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*Research article*

# Some new generalized $\kappa$–fractional Hermite–Hadamard–Mercer type integral inequalities and their applications

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**Abstract:** In this paper, we have established some new Hermite–Hadamard–Mercer type of inequalities by using $\kappa$–Riemann–Liouville fractional integrals. Moreover, we have derived two new integral identities as auxiliary results. From the applied identities as auxiliary results, we have obtained some new variants of Hermite–Hadamard–Mercer type via $\kappa$–Riemann–Liouville fractional integrals. Several special cases are deduced in detail and some know results are recaptured as well. In order to illustrate the efficiency of our main results, some applications regarding special means of positive real numbers and error estimations for the trapezoidal quadrature formula are provided as well.

**Keywords:** Hermite–Hadamard inequality; Jensen–Mercer inequality; Hölder's inequality; power mean inequality; special means; error estimation
**Mathematics Subject Classification:** 26A33, 26A51, 26D07, 26D10, 26D15

## 1. Introduction and preliminaries

Theory of inequalities play pivotal role in almost all branches of pure and applied mathematics. Theory of convex functions has played vital role in the development of theory of inequalities. In modern analysis many inequalities are direct consequences of the applications of convexity property of the functions. One of the most extensively as well as intensively studied inequality pertaining to convexity property of the functions is Hermite–Hadamard's inequality. This inequality provides necessary and sufficient condition for a function to be convex. It reads as: Let $\Phi : I = [\mathfrak{b}_1, \mathfrak{b}_2] \subset \mathbb{R} \mapsto \mathbb{R}$ be a convex function on closed interval $[\mathfrak{b}_1, \mathfrak{b}_2]$, then

$$\Phi\left(\frac{\mathfrak{b}_1 + \mathfrak{b}_2}{2}\right) \leq \frac{1}{\mathfrak{b}_2 - \mathfrak{b}_1} \int_{\mathfrak{b}_1}^{\mathfrak{b}_2} \Phi(\tau) d\tau \leq \frac{\Phi(\mathfrak{b}_1) + \Phi(\mathfrak{b}_2)}{2}.$$

In recent years several successful attempts have been made in obtaining novel improvements and generalizations of Hermite–Hadamard's inequality, see [1–4]. Dragomir and Pearce [5] have written a very informative monograph on Hermite–Hadamard's inequality and its applications. Interested readers can find very useful details pertaining to these inequalities. Another remarkable inequality which has played significant role in theory of inequalities is Jensen's inequality, see [6]. It reads as: Let $\Phi$ be a convex function on $[\mathfrak{b}_1, \mathfrak{b}_2]$, then for all $x_i \in [\mathfrak{b}_1, \mathfrak{b}_2]$ and $\mu_i \in [0, 1]$, where $i = 1, 2, \ldots, n$, we have

$$\Phi\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \sum_{i=1}^{n} \mu_i \Phi(x_i).$$

Following inequality is known as Jensen–Mercer's inequality in the literature:

$$\Phi\left(\mathfrak{b}_1 + \mathfrak{b}_2 - \sum_{i=1}^{n} \mu_i x_i\right) \leq \Phi(\mathfrak{b}_1) + \Phi(\mathfrak{b}_2) - \sum_{i=1}^{n} \mu_i \Phi(x_i),$$

for $\mu_i \in [0, 1]$, where $\Phi$ is a convex function. For more details, see [7].

Pavić [8] presented the generalized version of Jensen–Mercer's inequality as: Assume that $\Phi : [\mathfrak{b}_1, \mathfrak{b}_2] \mapsto \mathbb{R}$ be a convex function, where $x_i \in [\mathfrak{b}_1, \mathfrak{b}_2]$ are $n$–points. Let $\alpha, \beta, \mu_i \in [0, 1]$, $\gamma \in [-1, 1]$ be coefficients of sums $\alpha + \beta + \gamma = \sum_{i=1}^{n} \mu_i = 1$, then

$$\Phi\left(\alpha \mathfrak{b}_1 + \beta \mathfrak{b}_2 + \gamma \sum_{i=1}^{n} \mu_i x_i\right) \leq \alpha \Phi(\mathfrak{b}_1) + \beta \Phi(\mathfrak{b}_2) + \gamma \sum_{i=1}^{n} \mu_i \Phi(x_i). \tag{1.1}$$

**Remark 1.1.** *Note that*

*1) If we take $\alpha = 1 = \beta$ and $\gamma = -1$ in (1.1), then we get Jensen–Mercer inequality.*
*2) If we choose $\alpha = 0 = \beta$ and $\gamma = 1$ in (1.1), then we obtain the well-known Jensen inequality.*

For some recent studies regarding Hermite-Hadamard-Mercer type inequalities, see [9, 10].

Fractional calculus is the branch of mathematics which deals with integrals and derivatives of any arbitrary real or complex order. The history of fractional calculus is old but in recent years it has received significant popularity and importance. This can be attributed mainly due to its great many applications in various fields of science and engineering. It provides many useful tools for solving differential equations, integral equations, and problems involving special functions of mathematical physics. Among several known forms of fractional integrals, the Riemann–Liouville fractional integral has been investigated extensively, which is defined as follows:

**Definition 1.1** ( [11]). *Let $\Phi \in L_1[\mathfrak{b}_1, \mathfrak{b}_2]$ (the set of all integrable functions on $[\mathfrak{b}_1, \mathfrak{b}_2]$). The Riemann–Liouville integrals $J^{\nu}_{\mathfrak{b}_1^+}\Phi$ and $J^{\nu}_{\mathfrak{b}_2^-}\Phi$ of order $\nu > 0$ are defined by*

$$J^{\nu}_{\mathfrak{b}_1^+}\Phi(x_1) = \frac{1}{\Gamma(\nu)} \int_{\mathfrak{b}_1}^{x_1} (x_1 - \tau)^{\nu-1}\Phi(\tau)d\tau, \quad x_1 > \mathfrak{b}_1,$$

*and*

$$J^{\nu}_{\mathfrak{b}_2^-}\Phi(x_1) = \frac{1}{\Gamma(\nu)} \int_{x_1}^{\mathfrak{b}_2} (\tau - x_1)^{\nu-1}\Phi(\tau)d\tau, \quad x_1 < \mathfrak{b}_2,$$

Mubeen and Habibullah [12] introduced the notion of $\kappa$-Riemann–Liouville fractional integrals as: Let $\Phi \in L_1[\mathfrak{b}_1, \mathfrak{b}_2]$, then

$$J^{\nu,\kappa}_{\mathfrak{b}_1^+}\Phi(x_1) = \frac{1}{\kappa\Gamma_\kappa(\nu)} \int_{\mathfrak{b}_1}^{x_1} (x_1 - \tau)^{\frac{\nu}{\kappa}-1}\Phi(\tau)d\tau, \quad x_1 > \mathfrak{b}_1,$$

$$J^{\nu,\kappa}_{\mathfrak{b}_2^-}\Phi(x_1) = \frac{1}{\kappa\Gamma_\kappa(\nu)} \int_{x_1}^{\mathfrak{b}_2} (\tau - x_1)^{\frac{\nu}{\kappa}-1}\Phi(\tau)d\tau, \quad x_1 < \mathfrak{b}_2,$$

where $\Gamma_\kappa(\nu) = \int_0^\infty \tau^{\nu-1}e^{\frac{\tau^\kappa}{\kappa}}d\tau$, $\Re(\nu) > 0$, $\kappa \in \mathbb{R}^+$ is the $\kappa$–gamma function which was introduced and studied in [13].

Sarikaya et al. [14] were the first to derive fractional analogue of Hermite–Hadamard's inequality. Since then blend of techniques both from fractional calculus and convex analysis have been used in obtaining various fractional analogues of classical inequalities. For more details, see [15–22].

Having inspiration from the ongoing research, we will establish some new Hermite–Hadamard–Mercer type of inequalities by using $\kappa$–Riemann–Liouville fractional integrals. Moreover, we will derive two new integral identities as auxiliary results. Applying two identities as auxiliary results, we will obtain some new variants of Hermite–Hadamard–Mercer type via $\kappa$–Riemann–Liouville fractional integrals. Several special cases will be deduce in details and some know results will be recaptured as well. In order to illustrate the efficiency of our main results, some applications regarding special means of positive real numbers and error estimations for trapezoidal quadrature formula will be provide as well.

