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Research article

Ulam-Hyers-Rassias stability for a class of nonlinear implicit Hadamard fractional integral boundary value problem with impulses

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Abstract: This paper considers a class of nonlinear implicit Hadamard fractional differential equations with impulses. By using Banach's contraction mapping principle, we establish some sufficient criteria to ensure the existence and uniqueness of solution. Furthermore, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of this system are obtained by applying nonlinear functional analysis technique. As applications, an interesting example is provided to illustrate the effectiveness of main results.

Keywords: Hadamard fractional integral BVP; existence and uniqueness; stability; contraction mapping principle
Mathematics Subject Classification: 34A08, 34B10, 34B37, 34D20

1. Introduction

Since the beginning of the 1695, a number of different types of fractional differential have been proposed after the exploration and research of many famous scholars such as L'Hôpital, Leibniz, J. Bernoulli, Euler, Fourier, Lagrange, De Morgen, Laplace, Ya Sonin, Lacroix, Abel, Cantor, Riemann, Liouville, Caputo, Weyl, Grünwald, Letnikov, and so forth. These famous and important fractional derivatives mainly include Riemann-Liouville fractional derivative, Grünwald-Letnikov fractional derivative, Caputo fractional derivative, Weyl fractional derivative, et al. Particularly, Hadamard [1] proposed a new kind of fractional derivative in 1892. This type of fractional derivative is defined by the logarithmic function of any index in the integral kernel. The theory of fractional calculus becomes more and more mature after people study it carefully and deeply. There have been many monographs (see [2–9]) that systematically introduce the theory and applications of fractional calculus.

One important reason for the flourishing development of fractional calculus theory is its wide application in many fields of science, technology and engineering such as physics, chemistry, aerodynamics, electrodynamics, capacitor theory, electrical circuits, biology, control theory, and so on. Compared with the integral calculus, the fractional calculus has great advantages and accuracy in characterizing the phenomena and processes with memory, heredity and viscoelasticity. For example, Hooke's law and Newton's law of fluid would no longer be applicable in viscoelastic mechanics. To solve this problem, Scott-Blair [10], Gerasimov [11] and other scholars proposed to use fractional derivative instead of integer derivative to describe the relationship between stress and strain. They established a kind of generalized high-order model as follows

$$\sum_{j=0}^{n} c_j D^{\alpha_j}\xi(t) = \sum_{j=0}^{m} d_j D^{\beta_j}\eta(t),\ 0 < \alpha_j,\ \beta_j < 1,$$

where $\xi(t)$ and $\eta(t)$ represent stress and strain in viscoelastic mechanics, respectively. D^{α_j} and D^{β_j} are the fractional derivative operator. c_j and d_j are the ratio coefficients. Recently, some new results have been obtained in the study of new fractional viscoelastic mechanics models [12, 13] and fractional partial differential equation [14–16]. Moreover, the boundary value problem of fractional differential equation is a powerful mathematical tool for describing dynamic processes such as blood flow problems, underground water flow, population dynamics, and so on. So many scholars have extensively studied the fractional order boundary value problem, especially the Hadamard fractional order boundary value problem (see [17–21]).

The stability of a system is one of the important problems that must be considered when describing and designing a system. The transportation systems, for example, are expected to be smooth and safe. These involve the stability of the system in mathematics. Therefore, it is very important to study the stability of the differential equation system. The Ulam-Hyers (UH) stability proposed by Ulam [22] and Hyers [23] is one of the most important. Many scholars begin to study the UH-stability of various systems and obtain some good results (see [21, 24–30]). In addition, the impulsive phenomenon is ubiquitous in many systems. Therefore, it is necessary to consider the effect of the impulses in the fractional order differential equation system.

Inspired by the above arguments, we shall consider the following impulsive nonlinear boundary value problem of Hadamard fractional system

$$\begin{cases} {}^{H}D^{\alpha}_{t_k}y(t) = \sum\limits_{i=1}^{m} f_i(t, y(t), {}^{H}D^{\alpha}_{t_k}y(t), {}^{H}D^{\beta_i}_{t_k}y(t)),\ t \in (t_k, t_{k+1}] \subset J,\quad 0 \le k \le n, \\ {}^{H}J^{1-\alpha}_{t_k}y(t_k^+) - {}^{H}J^{1-\alpha}_{t_{k-1}}y(t_k^-) = I_k(y(t_k)),\ 1 \le k \le n, \\ {}^{H}J^{1-\alpha}_{a}y(a) = \lambda \cdot {}^{H}J^{1-\alpha}_{t_n}y(T), \end{cases} \tag{1.1}$$

where $J = [a, T], 0 < a < T, 0 < \beta_i < \alpha < 1 (i = 1, 2, \dots, m)$ and $\lambda \in R$ are some constants. ${}^{H}D^{*}_{t_k}$ stands the left-sided Hadamard fractional derivatives of order $*$. ${}^{H}J^{1-\alpha}_{t_k}$ is the left-sided Hadamard fractional integrals of order $1 - \alpha$. $f_i \in C(J \times R^3, R)$, $I_k \in C(R, R)$. The impulsive point sequence $\{t_k\}_{k=1}^{n}$ satisfies $a = t_0 < t_1 < t_2 < t_3 < \dots < t_n < t_{n+1} = T$. ${}^{H}J^{1-\alpha}_{t_k}y(t_k^+)$ and ${}^{H}J^{1-\alpha}_{t_{k-1}}y(t_k^-)$ represent the right and left limits at $t = t_k$ and satisfy ${}^{H}J^{1-\alpha}_{t_{k-1}}y(t_k^-) = {}^{H}J^{1-\alpha}_{t_{k-1}}y(t_k)$, respectively.

In addition, the authors [21] enlightened us on models and research ideals. They studied the existence and UH and UHR stability of solutions for the next system described as

$$\begin{cases} {}^{c}D^{\alpha}_{t_k}y(t) = f(t, y(t), {}^{c}D^{\alpha}_{t_k}y(t)),\ t \in (t_k, t_{k+1}] \subset J,\quad 0 \le k \le m,\ 0 < \alpha \le 1, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\ 1 \le k \le m, \\ ay(1) + by(T) = c, \end{cases}$$

where $J = [1, T]$, ${}^cD^{\alpha}_{t_k}$ is the Caputo-Hadamard fractional derivative, $f : J \times R^2 \to R$ is a given function, $I_k : R \to R$, a, b and c are real constants with $a + b \neq 0$. The impulsive point satisfy $1 = t_0 < t_1 < t_2 < t_3 < \ldots < t_n < t_{n+1} = T$. $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits and satisfy the left continuity at $t = t_k$, respectively.

In this article, we mainly investigate the existence and UH and UHR stability of solutions for the problem (1.1). Through the study of the system (1.1), some ideas and inspirations are provided for the study of the implicit fractional order differential equation with complex impulsive condition and boundary value condition. The remainder of the paper is structured as follows. In Section 2, we state some useful concepts of fractional calculus and auxiliary results. In Section 3, we establish some sufficient criteria to ensure the existence, uniqueness and stability of problem (1.1). In Section 4, we apply an example to illustrate the effectiveness of our results. Finally, we give a brief summary of the research objects, methods and discussions in Section 5.

2. Preliminaries

Let $C(J, R)$ be the Banach space of all continuous functions from J into R with the norm $\|y\|_C = \sup_{t \in J} |y(t)|$. The space of piecewise continuous functions is defined by

$$PC(J,R) = \Big\{ y : J \to R \,\Big|\, y \in C((t_l, t_{l+1}], R), l = 0, 1, \ldots, n, \text{ and there exist } {}^HJ^{1-\alpha}_{t_k} y(t_k^+) \text{ and } {}^HJ^{1-\alpha}_{t_{k-1}} y(t_k^-) \text{ with } {}^HJ^{1-\alpha}_{t_{k-1}} y(t_k^-) = {}^HJ^{1-\alpha}_{t_{k-1}} y(t_k),\ 1 \le k \le n \Big\}.$$

Clearly, $PC(J, R)$ is a Banach space equipped with the norm

$$\|y\|_{PC} = \max \left\{ \sup_{t \in J} |y(t)|, \max_{0 \le k \le m} \sup_{t_k < t \le t_{k+1}} |{}^HD^{\alpha}_{t_k} y(t)|, \max_{0 \le k \le m} \sup_{t_k < t \le t_{k+1}} |{}^HD^{\beta_i}_{t_k} y(t)|, 1 \le i \le m \right\}.$$

Definition 2.1. *[2] For $a > 0$, the left-sided Hadamard fractional integral of order $\alpha > 0$ for a function $y : [0, \infty) \to R$ is defined by*

$$ {}^HJ^{\alpha}_a y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(log \frac{t}{s} \right)^{\alpha - 1} \frac{y(s)}{s} ds,$$

provided the integral exists, where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ and $log(\cdot) = log_e(\cdot)$.

