilde{x}_{21} &= (-0.5000+0.7000\alpha,\ 0.5000-0.3000\alpha), \\
\tilde{x}_{12} &= (-0.3000+0.4000\alpha,\ 0.5000-0.4000\alpha), \\
\tilde{x}_{22} &= (-1.0000+0.8000\alpha,\ 0.6000-0.8000\alpha).
\end{aligned}$$

Example 4.2. *It is a* 4×4 *order consistent fuzzy matrix equation as follow:*

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \\ \tilde{x}_{41} & \tilde{x}_{42} \end{pmatrix} = \begin{pmatrix} (-45+39\alpha,\ 33-39\alpha) & (-23+29\alpha,\ 35-29\alpha) \\ (-57+48\alpha,\ 39-48\alpha) & (-28+37\alpha,\ 46-37\alpha) \\ (-84+69\alpha,\ 54-69\alpha) & (-37+52\alpha,\ 67-52\alpha) \\ (-66+63\alpha,\ 60-63\alpha) & (-45+48\alpha,\ 51-48\alpha) \end{pmatrix}.$$

According to Algorithm 1, we have

$$A^{\textcircled{\dagger}} = \begin{pmatrix} 0.0342 & 0.0513 & 0.0855 & 0.0171 \\ 0.0513 & 0.0769 & 0.1282 & 0.0256 \\ 0.0855 & 0.1282 & 0.2137 & 0.0427 \\ 0.0171 & 0.0256 & 0.0427 & 0.0085 \end{pmatrix}.$$

Then,

$$S_F = \begin{bmatrix} 0.0342 & 0.0513 & 0.0855 & 0.0171 & 0 & 0 & 0 & 0 \\ 0.0513 & 0.0769 & 0.1282 & 0.0256 & 0 & 0 & 0 & 0 \\ 0.0855 & 0.1282 & 0.2137 & 0.0427 & 0 & 0 & 0 & 0 \\ 0.0171 & 0.0256 & 0.0427 & 0.0085 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0342 & 0.0513 & 0.0855 & 0.0171 \\ 0 & 0 & 0 & 0 & 0.0513 & 0.0769 & 0.1282 & 0.0256 \\ 0 & 0 & 0 & 0 & 0.0855 & 0.1282 & 0.2137 & 0.0427 \\ 0 & 0 & 0 & 0 & 0.0171 & 0.0256 & 0.0427 & 0.0085 \end{bmatrix}.$$

By formula $X^{\iota} = S_F Y$, *we obtain matrix as follow:*

$$X^{\iota} = \begin{pmatrix} -12.7737+10.7730\alpha & -6.1560+8.1567\alpha \\ -19.1502+16.1505\alpha & -9.2285+12.2282\alpha \\ -31.9239+26.9235\alpha & -15.3845+20.3849\alpha \\ -6.3765+5.3775\alpha & -3.0725+4.0715\alpha \\ -8.7723+10.7730\alpha & -10.1574+8.1567\alpha \\ -13.1508+16.1505\alpha & -15.2279+12.2282\alpha \\ -21.9231+26.9235\alpha & -25.3853+20.3849\alpha \\ -4.3785+5.3775\alpha & -5.0705+4.0715\alpha \end{pmatrix}.$$

Then, we obtain a strong fuzzy matrix $\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \\ \tilde{x}_{41} & \tilde{x}_{42} \end{pmatrix}$ *as follow:*

$$\tilde{X}^{\iota} = \begin{pmatrix} (-12.7737+10.7730\alpha,\ 8.7723-10.7730\alpha) & (-6.1560+8.1567\alpha,\ 10.1574-8.1567\alpha) \\ (-19.1502+16.1505\alpha,\ 13.1508-16.1505\alpha) & (-9.2285+12.2282\alpha,\ 15.2279-12.2282\alpha) \\ (-31.9239+26.9235\alpha,\ 21.9231-26.9235\alpha) & (-15.3845+20.3849\alpha,\ 25.3853-20.3849\alpha) \\ (-6.3765+5.3775\alpha,\ 4.3785-5.3775\alpha) & (-3.0725+4.0715\alpha,\ 5.0705-4.0715\alpha) \end{pmatrix}.$$