## 2. Main results

In this section, we discuss our main results.

**Theorem 2.1.** *Assume that $\Phi : [\mathfrak{b}_1, \mathfrak{b}_2] \mapsto \mathbb{R}$ be a convex function. Let $\alpha, \beta, \in [0, 1]$, $\gamma \in (0, 1]$ be coefficients of sums $\alpha + \beta + \gamma = 1$ and $\nu, \kappa > 0$, then*

$$\begin{aligned}
&\Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\frac{x_1 + x_2}{2}\right) \\
&\leq \frac{\Gamma_\kappa(\nu+\kappa)}{2\gamma^{\frac{\nu}{\kappa}}(x_2 - x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)^-}\Phi\right)(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1) + \left(J^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)^+}\Phi\right)(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_2)\right] \\
&\leq \alpha\Phi(\mathfrak{b}_1) + \beta\Phi(\mathfrak{b}_2) + \gamma\frac{\Phi(x_1) + \Phi(x_2)}{2},
\end{aligned}$$

*holds for all $x_1, x_2 \in [\mathfrak{b}_1, \mathfrak{b}_2]$ with $x_1 < x_2$.*

*Proof.* Consider

$$\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_{11}+x_{21}}{2}\right)=\Phi\left(\frac{\alpha\flat_1+\beta\flat_2+\gamma x_{11}+\alpha\flat_1+\beta\flat_2+\gamma x_{21}}{2}\right).$$

Using change of variable technique, for $\alpha\flat_1+\beta\flat_2+\gamma x_{11}=\tau(\alpha\flat_1+\beta\flat_2+\gamma x_1)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_2)$ and $\alpha\flat_1+\beta\flat_2+\gamma x_{21}=(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_1)+\tau(\alpha\flat_1+\beta\flat_2+\gamma x_2)$, we have

$$\begin{aligned}\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+x_2}{2}\right)\leq\frac{1}{2}&\left[\Phi(\tau(\alpha\flat_1+\beta\flat_2+\gamma x_1)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_2))\right.\\&\left.+\Phi(\tau(\alpha\flat_1+\beta\flat_2+\gamma x_2)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_1))\right].\end{aligned}$$

Multiplying both side of above inequality $\tau^{\frac{\nu}{\kappa}-1}$ and integrating with respect to $\tau$ on $[0,1]$, we get

$$\begin{aligned}&\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+x_2}{2}\right)\\&\leq\frac{\nu}{2\kappa}\left[\int_0^1\tau^{\frac{\nu}{\kappa}-1}\Phi(\tau(\alpha\flat_1+\beta\flat_2+\gamma x_1)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_2))d\tau\right.\\&\quad\left.+\int_0^1\tau^{\frac{\nu}{\kappa}-1}\Phi(\tau(\alpha\flat_1+\beta\flat_2+\gamma x_2)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_1))d\tau\right].\end{aligned}$$

After simplify, we obtain

$$\begin{aligned}&\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+x_2}{2}\right)\\&\leq\frac{\Gamma_\kappa(\nu+\kappa)}{2\kappa\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}\Gamma_\kappa(\nu)}\left[\int_{\alpha\flat_1+\beta\flat_2+\gamma x_1}^{\alpha\flat_1+\beta\flat_2+\gamma x_2}(\alpha\flat_1+\beta\flat_2+\gamma x_2-u)^{\frac{\nu}{\kappa}-1}\Phi(u)du\right.\\&\quad\left.+\int_{\alpha\flat_1+\beta\flat_2+\gamma x_1}^{\alpha\flat_1+\beta\flat_2+\gamma x_2}(u-(\alpha\flat_1+\beta\flat_2+\gamma x_1))^{\frac{\nu}{\kappa}-1}\Phi(u)du\right].\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+x_2}{2}\right)\\&\leq\frac{\Gamma_\kappa(\nu+\kappa)}{2\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{(\alpha\flat_1+\beta\flat_2+\gamma x_2)^-}\Phi\right)(\alpha\flat_1+\beta\flat_2+\gamma x_1)+\left(J^{\nu,\kappa}_{(\alpha\flat_1+\beta\flat_2+\gamma x_1)^+}\Phi\right)(\alpha\flat_1+\beta\flat_2+\gamma x_2)\right].\end{aligned}$$

To prove second inequality, from convexity of $\Phi$, we have

$$\begin{aligned}&\Phi(\tau(\alpha\flat_1+\beta\flat_2+\gamma x_1)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_2))\\&\leq\tau\Phi(\alpha\flat_1+\beta\flat_2+\gamma x_1)+(1-\tau)\Phi(\alpha\flat_1+\beta\flat_2+\gamma x_2),\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\Phi(\tau(\alpha\flat_1+\beta\flat_2+\gamma x_2)+(1-\tau)(\alpha\flat_1+\beta\flat_2+\gamma x_1))\\&\leq\tau\Phi(\alpha\flat_1+\beta\flat_2+\gamma x_2)+(1-\tau)\Phi(\alpha\flat_1+\beta\flat_2+\gamma x_1).\end{aligned}\tag{2.2}$$

Adding inequalities (2.1) and (2.2), and then multiplying both side of above inequality by $\tau^{\frac{\nu}{\kappa}-1}$, and integrating with respect to $\tau$ on $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi(\tau(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)+(1-\tau)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2))d\tau\\
&+\int_0^1 \tau^{\frac{\nu}{\kappa}-1}\Phi(\tau(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)+(1-\tau)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1))d\tau\\
&\leq 2\frac{\nu}{\kappa}\left[\alpha\Phi(\mathfrak{b}_1)+\beta\Phi(\mathfrak{b}_2)+\gamma\frac{x_1+x_2}{2}\right].
\end{aligned}$$

After simple calculation, we obtain second part of our result. This completes our proof. □

**Corollary 2.1.** *If we choose $\alpha=0=\beta$ and $\gamma=1$ in Theorem 2.1, then*

$$\Phi\left(\frac{x_1+x_2}{2}\right)\leq\frac{\Gamma_\kappa(\nu+\kappa)}{2(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{x_2^-}\Phi\right)(x_1)+\left(J^{\nu,\kappa}_{x_1^+}\Phi\right)(x_2)\right]\leq\frac{\Phi(x_1)+\Phi(x_2)}{2},$$

*holds for all $x_1,x_2\in[\mathfrak{b}_1,\mathfrak{b}_2]$ with $x_1<x_2$, see [23].*

**Theorem 2.2.** *Assume that $\Phi:[\mathfrak{b}_1,\mathfrak{b}_2]\mapsto\mathbb{R}$ be a convex function. Let $\alpha,\beta\in[0,1]$, $\gamma\in(0,1]$ be coefficients of sums $\alpha+\beta+\gamma=1$ and $\nu,\kappa>0$, then*

$$\begin{aligned}
&\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+x_2}{2}\right)\\
&\leq\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}}}{2\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)+\left(J^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)\right]\\
&\leq\alpha\Phi(\mathfrak{b}_1)+\beta\Phi(\mathfrak{b}_2)+\gamma\frac{\Phi(x_1)+\Phi(x_2)}{2},
\end{aligned}$$

*holds for all $x_1,x_2\in[\mathfrak{b}_1,\mathfrak{b}_2]$ with $x_1<x_2$, and $\omega\in\mathbb{N}$.*

*Proof.* Since $\Phi$ is convex function, then

$$\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_{11}+x_{12}}{2}\right)\leq\frac{1}{2}\left[\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_{11})+\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_{21})\right].$$

Using change of variable technique for $x_{11}=\frac{\tau}{\omega+1}x_1+\frac{\omega+1-\tau}{\omega+1}x_2$ and $x_{21}=\frac{\omega+1-\tau}{\omega+1}x_1+\frac{\tau}{\omega+1}x_2$, we have

$$\begin{aligned}
\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+x_2}{2}\right)\leq\frac{1}{2}\bigg[&\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\tau}{\omega+1}x_1+\frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\\
&+\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1+\frac{\tau}{\omega+1}x_2\right)\right)\bigg].
\end{aligned}$$