Definition 2.2. *[2] For $a > 0$, $y \in C^n([a, \infty))$, the left-sided Hadamard fractional derivative of order $\alpha > 0$ for a function $y : [0, \infty) \to R$ is defined as*

$$ {}^HD^{\alpha}_a y(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt} \right)^n \int_a^t \left(log \frac{t}{s} \right)^{n - \alpha - 1} \frac{y(s)}{s} ds,$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1. *[2] For $a > 0$, assume that $y \in C^n(a, T) \bigcap L(a, T)$ with a left-sided Hadamard fractional derivative of order $\alpha > 0$. Then*

$$ {}^HJ^{\alpha}_a ({}^HD^{\alpha}_a y(t)) = y(t) + c_1 \left(log \frac{t}{a} \right)^{\alpha - 1} + c_2 \left(log \frac{t}{a} \right)^{\alpha - 2} + \ldots + c_n \left(log \frac{t}{a} \right)^{\alpha - n},$$

where $c_i \in R$, $i = 1, ..., n - 1$, and $n = [\alpha] + 1$.

Lemma 2.2. *[2] Assume that $\alpha > 0$, $\beta > 0$ and $0 < a < \infty$. then the following properties hold:*

$$
\begin{aligned}
&{}^{H}D_a^{\alpha}\left(log\frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(log\frac{x}{a}\right)^{\beta-\alpha-1},\\
&{}^{H}J_a^{\alpha}\left(log\frac{t}{a}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(log\frac{x}{a}\right)^{\beta+\alpha-1},\\
&{}^{H}D_a^{\alpha}\left(log\frac{t}{a}\right)^{\alpha-j}(x) = 0,\ \ j = [\alpha]+1,\\
&{}^{H}D_a^{\alpha}({}^{H}J_a^{\beta}y(t)) = {}^{H}J_a^{\beta-\alpha}y(t),\ \ {}^{H}J_a^{\beta}({}^{H}J_a^{\alpha}y(t)) = {}^{H}J_a^{\alpha+\beta}y(t).
\end{aligned}
$$

Now we introduce some concepts of Ulam-Hyers stability and Ulam-Hyers-Rassias stability.

Let $z \in PC(J,R)$, $\epsilon, \psi > 0$, and $\varphi \in PC(J,R)$ be non-decreasing. Consider two inequalities as follows:

$$
\begin{cases}
|{}^{H}D_{t_k}^{\alpha}z(t) - \sum\limits_{i=1}^{m} f_i(t, z(t), {}^{H}D_{t_k}^{\alpha}z(t), {}^{H}D_{t_k}^{\beta_i}z(t))| \le \epsilon,\ \ t \in (t_k, t_{k+1}],\ \ 0 \le k \le n,\\
|{}^{H}J_{t_k}^{1-\alpha}z(t_k^+) - {}^{H}J_{t_k}^{1-\alpha}z(t_k^-) - I_k(z(t_k))| \le \epsilon,\ \ 1 \le k \le n,
\end{cases}
\tag{2.1}
$$

$$
\begin{cases}
|{}^{H}D_{t_k}^{\alpha}z(t) - \sum\limits_{i=1}^{m} f_i(t, z(t), {}^{H}D_{t_k}^{\alpha}z(t), {}^{H}D_{t_k}^{\beta_i}z(t))| \le \epsilon\varphi(t),\ \ t \in (t_k, t_{k+1}],\ \ 0 \le k \le n,\\
|{}^{H}J_{t_k}^{1-\alpha}z(t_k^+) - {}^{H}J_{t_k}^{1-\alpha}z(t_k^-) - I_k(z(t_k))| \le \epsilon\psi,\ \ 1 \le k \le n,
\end{cases}
\tag{2.2}
$$

Definition 2.3. *The system* (1.1) *is called Ulam-Hyers stable if there exists a real number $C_1 > 0$ such that for each $\epsilon > 0$ and each solution $z \in PC(J,R)$ of inequality* (2.1)*, there exists a solution $y \in PC(J,R)$ of system* (1.1) *satisfying*

$$\|z(t) - y(t)\|_{PC} \le C_1\epsilon.$$

Definition 2.4. *The system* (1.1) *is called Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists a real number $C_3 > 0$ such that for each $\epsilon > 0$ and each solution $z \in PC(J,R)$ of inequality* (2.2)*, there exists a solution $y \in PC(J,R)$ of system* (1.1) *satisfying*

$$\|z(t) - y(t)\|_{PC} \le C_3\epsilon(\varphi(t) + \psi).$$

Remark 2.1. *A function $z \in PC(J,R)$ is a solution of inequality* (2.1) *if and only if there exists a function $\phi \in PC(J,R)$ and a sequence $\{\phi_k\}_{k=1}^{n}$ such that*

(i) $|\phi(t)| \le \epsilon$, $t \in (t_l, t_{l+1}]$, $0 \le l \le n$, *and* $|\phi_k| \le \epsilon$, $1 \le k \le n$.

(ii) ${}^{H}D_{t_k}^{\alpha}z(t) = \sum\limits_{i=1}^{m} f_i(t, z(t), {}^{H}D_{t_k}^{\alpha}z(t), {}^{H}D_{t_k}^{\beta_i}z(t)) + \phi(t),\ \ t \in (t_k, t_{k+1}],\ \ 0 \le k \le n.$

(iii) ${}^{H}J_{t_k}^{1-\alpha}z(t_k^+) - {}^{H}J_{t_{k-1}}^{1-\alpha}z(t_k^-) = I_k(z(t_k)) + \phi_k,\ \ 1 \le k \le n.$

Remark 2.2. *A function $z \in PC(J,R)$ is a solution of inequality* (2.2) *if and only if there exists a function $\omega \in PC(J,R)$ and a sequence $\{\omega_k\}_{k=1}^{n}$ such that*

(i) $|\omega(t)| \le \epsilon\varphi(t)$, $t \in (t_l, t_{l+1}]$, $0 \le l \le n$, *and* $|\omega_k| \le \epsilon\psi$, $1 \le k \le n$.

(ii) ${}^{H}D_{t_k}^{\alpha}z(t) = \sum\limits_{i=1}^{m} f_i(t, z(t), {}^{H}D_{t_k}^{\alpha}z(t), {}^{H}D_{t_k}^{\beta_i}z(t)) + \omega(t),\ \ t \in (t_k, t_{k+1}],\ \ 0 \le k \le n.$

(iii) ${}^{H}J_{t_k}^{1-\alpha}z(t_k^+) - {}^{H}J_{t_{k-1}}^{1-\alpha}z(t_k^-) = I_k(z(t_k)) + \omega_k,\ \ 1 \le k \le n.$

Lemma 2.3. *(Banach's contraction mapping principle [31]) Let $\mathbb{E}$ be a non-empty closed subset of a Banach space $\mathbb{X}$. If $\mathcal{T} : \mathbb{E} \to \mathbb{E}$ is a contraction mapping, then $\mathcal{T}$ has a unique fixed point $x^* \in \mathbb{E}$.*

3. Existence, uniqueness and stability

In this section, we shall focus on the existence, uniqueness, UH and UHR stability of the solutions for system (1.1). To do this, we need to prove the following important lemma.

