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Research article

Codimension two 1:1 strong resonance bifurcation in a discrete predator-prey model with Holling IV functional response

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Abstract: In this paper we revisit a discrete predator-prey model with Holling IV functional response. By using the method of semidiscretization, we obtain new discrete version of this predator-prey model. Some new results, besides its stability of all fixed points and the transcritical bifurcation, mainly for codimension two 1:1 strong resonance bifurcation, are derived by using the center manifold theorem and bifurcation theory, showing that this system possesses complicate dynamical properties.

Keywords: predator-prey model; holling IV functional response; transcritical bifurcation; 1:1 strong resonance bifurcation; codimension two
Mathematics Subject Classification: 39A28, 3AA30

1. Introduction

Nowadays, the natural environment is becoming more and more bad. Due to overdevelopment, many problems have arisen in the ecosystem, such as pollution, virus spread, etc. How to improve the environment? Mathematical modelling may be used to describe the change trend of the natural environment. This leads more and more scholars to study ecological balance by the method of mathematical modelling. The Lotka-Volterra predator-prey model is one of the most important mathematical models for studying ecological balance, which is respectively proposed by Lotka [1] and Volterra [2], and used to describe the dynamical relationship between the predator and the prey. However, the model ignores the actual effective factors. Firstly, the population growth should be limited by the environmental carrying capacity. Then, there are many factors which can influence the changes of population density and quantity in the predator-prey system, such as the birth rate and the growth rate of the prey [3]. The Lotka-Volterra predator-prey model adopts the most typical logistic growth model for the birth or growth of the prey. Of course, the predator-prey interaction is also affected by the functional response, which means, with the change of the number of prey, the predation rate of predator to prey will also change. This may be considered as a predator response to the prey [4]. Therefore, the Lotka-Volterra predator-prey model can be improved as

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{K}) - p(x,y)y, \\ \frac{dy}{dt} = y(-d + cp(x,y)x), \end{cases} \tag{1.1}$$

where r is the intrinsic growth rate, K is the carrying capacity of environment w.r.t the prey x, c is the growth rate of the predator y, d is the predator death rate, and $p(x,y)$ is the predator functional response, which plays an important role in determining the dynamical behaviors, such as the steady states, the oscillations, the bifurcations, the chaos phenomena and the limit cycles, etc.

To the best of our knowledge, there are several common forms of functional response $p(x,y)$ in population dynamics [5].

(1) $p(x,y)$ depends on x only (meaning $p(x,y) = p(x)$).

a. Holling type I [6–8]:

$$p(x) = mx;$$

b. Holling type II [9–12]:

$$p(x) = \frac{mx}{a + x};$$

c. Holling type III [13–16]:

$$p(x) = \frac{mx^2}{a + x^2};$$

d. Holling type IV [17–20]:

$$p(x) = \frac{mx}{a + x^2}.$$

(2) $p(x,y)$ depends on both x and y.

a. Ratio-dependent type [21]:

$$p(x,y) = \frac{mx}{x + ay};$$

b. Beddington-DeAngelis type [22,23]:

$$p(x,y) = \frac{mx}{a + bx + cy};$$

c. Hassell-Varley type [24,25]:

$$p(x,y) = \frac{mx}{y^\gamma + ax}, \gamma = \frac{1}{2}, \frac{1}{3}.$$

Here, the above parameters m, a, b, c, γ are all positive constants, and they have specific biological meanings in these functional responses.

Recently, Ruan and Xiao in [20] studied the following continuous-time predator-prey system with Holling type IV functional response

$$\begin{cases} \frac{dX}{dT} = RX(1 - \frac{X}{K}) - \frac{MXY}{A+X^2}, \\ \frac{dY}{dT} = Y[-D + \frac{CX}{A+X^2}]. \end{cases} \tag{1.2}$$

The authors in [26], by applying the forward Euler scheme to (1.2) and letting the step length $h = 1$ and the parameter $D = 1$, got the discrete version of (1.2) as follows

$$\begin{cases} x_{t+1} = (R+1)x_t[1 - \frac{R}{K(1+R)}x_t] - \frac{Mx_ty_t}{A+x_t^2}, \\ y_{t+1} = \frac{Cx_ty_t}{A+x_t^2}. \end{cases} \tag{1.3}$$

Letting again

$$\frac{R}{K(R+1)}x_t \to x_t,$$

and

$$a = R+1, b = \frac{M}{A}, d = \frac{CK(1+R)}{AR}, \epsilon = \frac{K^2(1+R)^2}{AR^2},$$

rewriting (1.3) as a map, the authors of [26] obtained and studied the following discrete-time map with non-monotonic functional response