Multiplying both side of above inequality $\tau^{\frac{\nu}{\kappa}-1}$ and integrating with respect to $\tau$ on $[0,1]$, we get

$$\begin{aligned}
&\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+x_2}{2}\right)\\
&\leq\frac{\nu}{2\kappa}\left[\int_0^1\tau^{\frac{\nu}{\kappa}-1}\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\tau}{\omega+1}x_1+\frac{\omega+1-\tau}{\omega+1}x_2\right)\right)d\tau\right.
\end{aligned}$$

$$+\int_0^1 \tau^{\frac{\nu}{\kappa}-1}\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1+\frac{\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau\Bigg]$$

$$=\frac{\nu(\omega+1)^{\frac{\nu}{\kappa}}}{2\kappa\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\int_{\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}}^{\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2}(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2-u)^{\frac{\nu}{\kappa}-1}\Phi(u)\mathrm{d}u\right.$$

$$\left.+\int_{\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1}^{\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}}(u-(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1))^{\frac{\nu}{\kappa}-1}\Phi(u)\mathrm{d}u\right]$$

$$=\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}}}{2\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(\mathrm{J}^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)+\left(\mathrm{J}^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)\right].$$

The first inequality is proved. To prove second inequality, from convexity of property of $\Phi$, we have

$$\begin{aligned}&\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\tau}{\omega+1}x_1+\frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\\&\le\alpha\Phi(\mathfrak{b}_1)+\beta\Phi(\mathfrak{b}_2)+\gamma\left(\frac{\tau}{\omega+1}\Phi(x_1)+\frac{\omega+1-\tau}{\omega+1}\Phi(x_2)\right),\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1+\frac{\tau}{\omega+1}x_2\right)\right)\\&\le\alpha\Phi(\mathfrak{b}_1)+\beta\Phi(\mathfrak{b}_2)+\gamma\left(\frac{\tau}{\omega+1}\Phi(x_2)+\frac{\omega+1-\tau}{\omega+1}\Phi(x_1)\right).\end{aligned}\tag{2.4}$$

Adding inequalities (2.3) and (2.4), multiplying both side by $\tau^{\frac{\nu}{\kappa}-1}$, and then integrating with respect to $\tau$ on $[0,1]$, we obtain second inequality. This completes the proof. □

**Corollary 2.2.** *If we choose $\alpha=0=\beta$ and $\gamma=1$ in Theorem 2.2, then*

$$\begin{aligned}\Phi\left(\frac{x_1+x_2}{2}\right)&\le\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}}}{2(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(\mathrm{J}^{\nu,\kappa}_{\left(\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(x_2)+\left(\mathrm{J}^{\nu,\kappa}_{\left(\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(x_1)\right]\\&\le\frac{\Phi(x_1)+\Phi(x_2)}{2},\end{aligned}$$

*holds for all $x_1,x_2\in[\mathfrak{b}_1,\mathfrak{b}_2]$ with $x_1<x_2$, and $\omega\in\mathbb{N}$.*

## 3. Further results

In this section, we derive two new auxiliary identities, which will be used in obtaining our further results.

**Lemma 3.1.** *Let $\Phi:[\mathfrak{b}_1,\mathfrak{b}_2]\mapsto\mathbb{R}$ be a differentiable function on $(\mathfrak{b}_1,\mathfrak{b}_2)$ with $\mathfrak{b}_1<\mathfrak{b}_2$. If $\Phi'\in L_1[\mathfrak{b}_1,\mathfrak{b}_2]$ and $\alpha,\beta\in[0,1]$, $\gamma\in(0,1]$ be coefficients of sums $\alpha+\beta+\gamma=1$ and $\nu,\kappa>0$, then*

$$\frac{\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)+\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)}{2}-\frac{\Gamma_\kappa(\nu+\kappa)}{2\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}$$

$$\times\left[\left(J^{\alpha,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)^-}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)+\left(J^{\alpha,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)^+}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)\right]$$
$$=\frac{\gamma(x_2-x_1)}{2}\left[\int_0^1(1-\tau)^{\frac{\nu}{\kappa}}\Phi'(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau\right.$$
$$\left.-\int_0^1\tau^{\frac{\nu}{\kappa}}\Phi'(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau\right],$$

*holds for all* $x_1, x_2 \in [\mathfrak{b}_1, \mathfrak{b}_2]$ *with* $x_1 < x_2$.

*Proof.* Consider

$$\mathcal{I}:=\frac{\gamma(x_2-x_1)}{2}\left[\int_0^1(1-\tau)^{\frac{\nu}{\kappa}}\Phi'(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau\right.$$
$$\left.-\int_0^1\tau^{\frac{\nu}{\kappa}}\Phi'(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau\right]$$
$$=\frac{\gamma(x_2-x_1)}{2}[\mathcal{I}_1-\mathcal{I}_2],$$

where

$$\mathcal{I}_1:=\int_0^1(1-\tau)^{\frac{\nu}{\kappa}}\Phi'(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau$$
$$=-\frac{(1-\tau)^{\frac{\nu}{\kappa}}\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))}{\gamma(x_2-x_1)}\Big|_0^1$$
$$-\frac{\nu}{\kappa\gamma(x_2-x_1)}\int_0^1(1-\tau)^{\frac{\nu}{\kappa}-1}\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau$$
$$=\frac{\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)}{\gamma(x_2-x_1)}-\frac{\Gamma_\kappa(\nu+\kappa)}{\gamma^{\frac{\nu}{\kappa}+1}(x_2-x_1)^{\frac{\nu}{\kappa}+1}}\left(J^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)^-}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1),$$

and

$$\mathcal{I}_2:=\int_0^1\tau^{\frac{\nu}{\kappa}}\Phi'(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau$$
$$=-\frac{\tau^{\frac{\nu}{\kappa}}\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))}{\gamma(x_2-x_1)}\Big|_0^1$$
$$+\frac{\nu}{\kappa\gamma(x_2-x_1)}\int_0^1\tau^{\frac{\nu}{\kappa}-1}\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma(\tau x_1+(1-\tau)x_2))\mathrm{d}\tau$$
$$=-\frac{\Phi(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)}{\gamma(x_2-x_1)}+\frac{\Gamma_\kappa(\nu+\kappa)}{\gamma^{\frac{\nu}{\kappa}+1}(x_2-x_1)^{\frac{\nu}{\kappa}+1}}\left(J^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)^+}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2).$$

Substituting the values of $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathcal{I}$, we obtain our required result. □

**Corollary 3.1.** *If we choose* $\alpha = 0 = \beta$ *and* $\gamma = 1$ *in Lemma 3.1, then*

$$\frac{\Phi(x_1)+\Phi(x_2)}{2}-\frac{\Gamma_\kappa(\nu+\kappa)}{2(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{x_2^-}\Phi\right)(x_1)+\left(J^{\nu,\kappa}_{x_1^+}\Phi\right)(x_2)\right]$$
$$=\frac{(x_2-x_1)}{2}\left[\int_0^1(1-\tau)^{\frac{\nu}{\kappa}}\Phi'(\tau x_1+(1-\tau)x_2)\mathrm{d}\tau-\int_0^1\tau^{\frac{\nu}{\kappa}}\Phi'(\tau x_1+(1-\tau)x_2)\mathrm{d}\tau\right].$$

**Lemma 3.2.** *Let $\Phi : [\mathfrak{b}_1, \mathfrak{b}_2] \mapsto \mathbb{R}$ be a differentiable function on $(\mathfrak{b}_1, \mathfrak{b}_2)$ with $\mathfrak{b}_1 < \mathfrak{b}_2$. If $\Phi' \in L_1[\mathfrak{b}_1, \mathfrak{b}_2]$ and $\alpha, \beta \in [0, 1]$, $\gamma \in (0, 1]$ be coefficients of sums $\alpha + \beta + \gamma = 1$ and $\nu, \kappa > 0$, then*

$$\begin{aligned}
&\frac{\Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\frac{\omega x_1 + x_2}{\omega+1}\right) + \Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\frac{x_1 + \omega x_2}{\omega+1}\right)}{\omega + 1} \\
&\quad - \frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2 - x_1)^{\frac{\nu}{\kappa}}}\Bigg[\left(\mathrm{J}^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1})^-}\Phi\right)(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1) \\
&\quad + \left(\mathrm{J}^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1})^+}\Phi\right)(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_2)\Bigg] \\
&= \frac{\gamma(x_2 - x_1)}{(\omega+1)^2}\left[\int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi'\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1 + \frac{\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau\right. \\
&\quad \left. - \int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi'\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\tau}{\omega+1}x_1 + \frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau\right],
\end{aligned}$$