Lemma 3.1. *Let* $0 < \alpha < 1$, $\lambda \in R$ *with* $\lambda \neq 1$, $\sigma_l \in C(J, R)$ $(l = 0, 1, 2, \ldots, n)$ *and* $I_k \in C(R, R)$ $(k = 1, 2, \ldots, n)$. *Then a function* $y(t) \in PC(J, R)$ *is a solution of the following impulsive linear fractional differential equation*

$$\begin{cases} {}^{H}D^{\alpha}_{t_k} y(t) = \sigma_k(t), \ t \in (t_k, t_{k+1}] \subset J, \ 0 \leq k \leq n, \\ {}^{H}J^{1-\alpha}_{t_k} y(t_k^+) - {}^{H}J^{1-\alpha}_{t_{k-1}} y(t_k^-) = I_k(y(t_k)), \ 1 \leq k \leq n, \\ {}^{H}J^{1-\alpha}_{a} y(a) = \lambda \cdot {}^{H}J^{1-\alpha}_{t_n} y(T), \end{cases} \tag{3.1}$$

if and only if $y(t) \in PC(J, R)$ *is a solution of the impulsive fractional integral equation as follows:*

$$y(t) = \begin{cases} {}^{H}J^{\alpha}_{a}\sigma_0(t) + c_1 \left(log\frac{t}{a}\right)^{\alpha-1}, \ t \in [a, t_1], \\ {}^{H}J^{\alpha}_{t_k}\sigma_k(t) + c_{k+1} \left(log\frac{t}{t_k}\right)^{\alpha-1}, \ t \in (t_k, t_{k+1}], \ 1 \leq k \leq n, \end{cases} \tag{3.2}$$

where

$$c_1 = \frac{\lambda}{(1-\lambda)\Gamma(\alpha)} \Big[\sum_{j=1}^{n} I_j(y(t_j)) + \sum_{j=1}^{n} {}^{H}J^{1}_{t_{j-1}}\sigma_{j-1}(t_j) + {}^{H}J^{1}_{t_n}\sigma_n(T) \Big],$$

$$c_{k+1} = \frac{1}{(1-\lambda)\Gamma(\alpha)} \Big[\sum_{j=1}^{k} I_j(y(t_j)) + \sum_{j=1}^{k} {}^{H}J^{1}_{t_{j-1}}\sigma_{j-1}(t_j) + \lambda \sum_{j=k+1}^{n} I_j(y(t_j))$$
$$+ \lambda \sum_{j=k+1}^{n} {}^{H}J^{1}_{t_{j-1}}\sigma_{j-1}(t_j) + \lambda\, {}^{H}J^{1}_{t_n}\sigma_n(T) \Big], \ k = 1, 3, \ldots, n.$$

Proof. Assume that $y(t) \in PC(J, R)$ is a solution of (3.1). When $t \in [t_0, t_1] = [a, t_1]$, integrating at both ends of of the first equation in (3.1) and applying Lemmas 2.1 and 2.2, we have

$$y(t) = {}^{H}J^{\alpha}_{t_0}\sigma_0(t) + c_1 \left(log\frac{t}{t_0}\right)^{\alpha-1}, \ {}^{H}J^{1-\alpha}_{t_0} y(t) = {}^{H}J^{1}_{t_0}\sigma_0(t) + c_1\Gamma(\alpha). \tag{3.3}$$

We derive from (3.3) that

$${}^{H}J^{1-\alpha}_{t_0} y(a) = c_1\Gamma(\alpha), \ {}^{H}J^{1-\alpha}_{t_0} y(t_1^-) = {}^{H}J^{1}_{t_0}\sigma_0(t_1) + c_1\Gamma(\alpha). \tag{3.4}$$

When $t \in (t_1, t_2]$, similar to (3.3) and (3.4), we get

$$y(t) = {}^{H}J^{\alpha}_{t_1}\sigma_1(t) + c_2 \left(log\frac{t}{t_1}\right)^{\alpha-1}, \ {}^{H}J^{1-\alpha}_{t_1} y(t) = {}^{H}J^{1}_{t_1}\sigma_1(t) + c_2\Gamma(\alpha), \tag{3.5}$$

$${}^{H}J^{1-\alpha}_{t_1} y(t_1^+) = c_2\Gamma(\alpha), \ {}^{H}J^{1-\alpha}_{t_1} y(t_2^-) = {}^{H}J^{1}_{t_1}\sigma_1(t_2) + c_2\Gamma(\alpha). \tag{3.6}$$

In the light of (3.4), (3.6) and the impulsive conditions of (3.1), we obtain

$$c_2 - c_1 = \frac{1}{\Gamma(\alpha)}\Big[I_1(y(t_1)) + {}^H J^1_{t_0}\sigma_0(t_1)\Big]. \tag{3.7}$$

When $t \in (t_k, t_{k+1}]$, $k = 2, 3, \ldots, n$, applying the mathematical induction, we find that

$$y(t) = {}^H J^\alpha_{t_k}\sigma_k(t) + c_{k+1}\left(log\frac{t}{t_k}\right)^{\alpha-1}, \ {}^H J^{1-\alpha}_{t_k} y(t) = {}^H J^1_{t_k}\sigma_k(t) + c_{k+1}\Gamma(\alpha), \tag{3.8}$$

$${}^H J^{1-\alpha}_{t_k} y(t_k^+) = c_{k+1}\Gamma(\alpha), \ {}^H J^{1-\alpha}_{t_k} y(t_{k+1}^-) = {}^H J^1_{t_k}\sigma_k(t_{k+1}) + c_{k+1}\Gamma(\alpha), \tag{3.9}$$

$$c_{k+1} - c_k = \frac{1}{\Gamma(\alpha)}\Big[I_k(y(t_k)) + {}^H J^1_{t_{k-1}}\sigma_{k-1}(t_k)\Big]. \tag{3.10}$$

In view of (3.7) and (3.10), we get for $k = 2, 3, \ldots, n$,

$$c_{k+1} - c_1 = \sum_{j=1}^{k}(c_{j+1} - c_j) = \frac{1}{\Gamma(\alpha)}\Big[\sum_{j=1}^{k} I_j(y(t_j)) + \sum_{j=1}^{k} {}^H J^1_{t_{j-1}}\sigma_{j-1}(t_j)\Big]. \tag{3.11}$$

From (3.4), (3.8), (3.11) and the boundary value condition of (3.1), we have

$$\begin{cases} c_1\Gamma(\alpha) = \lambda[{}^H J^1_{t_n}\sigma_n(T) + c_{n+1}\Gamma(\alpha)], \\ c_{n+1}\Gamma(\alpha) - c_1\Gamma(\alpha) = \sum_{j=1}^{n} I_j(y(t_j)) + \sum_{j=1}^{n} {}^H J^1_{t_{j-1}}\sigma_{j-1}(t_j). \end{cases} \tag{3.12}$$

By solving the Eq (3.12), we get

$$c_1 = \frac{\lambda}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{n} I_j(y(t_j)) + \sum_{j=1}^{n} {}^H J^1_{t_{j-1}}\sigma_{j-1}(t_j) + {}^H J^1_{t_n}\sigma_n(T)\Big], \tag{3.13}$$

$$c_{n+1} = \frac{1}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{n} I_j(y(t_j)) + \sum_{j=1}^{n} {}^H J^1_{t_{j-1}}\sigma_{j-1}(t_j) + \lambda\, {}^H J^1_{t_n}\sigma_n(T)\Big]. \tag{3.14}$$

Bring (3.13) into (3.11), we derive

$$\begin{aligned} c_{k+1} = &\frac{1}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{k} I_j(y(t_j)) + \sum_{j=1}^{k} {}^H J^1_{t_{j-1}}\sigma_{j-1}(t_j) + \lambda \sum_{j=k+1}^{n} I_j(y(t_j)) \\ &+ \lambda \sum_{j=k+1}^{n} {}^H J^1_{t_{j-1}}\sigma_{j-1}(t_j) + \lambda\, {}^H J^1_{t_n}\sigma_n(T)\Big], \ k = 1, 2, \ldots, n. \end{aligned} \tag{3.15}$$