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} ax(1-x) - \frac{bxy}{1+\epsilon x^2} \\ \frac{dxy}{1+\epsilon x^2} \end{pmatrix}. \tag{1.4}$$

Here, we think there are two problems among their derivation of the map (1.4). The one is that the map (1.4) is not the simplest equivalent mathematical map. In fact, letting $by \to y$, the map (1.4) may be reduced into

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} ax(1-x) - \frac{xy}{1+\epsilon x^2} \\ \frac{dxy}{1+\epsilon x^2} \end{pmatrix}. \tag{1.5}$$

The four parameters in (1.4) has been reduced to three ones in (1.5). Then we point out that it is unnecessary to take $D = 1$. Indeed, letting $\frac{X}{K} \to x$, $RT \to t$, $\frac{MY}{RK^2} \to y$, the system (1.2) is changed into the following form

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{xy}{a+x^2}, \\ \frac{dy}{dt} = y[-d + \frac{bx}{a+x^2}], \end{cases} \tag{1.6}$$

where $a = \frac{A}{K^2}$, $b = \frac{C}{RK}$, $d = \frac{D}{R}$. Obviously, the system (1.6) is completely equivalent to (1.2), but the five parameters in (1.2) is reduced to three ones in (1.6). It is relatively easy to consider the system (1.6) without assuming $d = 1$. The other is to violate the accuracy requirement when they used the forward Euler method to (1.2). In fact, it is well known that $\dot{x}(t) = \lim\limits_{h\to 0}\frac{x(t+h)-x(t)}{h}$. So, the discreteness of autonomous differential equation $\dot{x}(t) = f(x)$ is generally

$$\frac{x(t_{n+1}) - x(t_n)}{h} = f(x(t_n)),$$

where $t_n = t_0 + nh$ and h is a step length, requiring $0 < h \ll 1$. Denote $x_n = x(t_n)$. Then $x_{n+1} = x_n + hf(x_n)$. Back to (1.2) and letting $h = 1$ gives rise to (1.3). However, taking $h = 1$ violates the requirement of $0 < h \ll 1$. Thus, their investigations for (1.3), despite having mathematical meanings, do not have the same biological meanings as (1.2). So, the discretization of (1.2) is worth further investigating.

Now we use the semidiscretization method to (1.2). Since the system (1.6) is equivalent to (1.2) and simpler than (1.2), it is sufficient for us to consider the discretization of (1.6). Suppose that $[t]$ denotes the greatest integer not exceeding t. Consider the average change rate of the system (1.6) at integer number points

$$\begin{cases} \frac{1}{x(t)}\frac{dx(t)}{dt} = 1 - x([t]) - \frac{y([t])}{a+x([t])^2}, \\ \frac{1}{y(t)}\frac{dy(t)}{dt} = -d + \frac{bx([t])}{a+x([t])^2}. \end{cases} \tag{1.7}$$

Obviously, the system (1.7) has piecewise constant arguments, and for $t \in [0, +\infty)$, a solution $(x(t), y(t))$ of the system (1.7) possesses the following features:

(1) $x(t)$ and $y(t)$ are continuous on $[0, +\infty)$;

(2) $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist everywhere when $t \in [0, +\infty)$ except for the points $t \in \{0, 1, 2, 3, \cdots\}$;

(3) The system (1.7) is true in each interval $[n, n+1)$ with $n = 0, 1, 2, 3, \cdots$.

For any $t \in [n, n+1)$ with $n = 0, 1, 2, \cdots$, integrating (1.7) from n to t, one obtains the following system

$$\begin{cases} x(t) = x_n e^{1-x_n-\frac{y_n}{a+x_n^2}}(t-n), \\ y(t) = y_n e^{-d+\frac{bx_n}{a+x_n^2}}(t-n), \end{cases} \tag{1.8}$$

where $x_n = x(n)$ and $y_n = y(n)$.

Letting $t \to (n+1)^-$ in (1.8) leads to

$$\begin{cases} x_{n+1} = x_n e^{1-x_n-\frac{y_n}{a+x_n^2}}, \\ y_{n+1} = y_n e^{-d+\frac{bx_n}{a+x_n^2}}. \end{cases} \tag{1.9}$$

In this paper, our main aim is to consider the dynamics of (1.9), mainly for its bifurcation problems besides its stability.