The solution matrix for equation $A\Lambda = 0$ *is* $\begin{pmatrix} 2f(\alpha) & 2f(\alpha) \\ f(\alpha) & f(\alpha) \\ f(\alpha) & f(\alpha) \\ -4f(\alpha) & -4f(\alpha) \end{pmatrix}$. *Let* $f(\alpha) \in \mathcal{F}^{\iota}$, *where* $\mathcal{F}^{\iota}$ *(depends on* $\tilde{X}^{\iota}$*) denotes the class of functions on the unite interval* $y = f(\alpha)$, *such that the adequate functions* $\tilde{X}^{\iota\Lambda}$ *is* *monotonic and continuous:*

$$\tilde{X}^{\iota\Lambda} = \begin{pmatrix} (-12.7737+10.7730\alpha+f(\alpha),\ 8.7723-10.7730\alpha+f(\alpha)) & (-6.1560+8.1567\alpha+f(\alpha),\ 10.1574-8.1567\alpha+f(\alpha)) \\ (-19.1502+16.1505\alpha+\frac{1}{2}f(\alpha),\ 13.1508-16.1505\alpha+\frac{1}{2}f(\alpha)) & (-9.2285+12.2282\alpha+\frac{1}{2}f(\alpha),\ 15.2279-12.2282\alpha+\frac{1}{2}f(\alpha)) \\ (-31.9239+26.9235\alpha+\frac{1}{2}f(\alpha),\ 21.9231-26.9235\alpha+\frac{1}{2}f(\alpha)) & (-15.3845+20.3849\alpha+\frac{1}{2}f(\alpha),\ 25.3853-20.3849\alpha+\frac{1}{2}f(\alpha)) \\ (-6.3765+5.3775\alpha-2f(\alpha),\ 4.3785-5.3775\alpha-2f(\alpha)) & (-3.0725+4.0715\alpha-2f(\alpha),\ 5.0705-4.0715\alpha-2f(\alpha)) \end{pmatrix}.$$

By formula $W = \underline{Y} - \underline{S}X^{\iota}$ for each Λ, we have

$$W = \begin{pmatrix} 14.8464 - 14.8470\alpha & 11.7704 - 11.7698\alpha \\ 0.4506 - 0.4515\alpha & -0.3145 + 0.3154\alpha \\ 11.7510 - 11.7525\alpha & 9.1425 - 9.1410\alpha \\ 6.9972 - 6.9975\alpha & 4.9983 - 4.9980\alpha \end{pmatrix}.$$

By formula $|A|\Theta = W$, we have

$$\Theta = \begin{pmatrix} 1.7730 - 1.7730\alpha & 1.1567 - 1.1567\alpha \\ 4.1499 - 4.1505\alpha & 3.2288 - 3.2282\alpha \\ 8.9235 - 8.9235\alpha & 7.3849 - 7.3849\alpha \\ -9.6222 + 9.6225\alpha & -7.9288 + 7.9285\alpha \end{pmatrix}.$$

Then, we obtain the general strong fuzzy matrix solutions as follow:

$$\tilde{X} = \begin{pmatrix} (-11.0007+9.0000\alpha+f(\alpha),\ 6.9993-9.0000\alpha+f(\alpha)) & (-4.9993+7.0000\alpha+f(\alpha),\ 9.0007-7.0000+f(\alpha)) \\ (-15.0003+12.0000\alpha+\frac{1}{2}f(\alpha),\ 9.0009-12.0000\alpha+\frac{1}{2}f(\alpha)) & (-5.9997+9.0000\alpha+\frac{1}{2}f(\alpha),\ 11.9991-9.0000\alpha+\frac{1}{2}f(\alpha)) \\ (-23.0004+18.0000\alpha+\frac{1}{2}f(\alpha),\ 12.9996-18.0000\alpha+\frac{1}{2}f(\alpha)) & (-7.9996+13.0000\alpha+\frac{1}{2}f(\alpha),\ 18.0004-13.0000\alpha+\frac{1}{2}f(\alpha)) \\ (-15.9987+15.0000\alpha-2f(\alpha),\ 14.0007-15.0000\alpha-2f(\alpha)) & (-11.0013+12.0000\alpha-2f(\alpha),\ 12.9993-12.0000\alpha-2f(\alpha)) \end{pmatrix}.$$

Example 4.3. *It is a 4×4 order consistent fuzzy matrix equation with $X \in R(A^2)$ as follow:*

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \\ \tilde{x}_{41} & \tilde{x}_{42} \end{pmatrix} = \begin{pmatrix} (-45 + 39\alpha,\ 33 - 39\alpha) & (-23 + 29\alpha,\ 35 - 29\alpha) \\ (-57 + 48\alpha,\ 39 - 48\alpha) & (-28 + 37\alpha,\ 46 - 37\alpha) \\ (-84 + 69\alpha,\ 54 - 69\alpha) & (-37 + 52\alpha,\ 67 - 52\alpha) \\ (-66 + 63\alpha,\ 60 - 63\alpha) & (-45 + 48\alpha,\ 51 - 48\alpha) \end{pmatrix}.$$