From (3.3), (3.5), (3.8) and (3.13)–(3.15), we obtain (3.2). That is, $y(t) \in PC(J, \mathbb{R})$ is a solution of (3.2). Contrariwise, assume that $y(t) \in PC(J, R)$ is a solution of (3.2), according to Lemma 2.2, we have

$${}^H D^\alpha_a y(t) = {}^H D^\alpha_a\Big[{}^H J^\alpha_a\sigma_0(t) + c_1\left(log\frac{t}{a}\right)^{\alpha-1}\Big] = \sigma_0(t), \ t \in [a, t_1], \tag{3.16}$$

and

$$ {}^{H}D_a^{\alpha}y(t) = {}^{H}D_a^{\alpha}\Big[{}^{H}J_{t_k}^{\alpha}\sigma_k(t) + c_{k+1}\left(log\frac{t}{t_k}\right)^{\alpha-1} \Big] = \sigma_k(t),\ t \in (t_k, t_{k+1}],\ 1 \le k \le n. \tag{3.17} $$

By (3.9) and (3.10), we get

$$ {}^{H}J_{t_k}^{1-\alpha}y(t_k^+) - {}^{H}J_{t_{k-1}}^{1-\alpha}y(t_k^-) = I_k(y(t_k)),\ 1 \le k \le n. \tag{3.18} $$

In view of (3.4) and (3.12), we derive

$$ {}^{H}J_a^{1-\alpha}y(a) = \lambda \cdot {}^{H}J_{t_n}^{1-\alpha}y(T). \tag{3.19} $$

From (3.16)–(3.19), we know that $y(t) \in PC(J, R)$ satifies (3.1), namely, $y(t) \in PC(J, R)$ is also a solution of (3.1). The proof is completed. □

Based upon Lemma 3.1, we have the next significant lemma on system (1.1).

Lemma 3.2. *Let* $0 < \alpha < 1$, $\lambda \in R$ *with* $\lambda \neq 1$, $f_i \in C(J \times R^3, R)$ $(i = 1, 2, \dots, m)$ *and* $I_k \in C(R, R)$ $(k = 1, 2, \dots, n)$*. Then the function* $y(t) \in PC(J, R)$ *is a solution of system* (1.1) *if and only if* $y(t) \in PC(J, R)$ *is a solution of the following impulsive fractional integral equation*

$$ y(t) = \begin{cases} \sum\limits_{i=1}^{m} {}^{H}J_a^{\alpha}\left[f_i(t, y(t), {}^{H}D_a^{\alpha}y(t), {}^{H}D_a^{\beta_i}y(t))\right] + C_{1,y}^{*}\left(log\frac{t}{a}\right)^{\alpha-1},\ t \in [a, t_1], \\ \sum\limits_{i=1}^{m} {}^{H}J_{t_k}^{\alpha}\left[f_i(t, y(t), {}^{H}D_{t_k}^{\alpha}y(t), {}^{H}D_{t_k}^{\beta_i}y(t))\right] + C_{k+1,y}^{*}\left(log\frac{t}{t_k}\right)^{\alpha-1},\ t \in (t_k, t_{k+1}], \end{cases} \tag{3.20} $$

where $k = 1, 2, \dots, n$,

$$ \begin{aligned} C_{1,y}^{*} =& \frac{\lambda}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{n} I_j(y(t_j)) + \sum_{j=1}^{n}\sum_{i=1}^{m} {}^{H}J_{t_{j-1}}^{1}\left[f_i(t_j, y(t_j), {}^{H}D_{t_{j-1}}^{\alpha}y(t_j), {}^{H}D_{t_{j-1}}^{\beta_i}y(t_j))\right] \\ &+ \sum_{i=1}^{m} {}^{H}J_{t_n}^{1}\left[f_i(T, y(T), {}^{H}D_{t_n}^{\alpha}y(T), {}^{H}D_{t_n}^{\beta_i}y(T))\right]\Big], \end{aligned} \tag{3.21} $$

$$ \begin{aligned} C_{k+1,y}^{*} =& \frac{1}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{k} I_j(y(t_j)) + \sum_{j=1}^{k}\sum_{i=1}^{m} {}^{H}J_{t_{j-1}}^{1}\left[f_i(t_j, y(t_j), {}^{H}D_{t_{j-1}}^{\alpha}y(t_j), {}^{H}D_{t_{j-1}}^{\beta_i}y(t_j))\right] \\ &+ \lambda\sum_{j=k+1}^{n} I_j(y(t_j)) + \lambda\sum_{j=k+1}^{n}\sum_{i=1}^{m} {}^{H}J_{t_{j-1}}^{1}\left[f_i(t_j, y(t_j), {}^{H}D_{t_{j-1}}^{\alpha}y(t_j), {}^{H}D_{t_{j-1}}^{\beta_i}y(t_j))\right] \\ &+ \lambda\sum_{i=1}^{m} {}^{H}J_{t_n}^{1}\left[f_i(T, y(T), {}^{H}D_{t_n}^{\alpha}y(T), {}^{H}D_{t_n}^{\beta_i}y(T))\right]\Big],\ k = 1, 2, \dots, n. \end{aligned} \tag{3.22} $$

For the sake of discussion on the existence of solution for system (1.1), we define an operator $\mathcal{T} : PC(J, R) \to PC(J, R)$ according to Lemma 3.2 as follows:

$$ (\mathcal{T}y)(t) = \begin{cases} \sum\limits_{i=1}^{m} {}^{H}J_a^{\alpha}\left[f_i(t, y(t), {}^{H}D_a^{\alpha}y(t), {}^{H}D_a^{\beta_i}y(t))\right] + C_{1,y}^{*}\left(log\frac{t}{a}\right)^{\alpha-1},\ t \in [a, t_1], \\ \sum\limits_{i=1}^{m} {}^{H}J_{t_k}^{\alpha}\left[f_i(t, y(t), {}^{H}D_{t_k}^{\alpha}y(t), {}^{H}D_{t_k}^{\beta_i}y(t))\right] + C_{k+1,y}^{*}\left(log\frac{t}{t_k}\right)^{\alpha-1},\ t \in (t_k, t_{k+1}], \end{cases} \tag{3.23} $$

where $C^*_{1,y}$ and $C^*_{k+1,y}$ are given as (3.21) and (3.22), $k = 1, 2, \ldots, n$. In this way, the existence of the solutions of system (1.1) is equivalent to the existence of the fixed point of operator $\mathbb{F}$ defined as (3.23).

In the main results of this paper, we need the following basic assumptions.

(A_1) Assume that $f_i \in C(J \times R^3, R)$, $i = 1, 2, \ldots, m$, $I_k \in C(R, R)$, $k = 1, 2, \ldots, n$, and for all $t \in J$, $u, v, w, \bar{u}, \bar{v}, \bar{w} \in R$, there exist some constants $L_i, M_i, N_i > 0$ such that

$$|f_i(t, u, v, w) - f_i(t, \bar{u}, \bar{v}, \bar{w})| \leq L_i|u - \bar{u}| + M_i|v - \bar{v}| + N_i|w - \bar{w}|.$$

(A_2) Assume that λ, α and β_i ($i = 1, 2, \ldots, m$) are some real constants fatisfying $\lambda \neq 1$, $0 < \beta_i < \alpha < 1$, and $0 < \rho < 1$, where $\rho = \max\{\kappa, \rho_k, \theta_{i,k}, i = 1, 2, \ldots, m, k = 0, 1, 2, \ldots, n\}$,

$$\kappa = \sum_{i=1}^{m}(L_i + M_i + N_i),\ \ \rho_k = \frac{1}{\Gamma(\alpha + 1)}\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha}\sum_{i=1}^{m}(L_i + M_i + N_i),$$

$$\theta_{i,k} = \frac{1}{\Gamma(\alpha - \beta_i + 1)}\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha-\beta_i}\sum_{i=1}^{m}(L_i + M_i + N_i).$$

Theorem 3.1. *If the conditions* (A_1) *and* (A_2) *hold, then we have the next two assertions:*

(1) *the system* (1.1) *exists a unique solution* $y^*(t) \in PC(J, R)$.