The most difference between our results and the known ones is for us to find codim 2 strong resonance bifurcation in the system (1.9), showing that the system (1.9) possesses complicate dynamics: The existence for the fold bifurcation, Neimark-Sacker bifurcation and homoclinic bifurcation.

The rest of the paper is organized as follows. In Section 2, we give our results for the existence and stability of all nonnegative fixed points of the system (1.9). In Section 3, we choose the parameters a and b as bifurcation parameter to discuss its bifurcation problems at the fixed points A and E_0 respectively, including codim 1 transcritical bifurcation and codim 2 1:1 strong resonance bifurcation. In Section 4, we draw some conclusions and discussions.

2. Existence and stability of fixed points

In this section, we consider the existence and stability of fixed points of the system (1.9). To do this, we need a definition and a key lemma. For readers' convenience, we list them in the appendix of this paper.

The fixed points of the system (1.9) satisfy

$$x = xe^{1-x-\frac{y}{a+x^2}}, \quad y = ye^{-d+\frac{bx}{a+x^2}}.$$

Considering the biological meanings of the system (1.9), one only takes into account nonnegative fixed points. Thereout, one finds that the system (1.9) has at most four fixed points under different conditions:

The trivial fixed point $O(0,0)$, a semi-trivial fixed point $A(1,0)$, and two positive fixed points $E_1(x_1,y_1)$ and $E_2(x_2,y_2)$, when $b^2-4ad^2>0$, where

$$x_{1,2}=\frac{b\mp\sqrt{b^2-4ad^2}}{2d},$$
$$y_{1,2}=(1-x_{1,2})(a+x_{1,2}^2).$$

When $b^2-4ad^2=0$, the two positive fixed points coalesce into a unique positive fixed point

$$E_0(x_0,y_0)=E_0(\sqrt{a},2a(1-\sqrt{a})),\quad (0<a<1).$$

Theorem 2.1. *The existence conditions for all nonnegative fixed points of the system* (1.9) *are summarized in Table 1.*

Table 1. the existence of fixed points.

Conditions		Existing fixed points
$a<\frac{b}{d}-1$	$b>0,d>0$	O,A,E_1
$a=\frac{b}{d}-1$	$b\geqslant 2d$	O,A
	$b<2d$	O,A,E_1
$\frac{b}{d}-1<a<\frac{b^2}{4d^2}$	$b>2d$	O,A
	$b<2d$	O,A,E_1,E_2
$a=\frac{b^2}{4d^2}$	$b<2d$	O,A,E_0
	$b\geqslant 2d$	O,A
$a>\frac{b^2}{4d^2}$	$b>0,d>0$	O,A

Proof. Clearly, the system (1.9) always has the fixed points $O(0,0)$ and $A(1,0)$. We now discuss the existence of possible positive fixed points. The positive fixed points of the system (1.9) satisfy

$$\begin{cases} 1-x-\frac{y}{a+x^2}=0,\\ -d+\frac{bx}{a+x^2}=0,\end{cases}\tag{2.1}$$

where x is (are) the positive root (roots) of the equation $dx^2-bx+ad=0$, requiring $x<1$, whereas $y=(1-x)(a+x^2)$.

(1) When $a<\frac{b^2}{4d^2}$, one can see $x_i(i=1,2)$ exist, and $0<x_1<\sqrt{a}<x_2$.

a. If $x_1\geqslant 1$, one can deduce $x_2>1$, $y_1\leqslant 0$ and $y_2<0$, so both of positive fixed points E_1 and E_2 do not exist. One can derive $x_1\geqslant 1\Longleftrightarrow\frac{b-\sqrt{b^2-4ad^2}}{2d}\geqslant 1\Longleftrightarrow b>2d$, $\frac{b}{d}-1\leqslant a<\frac{b^2}{4d^2}$.

b. If $x_1<1\leqslant x_2$, one can see the positive fixed point E_1 exists while the positive fixed point E_2 does not exist. Also,

$$x_1<1\leqslant x_2\Leftrightarrow\begin{cases}\frac{b-\sqrt{b^2-4ad^2}}{2d}<1\\ \frac{b+\sqrt{b^2-4ad^2}}{2d}\geqslant 1\end{cases}\Leftrightarrow\begin{cases} b\leq 2d\ or\ b>2d, a<\frac{b}{d}-1\\ b\geq 2d\ or\ b\leqslant 2d, a\leqslant\frac{b}{d}-1\end{cases}.$$

When $b=2d$, one can know $\frac{b}{d}-1=\frac{b^2}{4d^2}$. So, under the condition $a<\frac{b^2}{4d^2}$, one has $a<\frac{b}{d}-1$.

c. If $x_2 < 1$, meaning $x_1 < x_2 < 1$, both positive fixed points E_1 and E_2 exist. One can see $x_2 < 1 \Longleftrightarrow \frac{b+\sqrt{b^2-4ad^2}}{2d} < 1 \Longleftrightarrow b < 2d, a > \frac{b}{d} - 1$.