The model fuzzy matrix equation is as follows:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ \underline{x}_{31} & \underline{x}_{32} \\ \underline{x}_{31} & \underline{x}_{42} \\ -\bar{x}_{11} & -\bar{x}_{12} \\ -\bar{x}_{21} & -\bar{x}_{22} \\ -\bar{x}_{31} & -\bar{x}_{32} \\ -\bar{x}_{41} & -\bar{x}_{42} \end{pmatrix} = \begin{pmatrix} -45 + 39\alpha & -23 + 29\alpha \\ -57 + 48\alpha & -28 + 37\alpha \\ -84 + 69\alpha & -37 + 52\alpha \\ -66 + 63\alpha & -45 + 48\alpha \\ -33 + 39\alpha & -35 + 29\alpha \\ -39 + 48\alpha & -46 + 37\alpha \\ -54 + 69\alpha & -67 + 52\alpha \\ -60 + 63\alpha & -51 + 48\alpha \end{pmatrix}.$$

According to Algorithm 1, we have

$$S^{\textcircled{\dagger}} = \begin{pmatrix} 0.2671 & 0.2756 & -0.2073 & 0.0085 & 0.2329 & 0.2244 & -0.2927 & -0.0085 \\ 0.2756 & 0.2885 & 0.0641 & -0.2372 & 0.2244 & 0.2115 & -0.0641 & -0.2628 \\ 0.0427 & -0.4359 & 0.3568 & 0.2714 & -0.0427 & -0.5641 & 0.1432 & 0.2286 \\ -0.7415 & -0.2372 & 0.2714 & 0.5043 & -0.7585 & -0.2628 & 0.2286 & 0.4957 \\ 0.2329 & 0.2244 & -0.2927 & -0.0085 & 0.2671 & 0.2756 & -0.2073 & 0.0085 \\ 0.2244 & 0.2115 & -0.0641 & -0.2628 & 0.2756 & 0.2885 & 0.0641 & -0.2372 \\ -0.0427 & -0.5641 & 0.1432 & 0.2286 & 0.0427 & -0.4359 & 0.3568 & 0.2714 \\ -0.7585 & -0.2628 & 0.2286 & 0.4957 & -0.7415 & -0.2372 & 0.2714 & 0.5043 \end{pmatrix}.$$

By formula $X = S^{\textcircled{\dagger}}Y$, we obtain the unique matrix solution as follow:

$$X(\alpha) = \begin{pmatrix} -11+9\alpha & -5+7\alpha \\ -15+12\alpha & -6+9\alpha \\ -23+18\alpha & -8+13\alpha \\ -16+15\alpha & -11+12\alpha \\ -7+9\alpha & -9+7\alpha \\ -9+12\alpha & -12+9\alpha \\ -13+18\alpha & -18+13\alpha \\ -14+15\alpha & -13+12\alpha \end{pmatrix}.$$

Then, we obtain a strong fuzzy matrix solution $\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \\ \tilde{x}_{31} & \tilde{x}_{32} \\ \tilde{x}_{41} & \tilde{x}_{42} \end{pmatrix}$ as follow:

$$\tilde{X} = \begin{pmatrix} (-11+9\alpha,\ 7-9\alpha) & (-5+7\alpha,\ 9-7\alpha) \\ (-15+12\alpha,\ 9-12\alpha) & (-6+9\alpha,\ 12-9\alpha) \\ (-23+18\alpha,\ 13-18\alpha) & (-8+13\alpha,\ 18-13\alpha) \\ (-16+15\alpha,\ 14-15\alpha) & (-11+12\alpha,\ 13-12\alpha) \end{pmatrix}.$$

5. Conclusions

In this paper, the Algorithm 2 is proposed to solve the GFME (2.5) whose the coefficient matrix is a real matrix. We build the Algorithm 1 for getting the Core-EP inverse, and the numerical Algorithm 2 for finding an arbitrary solution of the GFME (2.5). The method is also connected to the original Gong and Guo. approach from [13]. Moreover, If inconsistent (2.6) satisfies $X \in R(S^k)$, the unique least squares solution of inconsistent general fuzzy matrix equation are given by Core-EP inverse. For future work, we try to solve "inconsistent GFME (2.5)" and discuss about their general least squares solution sets.

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Conflict of interest

The authors declare no conflict of interest.

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