(2) *the system* (1.1) *is Ulam-Hyers stable, that is, if* $z(t) \in PC(J, R)$ *is a solution of the inequality* (2.1), *and* $y^*(t) \in PC(J, R)$ *is a unique solution of system* (1.1), *then*

$$\|z(t) - y^*(t)\|_{PC} \leq \frac{\nu}{1 - \rho}\epsilon,$$

where $\nu = \max\{1, \eta_0, \eta_1, \eta_2, \ldots, \eta_n\}$, $\eta_k = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha}$, $k = 0, 1, 2, \ldots, n$.

Proof. We firstly show that the assertion (1) of Theorem 3.1 holds. Define the operator $\mathcal{T} : PC(J, R) \to PC(J, R)$ as (3.23). Now we verify that $\mathcal{T} : PC(J, R) \to PC(J, R)$ is contractive. In fact, for all $y(t) \in PC(J, R)$, associated with $f_i \in C(J \times R^3, R)$, $I_k \in C(R, R)$, (3.23) and Definition 2.1, we know that $(\mathcal{T}y)(t) \in PC(J, R)$. For all $t \in J = [0, T]$, $y(t), \bar{y}(t) \in PC(J, R)$, when $t \in [a, t_1] = [t_0, t_1]$, in the light of (3.21), (3.23), (A_1), (A_2) and Lemma 2.2, we have

$$\begin{aligned}
\left|(\mathcal{T}y)(t) - (\mathcal{T}\bar{y})(t)\right| &= \Bigg|\sum_{i=1}^{m} {}^{H}J^{\alpha}_{a}[f_i(t, y(t),\ {}^{H}D^{\alpha}_{a}y(t), {}^{H}D^{\beta_i}_{a}y(t)) \\
&\quad - f_i(t, \bar{y}(t),\ {}^{H}D^{\alpha}_{a}\bar{y}(t), {}^{H}D^{\beta_i}_{a}\bar{y}(t))] + (C^*_{1,y} - C^*_{1,\bar{y}})\left(log\frac{t}{a}\right)^{\alpha-1}\Bigg| \\
&\leq \sum_{i=1}^{m} {}^{H}J^{\alpha}_{a}\left|f_i(t, y(t),\ {}^{H}D^{\alpha}_{a}y(t), {}^{H}D^{\beta_i}_{a}y(t)) - f_i(t, \bar{y}(t),\ {}^{H}D^{\alpha}_{a}\bar{y}(t),\ {}^{H}D^{\beta_i}_{a}\bar{y}(t))\right| \\
&\quad + \left|C^*_{1,y} - C^*_{1,\bar{y}}\right|\left(log\frac{a}{a}\right)^{\alpha-1} \\
&= \sum_{i=1}^{m} {}^{H}J^{\alpha}_{a}\left|f_i(t, y(t),\ {}^{H}D^{\alpha}_{a}y(t), {}^{H}D^{\beta_i}_{a}y(t)) - f_i(t, \bar{y}(t),\ {}^{H}D^{\alpha}_{a}\bar{y}(t),\ {}^{H}D^{\beta_i}_{a}\bar{y}(t))\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} {}^{H}J_{a}^{\alpha}\Big[L_i|y(t)-\bar{y}(t)| + M_i\Big|{}^{H}D_{a}^{\alpha}[y(t)-\bar{y}(t)]\Big| + N_i\Big|{}^{H}D_{a}^{\beta_i}[y(t)-\bar{y}(t)]\Big|\Big]\\
&\leq \sum_{i=1}^{m} {}^{H}J_{a}^{\alpha}\Big[(L_i+M_i+N_i)\|y(t)-\bar{y}(t)\|_{PC}\Big]\\
&=\Big[\frac{1}{\Gamma(\alpha+1)}\Big(log\frac{t}{a}\Big)^{\alpha}\sum_{i=1}^{m}(L_i+M_i+N_i)\Big]\|y(t)-\bar{y}(t)\|_{PC}\\
&\leq\Big[\frac{1}{\Gamma(\alpha+1)}\Big(log\frac{t_1}{a}\Big)^{\alpha}\sum_{i=1}^{m}(L_i+M_i+N_i)\Big]\|y(t)-\bar{y}(t)\|_{PC}\\
&=\rho_1\|y(t)-\bar{y}(t)\|_{PC},
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
&\Big|{}^{H}D_{a}^{\alpha}[(\mathcal{T}y)(t)-(\mathcal{T}\bar{y})(t)]\Big| = \Big|\sum_{i=1}^{m} {}^{H}D_{a}^{\alpha}\Big({}^{H}J_{a}^{\alpha}[f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t))\\
&\quad - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),{}^{H}D_{a}^{\beta_i}\bar{y}(t))]\Big) + (C_{1,y}^{*}-C_{1,\bar{y}}^{*})\,{}^{H}D_{a}^{\alpha}\Big(log\frac{t}{a}\Big)^{\alpha-1}\Big|\\
&=\Big|\sum_{i=1}^{m}[f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t)) - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),{}^{H}D_{a}^{\beta_i}\bar{y}(t))]\Big|\\
&\leq\sum_{i=1}^{m}\Big|f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t)) - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),{}^{H}D_{a}^{\beta_i}\bar{y}(t))\Big|\\
&\leq\sum_{i=1}^{m}\Big[L_i|y(t)-\bar{y}(t)| + M_i\Big|{}^{H}D_{a}^{\alpha}[y(t)-\bar{y}(t)]\Big| + N_i\Big|{}^{H}D_{a}^{\beta_i}[y(t)-\bar{y}(t)]\Big|\Big]\\
&\leq\Big[\sum_{i=1}^{m}(L_i+M_i+N_i)\Big]\|y(t)-\bar{y}(t)\|_{PC} = \kappa\|y(t)-\bar{y}(t)\|_{PC},
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\Big|{}^{H}D_{a}^{\beta_i}[(\mathcal{T}y)(t)-(\mathcal{T}\bar{y})(t)]\Big| = \Big|\sum_{i=1}^{m} {}^{H}J_{a}^{\alpha}[f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t))\\
&\quad - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),{}^{H}D_{a}^{\beta_i}\bar{y}(t))] + (C_{1,y}^{*}-C_{1,\bar{y}}^{*})\Big(log\frac{t}{a}\Big)^{\alpha-1}\Big|\\
&=\Big|\sum_{i=1}^{m} {}^{H}D_{a}^{\beta_i}\Big({}^{H}J_{a}^{\alpha}[f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t))\\
&\quad - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),{}^{H}D_{a}^{\beta_i}\bar{y}(t))]\Big) + (C_{1,y}^{*}-C_{1,\bar{y}}^{*})\,{}^{H}D_{a}^{\beta_i}\Big(log\frac{t}{a}\Big)^{\alpha-1}\Big|\\
&=\sum_{i=1}^{m} {}^{H}J_{a}^{\alpha-\beta_j}\Big|f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t)) - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),\\
&\quad {}^{H}D_{a}^{\beta_i}\bar{y}(t))\Big| + \Big|C_{1,y}^{*}-C_{1,\bar{y}}^{*}\Big|\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}\Big(log\frac{a}{t}\Big)^{\alpha-\beta_i-1}\\
&\leq\sum_{i=1}^{m} {}^{H}J_{a}^{\alpha-\beta_j}\Big|f_i(t,y(t),{}^{H}D_{a}^{\alpha}y(t),{}^{H}D_{a}^{\beta_i}y(t)) - f_i(t,\bar{y}(t),{}^{H}D_{a}^{\alpha}\bar{y}(t),
\end{aligned}
$$