*holds for all $x_1, x_2 \in [\mathfrak{b}_1, \mathfrak{b}_2]$ with $x_1 < x_2$, and $\omega \in \mathbb{N}$.*

*Proof.* Consider

$$\begin{aligned}
\mathrm{J} &:= \frac{\gamma(x_2 - x_1)}{(\omega+1)^2}\left[\int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi'\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1 + \frac{\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau\right. \\
&\quad \left. - \int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi'\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\tau}{\omega+1}x_1 + \frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau\right] \\
&= \frac{\gamma(x_2 - x_1)}{(\omega+1)^2}[\mathrm{J}_1 - \mathrm{J}_2],
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{J}_1 &:= \int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi'\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1 + \frac{\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau \\
&= (\omega+1)\frac{\tau^{\frac{\nu}{\kappa}}\Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1 + \frac{\tau}{\omega+1}x_2\right)\right)}{\gamma(x_2 - x_1)}\Bigg|_0^1 \\
&\quad - \frac{(\omega+1)\nu}{\kappa\gamma(x_2 - x_1)}\int_0^1 \tau^{\frac{\nu}{\kappa}-1}\Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1 + \frac{\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau \\
&= (\omega+1)\frac{\Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\frac{\omega x_1 + x_2}{\omega+1}\right)}{\gamma(x_2 - x_1)} \\
&\quad - \frac{\nu(\omega+1)^{\frac{\nu}{\kappa}+1}}{\kappa\gamma^{\frac{\nu}{\kappa}+1}(x_2 - x_1)^{\frac{\nu}{\kappa}+1}}\int_{\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1}^{\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}}(u - (\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1))^{\frac{\nu}{\kappa}-1}\Phi(u)\mathrm{d}u \\
&= (\omega+1)\frac{\Phi\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\frac{\omega x_1 + x_2}{\omega+1}\right)}{\gamma(x_2 - x_1)} - \frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}+1}}{\gamma^{\frac{\nu}{\kappa}+1}(x_2 - x_1)^{\frac{\nu}{\kappa}+1}}\left(\mathrm{J}^{\nu,\kappa}_{(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1})^-}\Phi\right)(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1),
\end{aligned}$$

and

$$\mathrm{J}_2 := \int_0^1 \tau^{\frac{\nu}{\kappa}}\Phi'\left(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma\left(\frac{\tau}{\omega+1}x_1 + \frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\mathrm{d}\tau$$

$$= -(\omega+1)\frac{\Phi\left(\alpha b_1+\beta b_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\gamma(x_2-x_1)}+\frac{(\omega+1)^{\frac{\nu}{\kappa}+1}\Gamma_\kappa(\nu+\kappa)}{\gamma^{\frac{\nu}{\kappa}+1}(x_2-x_1)^{\frac{\nu}{\kappa}+1}}\left(J^{\nu,\kappa}_{(\alpha b_1+\beta b_2+\gamma\frac{x_1+\omega x_2}{\omega+1})^+}\Phi\right)(\alpha b_1+\beta b_2+\gamma x_2).$$

Substituting the values of $J_1$ and $J_2$ in J and multiplying both sides by $\frac{\gamma(x_2-x_1)}{(\omega+1)^2}$, we obtain our required result. □

Now we derive some new results related to Hermite-Hadamard-Mercer type inequality using Lemma 3.1 and Lemma 3.2.

**Theorem 3.1.** *Under the assumptions of Lemma 3.1, if* $|\Phi'|$ *is a convex function, then*

$$\left|\frac{\Phi(\alpha b_1+\beta b_2+\gamma x_1)+\Phi(\alpha b_1+\beta b_2+\gamma x_2)}{2}-\frac{\Gamma_\kappa(\nu+\kappa)}{2\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(J^{\alpha,\kappa}_{(\alpha b_1+\beta b_2+\gamma x_2)^-}\Phi\right)(\alpha b_1+\beta b_2+\gamma x_1)+\left(J^{\alpha,\kappa}_{(\alpha b_1+\beta b_2+\gamma x_1)^+}\Phi\right)(\alpha b_1+\beta b_2+\gamma x_2)\right]\right|$$
$$\leq\frac{\gamma(x_2-x_1)}{2}\left[\left(\frac{2\kappa-\kappa\left(\frac{1}{2}\right)^{\frac{\nu}{\kappa}-1}}{\nu+\kappa}\right)(\alpha|\Phi'(b_1)|+\beta|\Phi'(b_2)|)+\frac{\kappa\gamma}{\nu+\kappa}\left(\kappa-\left(\frac{1}{2}\right)^{\frac{\nu}{\kappa}}\right)(|\Phi'(x_1)|+|\Phi'(x_2)|)\right].$$

*Proof.* Using Lemma 3.1, property of modulus, and convexity property of $|\Phi'|$, we have

$$\left|\frac{\Phi(\alpha b_1+\beta b_2+\gamma x_1)+\Phi(\alpha b_1+\beta b_2+\gamma x_2)}{2}-\frac{\Gamma_\kappa(\nu+\kappa)}{2\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(J^{\alpha,\kappa}_{(\alpha b_1+\beta b_2+\gamma x_2)^-}\Phi\right)(\alpha b_1+\beta b_2+\gamma x_1)+\left(J^{\alpha,\kappa}_{(\alpha b_1+\beta b_2+\gamma x_1)^+}\Phi\right)(\alpha b_1+\beta b_2+\gamma x_2)\right]\right|$$
$$\leq\frac{\gamma(x_2-x_1)}{2}\left[\int_0^1\left|(1-\tau)^{\frac{\nu}{\kappa}}-\tau^{\frac{\nu}{\kappa}}\right||\Phi'(\alpha b_1+\beta b_2+\gamma(\tau x_1+(1-\tau)x_2))|\,d\tau\right]$$
$$\leq\frac{\gamma(x_2-x_1)}{2}\left[\int_0^{\frac{1}{2}}[(1-\tau)^{\frac{\nu}{\kappa}}-\tau^{\frac{\nu}{\kappa}}]\,|\Phi'(\alpha b_1+\beta b_2+\gamma(\tau x_1+(1-\tau)x_2))|\,d\tau\right.$$
$$\left.+\int_{\frac{1}{2}}^1[\tau^{\frac{\nu}{\kappa}}-(1-\tau)^{\frac{\nu}{\kappa}}]\,|\Phi'(\alpha b_1+\beta b_2+\gamma(\tau x_1+(1-\tau)x_2))|\,d\tau\right]$$
$$\leq\frac{\gamma(x_2-x_1)}{2}\left[\int_0^{\frac{1}{2}}[(1-\tau)^{\frac{\nu}{\kappa}}-\tau^{\frac{\nu}{\kappa}}]\,[\alpha|\Phi'(b_1)|+\beta|\Phi'(b_2)|+\gamma(\tau|\Phi'(x_1)|+(1-\tau)|\Phi'(x_2)|)]\,d\tau\right.$$
$$\left.+\int_{\frac{1}{2}}^1[\tau^{\frac{\nu}{\kappa}}-(1-\tau)^{\frac{\nu}{\kappa}}]\,[\alpha|\Phi'(b_1)|+\beta|\Phi'(b_2)|+\gamma(\tau|\Phi'(x_1)|+(1-\tau)|\Phi'(x_2)|)]\,d\tau\right].$$