$$
\begin{aligned}
&{}^{H}D_a^{\beta_i}\bar{y}(t))\Big| + \left|C_{1,y}^* - C_{1,\bar{y}}^*\right| \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)}\left(log\frac{a}{a}\right)^{\alpha-\beta_i-1} \\
&= \sum_{i=1}^{m} {}^{H}J_a^{\alpha-\beta_i}\left| f_i(t, y(t), {}^{H}D_a^{\alpha}y(t), {}^{H}D_a^{\beta_i}y(t)) - f_i(t, \bar{y}(t), {}^{H}D_a^{\alpha}\bar{y}(t), {}^{H}D_a^{\beta_i}\bar{y}(t))\right| \\
&\leq \sum_{i=1}^{m} {}^{H}J_a^{\alpha-\beta_i}\Big[L_i|y(t)-\bar{y}(t)| + M_i\left|{}^{H}D_a^{\alpha}[y(t)-\bar{y}(t)]\right| + N_i\left|{}^{H}D_a^{\beta_i}[y(t)-\bar{y}(t)\right|\Big] \\
&\leq \sum_{i=1}^{m} {}^{H}J_a^{\alpha-\beta_i}\Big[(L_i + M_i + N_i)\|y(t)-\bar{y}(t)\|_{PC}\Big] \\
&= \left[\frac{1}{\Gamma(\alpha-\beta_i+1)}\left(log\frac{t}{a}\right)^{\alpha-\beta_i}\sum_{i=1}^{m}(L_i + M_i + N_i)\right]\|y(t)-\bar{y}(t)\|_{PC} \\
&\leq \left[\frac{1}{\Gamma(\alpha-\beta_i+1)}\left(log\frac{t_1}{a}\right)^{\alpha-\beta_i}\sum_{i=1}^{m}(L_i + M_i + N_i)\right]\|y(t)-\bar{y}(t)\|_{PC} \\
&= \theta_{i,1}\|y(t)-\bar{y}(t)\|_{PC}, \ i = 1, 2, \ldots, m. \qquad (3.26)
\end{aligned}
$$

When $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$, similar to (3.24)–(3.26), we derive

$$
\begin{aligned}
\left|(\mathcal{T}y)(t) - (\mathcal{T}\bar{y})(t)\right| &\leq \left[\frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha}\sum_{i=1}^{m}(L_i + M_i + N_i)\right]\|y(t)-\bar{y}(t)\|_{PC} \\
&= \rho_k\|y(t)-\bar{y}(t)\|_{PC}, \qquad (3.27)
\end{aligned}
$$

$$
\left|{}^{H}D_{t_k}^{\alpha}[(\mathcal{T}y)(t) - (\mathcal{T}\bar{y})(t)]\right| \leq \left[\sum_{i=1}^{m}(L_i + M_i + N_i)\right]\|y(t)-\bar{y}(t)\|_{PC} = \kappa\|y(t)-\bar{y}(t)\|_{PC}, \qquad (3.28)
$$

$$
\begin{aligned}
&\left|{}^{H}D_{t_k}^{\beta_i}[(\mathcal{T}y)(t) - (\mathcal{T}\bar{y})(t)]\right| \leq \left[\frac{1}{\Gamma(\alpha-\beta_i+1)}\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha-\beta_i}\sum_{i=1}^{m}(L_i + M_i + N_i)\right] \\
&\times \|y(t)-\bar{y}(t)\|_{PC} = \theta_{i,k}\|y(t)-\bar{y}(t)\|_{PC}, \ i = 1, 2, \ldots, m. \qquad (3.29)
\end{aligned}
$$

It follows from (3.24), (3.25) and (A_2) that

$$
\|(\mathcal{T}t)(t) - (\mathcal{T}\bar{y})(t)\|_{PC} \leq \rho\|y(t)-\bar{y}(t)\|_{PC}, \ \forall t \in J, y(t), \bar{y}(t) \in PC(J, R). \qquad (3.30)
$$

(3.30) indicates that $\mathcal{T} : PC(J, R) \to PC(J, R)$ is contractive. According to Lemma 2.3, we know that the operator $\mathcal{T} : PC(J, R) \to PC(J, R)$ exists a unique fixed point $y^*(t) \in PC(J, R)$. So the system (1.1) has a unique solution $y^*(t) \in PC(J, R)$.

Next we shall prove that (2) in Theorem 3.1 holds. Let $z(t) \in PC(J, R)$ be a solution of inequality (2.1), and $y^*(t) \in PC(J, R)$ be a unique solution of problem (1.1). Similar to Lemma 3.2, we get from Remark 2.1 that

$$
z(t) = \begin{cases} \sum\limits_{i=1}^{m} {}^{H}J_a^{\alpha}\left[f_i(t, z(t), {}^{H}D_a^{\alpha}z(t), {}^{H}D_a^{\beta_i}z(t)) + \phi(t)\right] + C_{1,z}^*\left(log\frac{t}{a}\right)^{\alpha-1}, \ t \in [a, t_1], \\ \sum\limits_{i=1}^{m} {}^{H}J_{t_k}^{\alpha}\left[f_i(t, z(t), {}^{H}D_{t_k}^{\alpha}z(t), {}^{H}D_{t_k}^{\beta_i}z(t)) + \phi(t)\right] + C_{k+1,z}^*\left(log\frac{t}{t_k}\right)^{\alpha-1}, \ t \in (t_k, t_{k+1}], \end{cases} \qquad (3.31)
$$

where $k = 1, 2, \ldots, n$,

$$C^*_{1,z} = \frac{\lambda}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{n}[I_j(z(t_j)) + \phi_j] + \sum_{j=1}^{n}\sum_{i=1}^{m}{}^HJ^1_{t_{j-1}}[f_i(t_j, z(t_j), {}^HD^\alpha_{t_{j-1}}z(t_j),$$

$${}^HD^{\beta_i}_{t_{j-1}}z(t_j)) + \phi(t_j)] + \sum_{i=1}^{m}{}^HJ^1_{t_n}[f_i(T, z(T), {}^HD^\alpha_{t_n}z(T), {}^HD^{\beta_i}_{t_n}z(T)) + \phi(T)]\Big], \tag{3.32}$$

$$C^*_{k+1,z} = \frac{1}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{k}[I_j(z(t_j)) + \phi_j] + \sum_{j=1}^{k}\sum_{i=1}^{m}{}^HJ^1_{t_{j-1}}[f_i(t_j, z(t_j), {}^HD^\alpha_{t_{j-1}}z(t_j),$$

$${}^HD^{\beta_i}_{t_{j-1}}z(t_j)) + \phi(t_j)] + \lambda\sum_{j=k+1}^{n}[I_j(z(t_j)) + \phi_j] + \lambda\sum_{j=k+1}^{n}\sum_{i=1}^{m}{}^HJ^1_{t_{j-1}}[f_i(t_j, z(t_j),$$

$${}^HD^\alpha_{t_{j-1}}z(t_j), {}^HD^{\beta_i}_{t_{j-1}}z(t_j)) + \phi(t_j)] + \lambda\sum_{i=1}^{m}{}^HJ^1_{t_n}[f_i(T, z(T), {}^HD^\alpha_{t_n}z(T), {}^HD^{\beta_i}_{t_n}z(T))$$

$$+\phi(T)]\Big],\ k = 1, 2, \ldots, n. \tag{3.33}$$

Noticing that $y^*(t)$ satisfies (3.20)–(3.22), Similar to the derivations of (3.24)–(3.29), we apply Remark 2.1 and (3.31)–(3.33) to obtain, for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, n$, $i = 1, 2, \ldots, m$,

$$|z(t) - y^*(t)| \le \rho_1\|z(t) - y^*(t)\|_{PC} + {}^HJ^\alpha_a|\phi(t)| \le \rho_k\|z(t) - y^*(t)\|_{PC} + \frac{\epsilon}{\Gamma(\alpha+1)}\left(log\frac{t_{k+1}}{t_k}\right)^\alpha, \tag{3.34}$$

$$|{}^HD^\alpha_a[z(t) - y^*(t)]| \le \kappa\|z(t) - y^*(t)\|_{PC} + |\phi(t)| \le \kappa\|z(t) - y^*(t)\|_{PC} + \epsilon,\ t \in (t_k, t_{k+1}], \tag{3.35}$$

$$\begin{aligned}|{}^HD^{\beta_i}_a[z(t) - y^*(t)]| &\le \theta_{i,1}\|z(t) - y^*(t)\|_{PC} + {}^HJ^{\alpha-\beta_i}_a|\phi(t)| \\ &\le \theta_{i,1}\|z(t) - y^*(t)\|_{PC} + \frac{\epsilon}{\Gamma(\alpha+1)}\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha-\beta_i}.\end{aligned} \tag{3.36}$$