After simple calculations, we obtain the required result. □

**Corollary 3.2.** *If we take* $\alpha=0=\beta$ *and* $\gamma=1$ *in Theorem 3.1, then*

$$\left|\frac{\Phi(x_1)+\Phi(x_2)}{2}-\frac{\Gamma_\kappa(\nu+\kappa)}{2(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{x_2^-}\Phi\right)(x_1)+\left(J^{\alpha,\kappa}_{x_1^+}\Phi\right)(x_2)\right]\right|$$
$$\leq\frac{(x_2-x_1)}{2}\left[\frac{\kappa}{\nu+\kappa}\left(\kappa-\left(\frac{1}{2}\right)^{\frac{\nu}{\kappa}}\right)(|\Phi'(x_1)|+|\Phi'(x_2)|)\right].$$

**Corollary 3.3.** *If we choose $\nu = 1 = \kappa$ in Theorem 3.1, then*

$$\left|\frac{\Phi(\alpha\flat_1+\beta\flat_2+\gamma x_1)+\Phi(\alpha\flat_1+\beta\flat_2+\gamma x_2)}{2}-\frac{1}{2\gamma(x_2-x_1)}\int_{\alpha\flat_1+\beta\flat_2+\gamma x_1}^{\alpha\flat_1+\beta\flat_2+\gamma x_2}\Phi(u)\mathrm{d}u\right|$$
$$\leq\frac{\gamma(x_2-x_1)}{2}\left[\frac{\alpha|\Phi'(\flat_1)|+\beta|\Phi'(\flat_2)|}{2}+\frac{\gamma(|\Phi'(x_1)|+|\Phi'(x_2)|)}{4}\right].$$

**Remark 3.1.** *Using Lemma 3.1, Hölder's inequality or power mean inequality, interested reader can obtain new interesting integral inequalities. We omit here their proofs.*

**Theorem 3.2.** *Under the assumptions of Lemma 3.2, if $|\Phi'|$ is a convex function, then*

$$\left|\frac{\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)+\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega+1}-\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(\mathrm{J}^{\nu,\kappa}_{(\alpha\flat_1+\beta\flat_2+\gamma\frac{\omega x_1+x_2}{\omega+1})^-}\Phi\right)(\alpha\flat_1+\beta\flat_2+\gamma x_1)+\left(\mathrm{J}^{\nu,\kappa}_{(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+\omega x_2}{\omega+1})^+}\Phi\right)(\alpha\flat_1+\beta\flat_2+\gamma x_2)\right]\right|$$
$$\leq\frac{2\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu+\kappa}\right)\left[\alpha|\Phi'(\flat_1)|+\beta|\Phi'(\flat_2)|+\gamma\frac{|\Phi'(x_1)|+|\Phi'(x_2)|}{2}\right].$$

*Proof.* Using Lemma 3.2, property of modulus, and convexity property of $|\Phi'|$, we have

$$\left|\frac{\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)+\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega+1}-\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(\mathrm{J}^{\nu,\kappa}_{(\alpha\flat_1+\beta\flat_2+\gamma\frac{\omega x_1+x_2}{\omega+1})^-}\Phi\right)(\alpha\flat_1+\beta\flat_2+\gamma x_1)+\left(\mathrm{J}^{\nu,\kappa}_{(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+\omega x_2}{\omega+1})^+}\Phi\right)(\alpha\flat_1+\beta\flat_2+\gamma x_2)\right]\right|$$
$$\leq\frac{\gamma(x_2-x_1)}{(\omega+1)^2}\int_0^1\tau^{\frac{\nu}{\kappa}}\left[\left|\Phi'\left(\alpha\flat_1+\beta\flat_2+\gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1+\frac{\tau}{\omega+1}x_2\right)\right)\right|\right.$$
$$\left.+\left|\Phi'\left(\alpha\flat_1+\beta\flat_2+\gamma\left(\frac{\tau}{\omega+1}x_1+\frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\right|\right]\mathrm{d}\tau$$
$$\leq\frac{\gamma(x_2-x_1)}{(\omega+1)^2}\int_0^1\tau^{\frac{\nu}{\kappa}}\left[\alpha|\Phi'(\flat_1)|+\beta|\Phi'(\flat_2)|+\gamma\left(\frac{\omega+1-\tau}{\omega+1}|\Phi'(x_1)|+\frac{\tau}{\omega+1}|\Phi'(x_2)|\right)\right.$$
$$\left.+\alpha|\Phi'(\flat_1)|+\beta|\Phi'(\flat_2)|+\gamma\left(\frac{\tau}{\omega+1}|\Phi'(x_1)|+\frac{\omega+1-\tau}{\omega+1}|\Phi'(x_2)|\right)\right]\mathrm{d}\tau$$
$$=\frac{2\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu+\kappa}\right)\left[\alpha|\Phi'(\flat_1)|+\beta|\Phi'(\flat_2)|+\gamma\frac{|\Phi'(x_1)|+|\Phi'(x_2)|}{2}\right].$$

This completes the proof. □

**Corollary 3.4.** *If we take $\nu = \omega = \kappa = 1$ in Theorem 3.2, then*

$$\left|\Phi\left(\alpha\flat_1+\beta\flat_2+\gamma\frac{x_1+x_2}{2}\right)-\frac{1}{\gamma(x_2-x_1)}\int_{\alpha\flat_1+\beta\flat_2+\gamma x_1}^{\alpha\flat_1+\beta\flat_2+\gamma x_2}\Phi(u)\mathrm{d}u\right|$$
$$\leq\frac{\gamma(x_2-x_1)}{4}\left[\alpha|\Phi'(\flat_1)|+\beta|\Phi'(\flat_2)|+\gamma\frac{|\Phi'(x_1)|+|\Phi'(x_2)|}{2}\right].$$

**Corollary 3.5.** *If we choose $\alpha = 0 = \beta$ and $\gamma = 1$ in Theorem 3.2, then*

$$\left|\frac{\Phi\left(\frac{\omega x_1+x_2}{\omega+1}\right)+\Phi\left(\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega+1}-\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(\mathrm{J}^{\nu,\kappa}_{\left(\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(x_1)+\left(\mathrm{J}^{\nu,\kappa}_{\left(\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(x_2)\right]\right|$$
$$\leq \frac{(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu+\kappa}\right)\left[|\Phi'(x_1)|+|\Phi'(x_2)|\right].$$

**Theorem 3.3.** *Under the assumptions of Lemma 3.2, if $|\Phi'|^q$ is a convex function, then*

$$\left|\frac{\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)+\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega+1}-\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(\mathrm{J}^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)+\left(\mathrm{J}^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)\right]\right|$$
$$\leq \frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu p+\kappa}\right)^{\frac{1}{p}}\left[\left(\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{1}{2(\omega+1)}|\Phi'(x_1)|^q+\frac{2(\omega+1)-1}{2(\omega+1)}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{1}{2(\omega+1)}|\Phi'(x_2)|^q+\frac{2(\omega+1)-1}{2(\omega+1)}|\Phi'(x_1)|^q\right)\right)^{\frac{1}{q}}\right],$$

*where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$.*

*Proof.* Using Lemma 3.2, property of modulus, Hölder's inequality and the convexity property of $|\Phi'|^q$, we have

$$\left|\frac{\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)+\Phi\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega+1}-\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2-x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(\mathrm{J}^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_1)+\left(\mathrm{J}^{\nu,\kappa}_{\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma x_2)\right]\right|$$
$$\leq \frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\int_0^1\tau^{\frac{\nu p}{\kappa}}\mathrm{d}\tau\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|\Phi'\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\omega+1-\tau}{\omega+1}x_1+\frac{\tau}{\omega+1}x_2\right)\right)\right|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\Phi'\left(\alpha\mathfrak{b}_1+\beta\mathfrak{b}_2+\gamma\left(\frac{\tau}{\omega+1}x_1+\frac{\omega+1-\tau}{\omega+1}x_2\right)\right)\right|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu p+\kappa}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left[\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{\omega+1-\tau}{\omega+1}|\Phi'(x_1)|^q+\frac{\tau}{\omega+1}|\Phi'(x_2)|^q\right)\right]\mathrm{d}\tau\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{\tau}{\omega+1}|\Phi'(x_1)|^q+\frac{\omega+1-\tau}{\omega+1}|\Phi'(x_2)|^q\right)\right]\mathrm{d}\tau\right)^{\frac{1}{q}}\right]$$
$$=\frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu p+\kappa}\right)^{\frac{1}{p}}\left[\left(\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{1}{2(\omega+1)}|\Phi'(x_1)|^q+\frac{2(\omega+1)-1}{2(\omega+1)}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{1}{2(\omega+1)}|\Phi'(x_2)|^q+\frac{2(\omega+1)-1}{2(\omega+1)}|\Phi'(x_1)|^q\right)\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