From (3.34)–(3.36) and noticing $\left(log\frac{t_{k+1}}{t_k}\right)^{\alpha-\beta_i} < \left(log\frac{t_{k+1}}{t_k}\right)^\alpha$, one has

$$\|z(t) - y^*(t)\|_{PC} \le \rho\|z(t) - y^*(t)\|_{PC} + \nu\epsilon,$$

which implies that

$$\|z(t) - y^*(t)\|_{PC} \le \frac{\nu}{1-\rho}\epsilon. \tag{3.37}$$

Thus, by (3.37) and Definition 2.3, we conclude that problem (1.1) is Ulam-Hyers stable. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that* (H_1) *and* (H_2) *hold, further assume that the following condition* (A_3) *also holds.*

(A_3) *There exist a nondecreasing function* $\varphi \in PC(J,(0,\infty))$*, and* $\vartheta > 0$ *such that*

$$^{H}J_a^{\alpha}\varphi(t) \le \vartheta\varphi(t),\ \ ^{H}J_a^{\alpha-\beta_i}\varphi(t) \le \vartheta\varphi(t),\ \ \forall t \in J,\ \ i = 1,2,\dots,m.$$

Then the problem (1.1) *is Ulam-Hyers-Rassias stable with respect to* (φ,ψ)*, namely, if* $z(t) \in PC(J,R)$ *is a solution of inequality* (2.2)*, and* $y^*(t) \in PC(J,R)$ *is a unique solution of system* (1.1)*, then*

$$\|z(t) - y^*(t)\|_{PC} \le \frac{\delta}{1-\rho}\epsilon(\varphi(t)+\psi),$$

where $\delta = \max\{1,\vartheta\}$.

Proof. Let $z(t) \in PC(J,R)$ be a solution of inequality (2.2), and $y^*(t) \in PC(J,R)$ be a unique solution of problem (1.1). Similar to Lemma 3.2, we get from Remark 2.2 that

$$z(t) = \begin{cases} \sum\limits_{i=1}^{m} {}^{H}J_a^{\alpha}\left[f_i(t,z(t),\ ^{H}D_a^{\alpha}z(t),^{H}D_a^{\beta_i}z(t)) + \omega(t)\right] + \overline{C}_{1,z}\left(log\frac{t}{a}\right)^{\alpha-1},\ t \in [a,t_1], \\ \sum\limits_{i=1}^{m} {}^{H}J_{t_k}^{\alpha}\left[f_i(t,z(t),\ ^{H}D_{t_k}^{\alpha}z(t),^{H}D_{t_k}^{\beta_i}z(t)) + \omega(t)\right] + \overline{C}_{k+1,z}\left(log\frac{t}{t_k}\right)^{\alpha-1},\ t \in (t_k,t_{k+1}], \end{cases} \tag{3.38}$$

where $k = 1,2,\dots,n$,

$$\overline{C}_{1,z} = \frac{\lambda}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{n}[I_j(z(t_j)) + \omega_j] + \sum_{j=1}^{n}\sum_{i=1}^{m} {}^{H}J_{t_{j-1}}^{1}[f_i(t_j,z(t_j),\ ^{H}D_{t_{j-1}}^{\alpha}z(t_j),$$

$$^{H}D_{t_{j-1}}^{\beta_i}z(t_j)) + \omega(t_j)] + \sum_{i=1}^{m} {}^{H}J_{t_n}^{1}[f_i(T,z(T),\ ^{H}D_{t_n}^{\alpha}z(T),^{H}D_{t_n}^{\beta_i}z(T)) + \omega(T)]\Big], \tag{3.39}$$

$$\overline{C}_{k+1,z} = \frac{1}{(1-\lambda)\Gamma(\alpha)}\Big[\sum_{j=1}^{k}[I_j(z(t_j)) + \omega_j] + \sum_{j=1}^{k}\sum_{i=1}^{m} {}^{H}J_{t_{j-1}}^{1}[f_i(t_j,z(t_j),^{H}D_{t_{j-1}}^{\alpha}z(t_j),$$

$$^{H}D_{t_{j-1}}^{\beta_i}z(t_j)) + \omega(t_j)] + \lambda\sum_{j=k+1}^{n}[I_j(z(t_j)) + \omega_j] + \lambda\sum_{j=k+1}^{n}\sum_{i=1}^{m} {}^{H}J_{t_{j-1}}^{1}[f_i(t_j,z(t_j),$$

$$^{H}D_{t_{j-1}}^{\alpha}z(t_j),^{H}D_{t_{j-1}}^{\beta_i}z(t_j)) + \omega(t_j)] + \lambda\sum_{i=1}^{m} {}^{H}J_{t_n}^{1}[f_i(T,z(T),\ ^{H}D_{t_n}^{\alpha}z(T),^{H}D_{t_n}^{\beta_i}z(T))$$

$$+\ \omega(T)]\Big],\ k = 1,2,\dots,n. \tag{3.40}$$

Noticing that $y^*(t)$ satisfies (3.20)–(3.22), similar to the derivations of (3.24)–(3.29), we apply Remark 2.2, (A_3) and (3.38)–(3.40) to obtain, for $t \in (t_k,t_{k+1}]$, $k = 0,1,\dots,n$, $i = 1,2,\dots,m$,

$$|z(t) - y^*(t)| \le \rho_1\|z(t) - y^*(t)\|_{PC} + {}^{H}J_a^{\alpha}|\omega(t)| \le \rho_k\|z(t) - y^*(t)\|_{PC} + \epsilon\vartheta\varphi(t), \tag{3.41}$$

$$|^{H}D_a^{\alpha}[z(t) - y^*(t)]| \le \kappa\|z(t) - y^*(t)\|_{PC} + |\phi(t)| \le \kappa\|z(t) - y^*(t)\|_{PC} + \epsilon\varphi(t),\ t \in (t_k,t_{k+1}], \tag{3.42}$$

$$|^{H}D_a^{\beta_i}[z(t) - y^*(t)]| \le \theta_{i,1}\|z(t) - y^*(t)\|_{PC} + {}^{H}J_a^{\alpha-\beta_i}|\phi(t)| \le \theta_{i,1}\|z(t) - y^*(t)\|_{PC} + \epsilon\vartheta\varphi(t). \tag{3.43}$$

From (3.41)–(3.43), we get

$$\|z(t) - y^*(t)\|_{PC} \leq \rho\|z(t) - y^*(t)\|_{PC} + \delta\epsilon\varphi(t),$$

which implies that

$$\|z(t) - y^*(t)\|_{PC} < \frac{\delta}{1-\rho}\epsilon(\varphi(t) + \psi). \tag{3.44}$$

Attributed to (3.44) and Definition 2.4, we know that problem (1.1) is Ulam-Hyers-Rassias stable. The proof is completed. □

4. An example

Consider the following nonlinear implicit Hadamard fractional integaral boundary value problem with impulses

$$\begin{cases} {}^HD^{\alpha}_{t_k}y(t) = \sum\limits_{i=1}^{m} f_i(t, y(t), {}^HD^{\alpha}_{t_k}y(t), {}^HD^{\beta_i}_{t_k}y(t)), \ t \in (t_k, t_{k+1}] \subset J, \quad 0 \leq k \leq n, \\ {}^HJ^{1-\alpha}_{t_k}y(t_k^+) - {}^HJ^{1-\alpha}_{t_{k-1}}y(t_k^-) = I_k(y(t_k)), \ 1 \leq k \leq n, \\ {}^HJ^{1-\alpha}_{a}y(a) = \lambda \cdot {}^HJ^{1-\alpha}_{t_n}y(T), \end{cases} \tag{4.1}$$

where $m = n = 2$, $J = [1, e]$, $a = t_0 = 1 < t_1 = \frac{5}{4} < t_2 = 2 < t_3 = e = T$, $\alpha = \frac{3}{4}$, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{1}{2}$, $\lambda = -\sqrt{2}$, $f_1(t, u, v, w) = \frac{t^2+u+\sin(3v)+e^{-2|w|}}{10(2+t^2)}$, $f_2(t, u, v, w) = \frac{\sqrt{t}+\arctan u+\cos(5v)+3we^{-|w|}}{20}$, $I_1(u) = u^3$, $I_2(u) = \sqrt[3]{u}$.

Obviously, $f_1, f_2 \in C(J \times R^3, R)$, $I_1, I_2 \in C(R, R)$. By calculations, we have

$$|f_1(t, u, v, w) - f_1(t, \bar{u}, \bar{v}, \bar{w})| \leq \frac{1}{20}|u - \bar{u}| + \frac{3}{20}|v - \bar{v}| + \frac{1}{10}|w - \bar{w}|,$$

$$|f_2(t, u, v, w) - f_2(t, \bar{u}, \bar{v}, \bar{w})| \leq \frac{1}{20}|u - \bar{u}| + \frac{1}{4}|v - \bar{v}| + \frac{3}{10}|w - \bar{w}|.$$

Thus we derive that $L_1 = L_2 = \frac{1}{20}$, $M_1 = \frac{3}{20}$, $M_2 = \frac{1}{4}$, $N_1 = \frac{1}{10}$, $N_2 = \frac{3}{10}$, and

$$\eta_0 = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_1}{t_0}\right)^{\alpha} \approx 0.3533, \ \eta_1 = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_2}{t_1}\right)^{\alpha} \approx 0.6176,$$

$$\eta_2 = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_3}{t_2}\right)^{\alpha} \approx 0.4486, \ \kappa = \sum_{i=1}^{2}(L_i + M_i + N_i) = 0.9,$$

$$\rho_0 = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_1}{t_0}\right)^{\alpha}\sum_{i=1}^{2}(L_i + M_i + N_i) \approx 0.3179,$$

$$\rho_1 = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_2}{t_1}\right)^{\alpha}\sum_{i=1}^{2}(L_i + M_i + N_i) \approx 0.5559,$$

$$\rho_2 = \frac{1}{\Gamma(\alpha+1)}\left(log\frac{t_3}{t_2}\right)^{\alpha}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.4037,$$

$$\theta_{1,0} = \frac{1}{\Gamma(\alpha-\beta_1+1)}\left(log\frac{t_1}{t_0}\right)^{\alpha-\beta_1}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.4797,$$

$$\theta_{1,1} = \frac{1}{\Gamma(\alpha-\beta_1+1)}\left(log\frac{t_2}{t_1}\right)^{\alpha-\beta_1}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.6962,$$

$$\theta_{1,2} = \frac{1}{\Gamma(\alpha-\beta_1+1)}\left(log\frac{t_3}{t_2}\right)^{\alpha-\beta_1}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.5626,$$

$$\theta_{2,0} = \frac{1}{\Gamma(\alpha-\beta_2+1)}\left(log\frac{t_1}{t_0}\right)^{\alpha-\beta_2}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.6824,$$

$$\theta_{2,1} = \frac{1}{\Gamma(\alpha-\beta_2+1)}\left(log\frac{t_2}{t_1}\right)^{\alpha-\beta_2}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.8221,$$

$$\theta_{2,2} = \frac{1}{\Gamma(\alpha-\beta_2+1)}\left(log\frac{t_3}{t_2}\right)^{\alpha-\beta_2}\sum_{i=1}^{2}(L_i+M_i+N_i) \approx 0.7390.$$

So $0 < \rho = \max\{\kappa, \rho_0, \rho_1, \rho_2, \theta_{1,0}, \theta_{1,1}, \theta_{1,2}, \theta_{2,0}, \theta_{2,1}, \theta_{2,2}\} = 0.9 < 1$, $0 < \nu = \max\{1, \eta_0, \eta_1, \eta_2\} = 1$. Thus all conditions of Theorem 3.1 hold. Therefore, according to Theorem 3.1, the problem (4.1) exists a unique solution $y^*(t) \in PC(J, R)$. Simultaneously, the problem (4.1) is Ulam-Hyers stable, that is, assume that $z(t) \in PC(J, R)$ is a solution of the inequality (2.1), then $\|z(t) - y^*(t)\|_{PC} \le \frac{\nu}{1-\rho}\epsilon = 10\epsilon$.

In addition, let $\psi = 1$, $\varphi(t) = e^t$, then for $t \in J = [1, e]$, $\varphi(t) = e^t > 0$, and

$$\begin{aligned}{}^{H}J_1^{\alpha}\varphi(t) = {}^{H}J_1^{\frac{3}{4}}\varphi(t) &= \frac{1}{\Gamma(\frac{3}{4})}\int_1^t\left(\log\frac{t}{s}\right)^{-\frac{1}{4}}\frac{e^s}{s}ds \le \frac{e^t}{\Gamma(\frac{3}{4})}\int_1^t\left(\log\frac{t}{s}\right)^{-\frac{1}{4}}\frac{ds}{s}\\ &=\frac{e^t}{\Gamma(\frac{7}{4})}(\log(t))^{\frac{3}{4}} \le \frac{1}{\Gamma(\frac{7}{4})}\varphi(t) \approx 1.0881\varphi(t),\end{aligned}$$

$$\begin{aligned}{}^{H}J_1^{\alpha-\beta_1}\varphi(t) = {}^{H}J_1^{\frac{1}{2}}\varphi(t) &= \frac{1}{\Gamma(\frac{1}{2})}\int_1^t\left(\log\frac{t}{s}\right)^{-\frac{1}{2}}\frac{e^s}{s}ds \le \frac{e^t}{\Gamma(\frac{1}{2})}\int_1^t\left(\log\frac{t}{s}\right)^{-\frac{1}{2}}\frac{ds}{s}\\ &=\frac{e^t}{\Gamma(\frac{3}{2})}\sqrt{\log(t)} \le \frac{1}{\Gamma(\frac{3}{2})}\varphi(t) \approx 1.1284\varphi(t),\end{aligned}$$

$$
\begin{aligned}
{}^{H}J_1^{\alpha-\beta_2}\varphi(t) = {}^{H}J_1^{\frac{1}{4}}\varphi(t) &= \frac{1}{\Gamma(\frac{1}{4})}\int_1^t \left(\log\frac{t}{s}\right)^{-\frac{3}{4}} \frac{e^s}{s}ds \le \frac{e^t}{\Gamma(\frac{1}{4})}\int_1^t \left(\log\frac{t}{s}\right)^{-\frac{3}{4}} \frac{ds}{s} \\
&= \frac{e^t}{\Gamma(\frac{5}{4})}(\log(t))^{\frac{1}{4}} \le \frac{1}{\Gamma(\frac{5}{4})}\varphi(t) \approx 1.1033\varphi(t).
\end{aligned}
$$

Take $\vartheta = 1.1284$, thus the condition (A_3) holds, and $\delta = \max\{1, \vartheta\} = 1.1284$. So it follows from Theorem 3.2 that the problem (4.1) is Ulam-Hyers-Rassias stable with respect to (φ, ψ), namely, if $z(t) \in PC(J, R)$ is a solution of inequality (2.2), then

$$
\|z(t) - y^*(t)\|_{PC} \le \frac{\delta}{1-\rho}\epsilon(\varphi(t) + \psi) \approx 11.284\epsilon(e^t + 1).
$$

5. Conclusions

For many practical systems, the stability of system is an important problem to be considered. Therefore, we mainly study the Ulam-Hyers and Ulam-Hyers-Rassias stability of a class of nonlinear implicit Hadamard fractional differential Eq (1.1) with integral boundary value condition and impulses in this paper. By using Banach's contraction mapping principle and inequality techniques, We obtain some sufficient criteria to guarantee the existence, uniqueness and stability of the solution. From the condition (A_2), we conclude that the value of impulse point has an important influence on the existence and stability of the system solution. So it is necessary to consider the effect of pulse in the study of some practical systems. Moreover, the mathematical methods and techniques used in this paper are very useful for dealing with similar problems.

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Conflict of interest

All authors declare that they have no competing interests.

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