**Corollary 3.6.** *If we take $\nu = \omega = \kappa = 1$ in Theorem 3.3, then*

$$\left|\Phi\left(\alpha b_1 + \beta b_2 + \gamma\frac{x_1 + x_2}{2}\right) - \frac{1}{\gamma(x_2 - x_1)}\int_{\alpha b_1+\beta b_2+\gamma x_1}^{\alpha b_1+\beta b_2+\gamma x_2} \Phi(u)\mathrm{d}u\right|$$
$$\leq \frac{\gamma(x_2 - x_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\alpha|\Phi'(b_1)|^q + \beta|\Phi'(b_2)|^q + \gamma\left(\frac{1}{4}|\Phi'(x_1)|^q + \frac{3}{4}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\alpha|\Phi'(b_1)|^q + \beta|\Phi'(b_2)|^q + \gamma\left(\frac{1}{4}|\Phi'(x_2)|^q + \frac{3}{4}|\Phi'(x_1)|^q\right)\right)^{\frac{1}{q}}\right].$$

**Corollary 3.7.** *If we choose $\alpha = 0 = \beta$ and $\gamma = 1$ in Theorem 3.3, then*

$$\left|\frac{\Phi\left(\frac{\omega x_1+x_2}{\omega+1}\right) + \Phi\left(\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega + 1} - \frac{\Gamma_\kappa(\nu + \kappa)(\omega + 1)^{\frac{\nu}{\kappa}-1}}{(x_2 - x_1)^{\frac{\nu}{\kappa}}}\left[\left(J^{\nu,\kappa}_{\left(\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(x_1) + \left(J^{\nu,\kappa}_{\left(\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(x_2)\right]\right|$$
$$\leq \frac{(x_2 - x_1)}{(\omega + 1)^2}\left(\frac{\kappa}{\nu p + \kappa}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2(\omega + 1)}|\Phi'(x_1)|^q + \frac{2(\omega + 1) - 1}{2(\omega + 1)}|\Phi'(x_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2(\omega + 1)}|\Phi'(x_2)|^q + \frac{2(\omega + 1) - 1}{2(\omega + 1)}|\Phi'(x_1)|^q\right)^{\frac{1}{q}}\right].$$

**Theorem 3.4.** *Under the assumptions of Lemma 3.2, if $|\Phi'|^q$ is a convex function for $q \geq 1$, then*

$$\left|\frac{\Phi\left(\alpha b_1 + \beta b_2 + \gamma\frac{\omega x_1+x_2}{\omega+1}\right) + \Phi\left(\alpha b_1 + \beta b_2 + \gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega + 1} - \frac{\Gamma_\kappa(\nu + \kappa)(\omega + 1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2 - x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(J^{\nu,\kappa}_{\left(\alpha b_1+\beta b_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(\alpha b_1 + \beta b_2 + \gamma x_1) + \left(J^{\nu,\kappa}_{\left(\alpha b_1+\beta b_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(\alpha b_1 + \beta b_2 + \gamma x_2)\right]\right|$$
$$\leq \frac{\gamma(x_2 - x_1)}{(\omega + 1)^2}\left(\frac{\kappa}{\nu + \kappa}\right)^{1-\frac{1}{q}}\left[\left(\frac{\kappa\alpha}{\nu + \kappa}|\Phi'(b_1)|^q + \frac{\kappa\beta}{\nu + \kappa}|\Phi'(b_2)|^q + \gamma\left(\frac{k\omega(\omega + 2\kappa) + \kappa^2}{(\omega + 1)(\nu + \kappa)(\nu + 2\kappa)}|\Phi'(x_1)|^q\right.\right.\right.$$
$$\left.\left.+\frac{\kappa}{(\omega + 1)(\nu + 2\kappa)}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}} + \left(\frac{\kappa\alpha}{\nu + \kappa}|\Phi'(b_1)|^q + \frac{\kappa\beta}{\nu + \kappa}|\Phi'(b_2)|^q\right.$$
$$\left.\left.+\gamma\left(\frac{\kappa}{(\omega + 1)(\nu + 2\kappa)}|\Phi'(x_1)|^q + \frac{k\omega(\nu + 2\kappa) + \kappa^2}{(\omega + 1)(\nu + \kappa)(\nu + 2\kappa)}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right].$$

*Proof.* Using Lemma 3.2, property of modulus, power mean inequality and the convexity property of $|\Phi'|^q$, we have

$$\left|\frac{\Phi\left(\alpha b_1 + \beta b_2 + \gamma\frac{\omega x_1+x_2}{\omega+1}\right) + \Phi\left(\alpha b_1 + \beta b_2 + \gamma\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega + 1} - \frac{\Gamma_\kappa(\nu + \kappa)(\omega + 1)^{\frac{\nu}{\kappa}-1}}{\gamma^{\frac{\nu}{\kappa}}(x_2 - x_1)^{\frac{\nu}{\kappa}}}\right.$$
$$\left.\times\left[\left(J^{\nu,\kappa}_{\left(\alpha b_1+\beta b_2+\gamma\frac{\omega x_1+x_2}{\omega+1}\right)^-}\Phi\right)(\alpha b_1 + \beta b_2 + \gamma x_1) + \left(J^{\nu,\kappa}_{\left(\alpha b_1+\beta b_2+\gamma\frac{x_1+\omega x_2}{\omega+1}\right)^+}\Phi\right)(\alpha b_1 + \beta b_2 + \gamma x_2)\right]\right|$$

$$
\leq \frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\int_0^1 \tau^{\frac{\nu}{\kappa}} \mathrm{d}\tau\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \tau^{\frac{\nu}{\kappa}}\left|\Phi'\left(\alpha \mathfrak{b}_1+\beta \mathfrak{b}_2+\gamma\left(\frac{\omega+1-\tau}{\omega+1} x_1+\frac{\tau}{\omega+1} x_2\right)\right)\right|^q \mathrm{d}\tau\right)^{\frac{1}{q}}\right.
$$

$$
\left.+\left(\int_0^1 \tau^{\frac{\nu}{\kappa}}\left|\Phi'\left(\alpha \mathfrak{b}_1+\beta \mathfrak{b}_2+\gamma\left(\frac{\tau}{\omega+1} x_1+\frac{\omega+1-\tau}{\omega+1} x_2\right)\right)\right|^q \mathrm{d}\tau\right)^{\frac{1}{q}}\right]
$$

$$
\leq \frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu+\kappa}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \tau^{\frac{\nu}{\kappa}}\left[\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{\omega+1-\tau}{\omega+1}|\Phi'(x_1)|^q+\frac{\tau}{\omega+1}|\Phi'(x_2)|^q\right)\right] \mathrm{d}\tau\right)^{\frac{1}{q}}\right.
$$

$$
\left.+\left(\int_0^1 \tau^{\frac{\nu}{\kappa}}\left[\alpha|\Phi'(\mathfrak{b}_1)|^q+\beta|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{\tau}{\omega+1}|\Phi'(x_1)|^q+\frac{\omega+1-\tau}{\omega+1}|\Phi'(x_2)|^q\right)\right] \mathrm{d}\tau\right)^{\frac{1}{q}}\right]
$$

$$
=\frac{\gamma(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu+\kappa}\right)^{1-\frac{1}{q}}\left[\left(\frac{\kappa\alpha}{\nu+\kappa}|\Phi'(\mathfrak{b}_1)|^q+\frac{\kappa\beta}{\nu+\kappa}|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{k\omega(\omega+2\kappa)+\kappa^2}{(\omega+1)(\nu+\kappa)(\nu+2\kappa)}|\Phi'(x_1)|^q\right.\right.\right.
$$

$$
\left.\left.+\frac{\kappa}{(\omega+1)(\nu+2\kappa)}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}+\left(\frac{\kappa\alpha}{\nu+\kappa}|\Phi'(\mathfrak{b}_1)|^q+\frac{\kappa\beta}{\nu+\kappa}|\Phi'(\mathfrak{b}_2)|^q\right.
$$

$$
\left.\left.+\gamma\left(\frac{\kappa}{(\omega+1)(\nu+2\kappa)}|\Phi'(x_1)|^q+\frac{k\omega(\nu+2\kappa)+\kappa^2}{(\omega+1)(\nu+\kappa)(\nu+2\kappa)}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right].
$$

This completes the proof. □

**Corollary 3.8.** *If we take $\nu=\omega=\kappa=1$ in Theorem 3.4, then*

$$
\left|\Phi\left(\alpha \mathfrak{b}_1+\beta \mathfrak{b}_2+\gamma \frac{x_1+x_2}{2}\right)-\frac{1}{\gamma(x_2-x_1)} \int_{\alpha \mathfrak{b}_1+\beta \mathfrak{b}_2+\gamma x_1}^{\alpha \mathfrak{b}_1+\beta \mathfrak{b}_2+\gamma x_2} \Phi(u) \mathrm{d}u\right|
$$

$$
\leq \frac{\gamma(x_2-x_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{\alpha}{2}|\Phi'(\mathfrak{b}_1)|^q+\frac{\beta}{2}|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{1}{3}|\Phi'(x_1)|^q+\frac{1}{6}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right.
$$

$$
\left.+\left(\frac{\alpha}{2}|\Phi'(\mathfrak{b}_1)|^q+\frac{\beta}{2}|\Phi'(\mathfrak{b}_2)|^q+\gamma\left(\frac{1}{6}|\Phi'(x_1)|^q+\frac{1}{3}|\Phi'(x_2)|^q\right)\right)^{\frac{1}{q}}\right].
$$

**Corollary 3.9.** *If we choose $\alpha=0=\beta$ and $\gamma=1$ in Theorem 3.4, then*

$$
\left|\frac{\Phi\left(\frac{\omega x_1+x_2}{\omega+1}\right)+\Phi\left(\frac{x_1+\omega x_2}{\omega+1}\right)}{\omega+1}-\frac{\Gamma_\kappa(\nu+\kappa)(\omega+1)^{\frac{\nu}{\kappa}-1}}{(x_2-x_1)^{\frac{\nu}{\kappa}}}\left[\left(\mathrm{J}_{\left(\frac{\omega x_1+x_2}{\omega+1}\right)^-}^{\nu,\kappa} \Phi\right)(x_1)+\left(\mathrm{J}_{\left(\frac{x_1+\omega x_2}{\omega+1}\right)^+}^{\nu,\kappa} \Phi\right)(x_2)\right]\right|
$$

$$
\leq \frac{(x_2-x_1)}{(\omega+1)^2}\left(\frac{\kappa}{\nu+\kappa}\right)^{1-\frac{1}{q}}\left[\left(\frac{k\omega(\omega+2\kappa)+\kappa^2}{(\omega+1)(\nu+\kappa)(\nu+2\kappa)}|\Phi'(x_1)|^q+\frac{\kappa}{(\omega+1)(\nu+2\kappa)}|\Phi'(x_2)|^q\right)^{\frac{1}{q}}\right.
$$

$$
\left.+\left(\frac{\kappa}{(\omega+1)(\nu+2\kappa)}|\Phi'(x_1)|^q+\frac{k\omega(\nu+2\kappa)+\kappa^2}{(\omega+1)(\nu+\kappa)(\nu+2\kappa)}|\Phi'(x_2)|^q\right)^{\frac{1}{q}}\right].
$$

## 4. Applications

In this section, we will discuss some applications regarding our results for special means and error estimations.

*4.1. Applications to special means*

Let recall the following two special means:

- The arithmetic mean is defined as

$$\mathcal{A}(x_1, x_2) := \frac{x_1 + x_2}{2}.$$

- The generalized log–mean is given by

$$\mathcal{L}_n(x_1, x_2) := \left[\frac{x_2^{n+1} - x_1^{n+1}}{(n+1)(x_2 - x_1)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\},$$

where $0 < x_1 < x_2$ are real numbers.

Using above special means we can establish some new inequalities as follows:

**Proposition 4.1.** *Let $x_1, x_2 \in [\mathfrak{b}_1, \mathfrak{b}_2]$ with $0 < \mathfrak{b}_1 < \mathfrak{b}_2$ and $\alpha, \beta, \in [0,1]$, $\gamma \in (0,1]$ be coefficients of sums $\alpha + \beta + \gamma = 1$, then for $n > 1$, we have*

$$\begin{aligned}&\left|\mathcal{A}\left((\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1)^{n+2}, (\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_2)^{n+2}\right) - \frac{1}{2}\mathcal{L}_{n+2}^{n+2}(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1, \alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_2)\right| \\ &\leq \frac{\gamma(n+2)(x_2 - x_1)}{2}\left[\mathcal{A}\left(\alpha\mathfrak{b}_1{}^{n+1}, \beta\mathfrak{b}_2{}^{n+1}\right) + \frac{\gamma}{2}\mathcal{A}\left(x_1^{n+1}, x_2^{n+1}\right)\right]. \qquad (4.1)\end{aligned}$$

*Proof.* The proof directly follows from Theorem 3.1 applying for $\Phi(x) = x^{n+2}$ and $\nu = 1 = \kappa$. □

**Proposition 4.2.** *Let $x_1, x_2 \in [\mathfrak{b}_1, \mathfrak{b}_2]$ with $0 < \mathfrak{b}_1 < \mathfrak{b}_2$ and $\alpha, \beta, \in [0,1]$, $\gamma \in (0,1]$ be coefficients of sums $\alpha + \beta + \gamma = 1$, then for $n > 1$, we have*

$$\begin{aligned}&\left|(2\mathcal{A}(\alpha\mathfrak{b}_1, \beta\mathfrak{b}_2) + \gamma\mathcal{A}(x_1, x_2))^{n+2} - \mathcal{L}_{n+2}^{n+2}(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1, \alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_2)\right| \\ &\leq \frac{\gamma(n+2)(x_2 - x_1)}{2}\left[\mathcal{A}\left(\alpha\mathfrak{b}_1{}^{n+1}, \beta\mathfrak{b}_2{}^{n+1}\right) + \frac{\gamma}{2}\mathcal{A}\left(x_1^{n+1}, x_2^{n+1}\right)\right]. \qquad (4.2)\end{aligned}$$

*Proof.* The proof directly follows from Theorem 3.2 applying for $\Phi(x) = x^{n+2}$ and $\nu = \omega = \kappa = 1$. □

**Proposition 4.3.** *Let $x_1, x_2 \in [\mathfrak{b}_1, \mathfrak{b}_2]$ with $0 < \mathfrak{b}_1 < \mathfrak{b}_2$ and $\alpha, \beta, \in [0,1]$, $\gamma \in (0,1]$ be coefficients of sums $\alpha + \beta + \gamma = 1$, then for $n > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$, we have*

$$\begin{aligned}&\left|(2\mathcal{A}(\alpha\mathfrak{b}_1, \beta\mathfrak{b}_2) + \gamma\mathcal{A}(x_1, x_2))^{n+2} - \mathcal{L}_{n+2}^{n+2}(\alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_1, \alpha\mathfrak{b}_1 + \beta\mathfrak{b}_2 + \gamma x_2)\right| \\ &\leq \frac{\gamma(n+2)(x_2 - x_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(2\mathcal{A}\left(\alpha\mathfrak{b}_1{}^{q(n+1)}, \beta\mathfrak{b}_2{}^{q(n+1)}\right) + \frac{\gamma}{2}\mathcal{A}\left(x_1^{q(n+1)}, 3x_2^{q(n+1)}\right)\right)^{\frac{1}{q}}\right. \\ &\left. + \left(2\mathcal{A}\left(\alpha\mathfrak{b}_1{}^{q(n+1)}, \beta\mathfrak{b}_2{}^{q(n+1)}\right) + \frac{\gamma}{2}\mathcal{A}\left(3x_1^{q(n+1)}, x_2^{q(n+1)}\right)\right)^{\frac{1}{q}}\right]. \qquad (4.3)\end{aligned}$$

*Proof.* The proof directly follows from Theorem 3.3 applying for $\Phi(x) = x^{n+2}$ and $\nu = \omega = \kappa = 1$. □

**Proposition 4.4.** *Let $x_1, x_2 \in [\flat_1, \flat_2]$ with $0 < \flat_1 < \flat_2$ and $\alpha, \beta, \in [0,1]$, $\gamma \in (0,1]$ be coefficients of sums $\alpha+\beta+\gamma = 1$, then for $n > 1$ and $q \geq 1$, we have*

$$\begin{aligned}&\left|(2\mathcal{A}(\alpha\flat_1, \beta\flat_2) + \gamma\mathcal{A}(x_1, x_2))^{n+2} - \mathcal{L}_{n+2}^{n+2}(\alpha\flat_1 + \beta\flat_2 + \gamma x_1, \alpha\flat_1 + \beta\flat_2 + \gamma x_2)\right| \\ &\leq \frac{\gamma(n+2)(x_2 - x_1)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\mathcal{A}\left(\alpha{\flat_1}^{q(n+1)}, \beta{\flat_2}^{q(n+1)}\right) + \frac{\gamma}{3}\mathcal{A}\left(2x_1^{q(n+1)}, x_2^{q(n+1)}\right)\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\mathcal{A}\left(\alpha{\flat_1}^{q(n+1)}, \beta{\flat_2}^{q(n+1)}\right) + \frac{\gamma}{3}\mathcal{A}\left(x_1^{q(n+1)}, 2x_2^{q(n+1)}\right)\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{4.4}$$

*Proof.* The proof directly follows from Theorem 3.4 applying for $\Phi(x) = x^{n+2}$ and $\nu = \omega = \kappa = 1$. □

**Remark 4.1.** *For suitable choices of function $\Phi$, many other interesting inequalities regarding new special means can be derived. We omit here their proofs and the details are left to the interested reader.*

*4.2. Trapezoidal quadrature formula*

Let consider some applications of the integral inequalities obtained above, to find new error bounds for the trapezoidal quadrature formula. First, we fix three parameters $\alpha, \beta, \in [0,1]$, $\gamma \in (0,1]$ such that $\alpha+\beta+\gamma = 1$.

For $\flat_2 > \flat_1 > 0$, let $\mathcal{U} : \flat_1 = \chi_0 < \chi_1 < \ldots < \chi_{n-1} < \chi_n = \flat_2$ be a partition of $[\flat_1, \flat_2]$ and $x_{i,1}, x_{i,2} \in [\chi_i, \chi_{i+1}]$ for all $i = 0, 1, 2, \ldots, n-1$.

We denote, respectively,

$$\mathcal{S}(\mathcal{U}, \Phi) := \gamma \sum_{i=0}^{n-1} \Phi\left(\alpha\chi_i + \beta\chi_{i+1} + \gamma\frac{x_{i,1} + x_{i,2}}{2}\right)\hbar_i,$$

and

$$\int_{\alpha\flat_1+\beta\flat_2+\gamma x_1}^{\alpha\flat_1+\beta\flat_2+\gamma x_2} \Phi(u)\mathrm{d}u := \mathcal{S}(\mathcal{U}, \Phi) + \mathcal{R}(\mathcal{U}, \Phi),$$

where $\mathcal{R}(\mathcal{U}, \Phi)$ is the remainder term and $\hbar_i = \chi_{i+1} - \chi_i$.

Using above notations, we are in position to prove the following error estimations.

**Proposition 4.5.** *Under the assumptions of Theorem 3.2, if we take $\nu = \omega = \kappa = 1$, then the following inequality holds:*

$$|\mathcal{R}(\mathcal{U}, \Phi)| \leq \frac{\gamma}{4} \sum_{i=0}^{n-1} \hbar_i^2 \left[\alpha|\Phi'(\chi_i)| + \beta|\Phi'(\chi_{i+1})| + \gamma\frac{|\Phi'(x_{i,1})| + |\Phi'(x_{i,2})|}{2}\right].$$

*Proof.* Using the Theorem 3.2 on subinterval $[\chi_i, \chi_{i+1}]$ of closed interval $[\flat_1, \flat_2]$ and choosing $\nu = \omega = \kappa = 1$, for all $i = 0, 1, 2, \ldots, n-1$, we have

$$\left|\gamma\Phi\left(\alpha\chi_i + \beta\chi_{i+1} + \gamma\frac{x_{i,1} + x_{i,2}}{2}\right)\hbar_i - \int_{\alpha\chi_i+\beta\chi_{i+1}+\gamma x_{i,1}}^{\alpha\chi_i+\beta\chi_{i+1}+\gamma x_{i,2}} \Phi(u)\mathrm{d}u\right| \tag{4.5}$$

$$\leq \frac{\gamma}{4}\hbar_i^2\left[\alpha|\Phi'(\chi_i)| + \beta|\Phi'(\chi_{i+1})| + \gamma\frac{|\Phi'(x_{i,1})| + |\Phi'(x_{i,2})|}{2}\right].$$

Summing inequality (4.5) over $i$ from 0 to $n-1$ and using the properties of the modulus, we obtain the desired inequality. □

**Proposition 4.6.** *Under the assumptions of Theorem 3.3, if we take* $\nu = \omega = \kappa = 1$, *then the following inequality holds:*

$$|\mathcal{R}(\mathcal{U},\Phi)| \leq \frac{\gamma}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}\hbar_i^2\left[\left(\alpha|\Phi'(\chi_i)|^q + \beta|\Phi'(\chi_{i+1})|^q + \gamma\left(\frac{1}{4}|\Phi'(x_{i,1})|^q + \frac{3}{4}|\Phi'(x_{i,2})|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\alpha|\Phi'(\chi_i)|^q + \beta|\Phi'(\chi_{i+1})|^q + \gamma\left(\frac{1}{4}|\Phi'(x_{i,2})|^q + \frac{3}{4}|\Phi'(x_{i,1})|^q\right)\right)^{\frac{1}{q}}\right].$$

*Proof.* Applying the same technique as in Proposition 4.5 but using Theorem 3.3 and choosing $\nu = \omega = \kappa = 1$. □

**Proposition 4.7.** *Under the assumptions of Theorem 3.4, if we take* $\nu = \omega = \kappa = 1$, *then the following inequality holds:*

$$|\mathcal{R}(\mathcal{U},\Phi)| \leq \frac{\gamma}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\sum_{i=0}^{n-1}\hbar_i^2\left[\left(\frac{\alpha}{2}|\Phi'(\chi_i)|^q + \frac{\beta}{2}|\Phi'(\chi_{i+1})|^q + \gamma\left(\frac{1}{3}|\Phi'(x_{i,1})|^q + \frac{1}{6}|\Phi'(x_{i,2})|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{\alpha}{2}|\Phi'(\chi_i)|^q + \frac{\beta}{2}|\Phi'(\chi_{i+1})|^q + \gamma\left(\frac{1}{6}|\Phi'(x_{i,1})|^q + \frac{1}{3}|\Phi'(x_{i,2})|^q\right)\right)^{\frac{1}{q}}\right].$$

*Proof.* Applying the same technique as in Proposition 4.5 but using Theorem 3.4 and choosing $\nu = \omega = \kappa = 1$. □

## 5. Conclusions

In this paper, we have established some new Hermite–Hadamard–Mercer type of inequalities by using $\kappa$–Riemann–Liouville fractional integrals. Moreover, we have derived two new integral identities as auxiliary results. From the applied identities as auxiliary results, we have obtained some new variants of Hermite–Hadamard–Mercer type via $\kappa$–Riemann–Liouville fractional integrals. Several special cases are deduced in details and some know results are recaptured as well. In order to illustrate the efficiency of our main results, some applications regarding special means of positive real numbers and error estimations for trapezoidal quadrature formula are provided as well. To the best of our knowledge these results are new in the literature. Since the class of convex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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**Conflict of interest**

The authors declare that they have no competing interests.

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