(2) When $a = \frac{b^2}{4d^2}$, $x_1 = x_2 = x_0 = \frac{b}{2d} = \sqrt{a} > 0$.

a. If $b < 2d$, *i.e.*, $a < 1$, then the positive fixed point E_1 coincides with E_2 and a unique positive fixed point $E_0(\sqrt{a}, 2a(1-\sqrt{a}))$ arises.

b. If $b \geqslant 2d$, *i.e.*, $a \geqslant 1$, the unique positive fixed point E_0 does not exist.

(3) When $a > \frac{b^2}{4d^2}$, x_1 and x_2 do not exist, then the system (1.9) only has boundary fixed points O and A.

Finally, we summarize all of the results in the Table 1. The proof is over.

Now we begin to analyze the stability of these fixed points. The Jacobian matrix of the linearized equation for the system (1.9) at a fixed point $E(x, y)$ is

$$J(E) = \begin{pmatrix} \left(\frac{2x^2y}{(a+x^2)^2} - x + 1\right)e^{1-x-\frac{y}{a+x^2}} & \frac{-x}{a+x^2}e^{1-x-\frac{y}{a+x^2}} \\ \frac{by(a-x^2)}{(a+x^2)^2}e^{-d+\frac{bx}{a+x^2}} & e^{-d+\frac{bx}{a+x^2}} \end{pmatrix}.$$

The characteristic polynomial of Jacobian matrix $J(E)$ reads as

$$F(\lambda) = \lambda^2 + B\lambda + C,$$

where

$$B = -Tr(J(E)), C = Det(J(E)).$$

Now, we formulate some results for the stability of the fixed points O, A, E_1, E_2 and E_0 in the following theorems.

Theorem 2.2. *The fixed point* $O = (0, 0)$ *of the system* (1.9) *is a saddle.*

Theorem 2.3. *The following statements about the fixed point* $A = (1, 0)$ *of the system* (1.9) *are true.*

(1) For $b < d(a + 1)$, *the fixed point A(1,0) is a sink;*
(2) For $b = d(a + 1)$, *the fixed point A(1,0) is non-hyperbolic;*
(3) For $b > d(a + 1)$, *the fixed point A(1,0) is a saddle.*

The proofs for Theorems 2.2 and 2.3 are simple, and omitted here.

Theorem 2.4. *For* $0 < a < \frac{b^2}{4d^2} < 1$, *the positive fixed point* $E_1 = (x_1, y_1)$ *of the system* (1.9) *occurs. Moreover, the following statements are valid.*

(1) If $d < \frac{x_1(a-2x_1+3x_1^2)}{(1-x_1)(a-x_1^2)}$, *then* E_1 *is a sink;*
(2) If $d = \frac{x_1(a-2x_1+3x_1^2)}{(1-x_1)(a-x_1^2)}$ *and* $a > x_1(2-3x_1)$, E_1 *is non-hyperbolic;*
(3) If $d > \frac{x_1(a-2x_1+3x_1^2)}{(1-x_1)(a-x_1^2)}$, E_1 *is a source.*

Proof. The Jacobian matrix of the linearized equation for the system (1.9) at $E_1 = (x_1, y_1)$ is given by

$$J(E_1) = \begin{pmatrix} \frac{2x_1^2(1-x_1)}{a+x_1^2} - x_1 + 1 & \frac{-x_1}{a+x_1^2} \\ \frac{d(1-x_1)(a-x_1^2)}{x_1} & 1 \end{pmatrix}.$$

The characteristic polynomial of Jacobian matrix $J(E_1)$ can be written as

$$F(\lambda) = \lambda^2 + B\lambda + C,$$

where

$$-B = Tr(J(E_1)) = 2 + \frac{d}{b}(2x_1 - 3x_1^2 - a),$$
$$C = Det(J(E_1)) = 1 + \frac{d}{b}(2x_1 - 3x_1^2 - a + \frac{d(1 - x_1)(a - x_1^2)}{x_1}).$$

It is easy to see that

$$F(1) = \frac{d^2(1 - x_1)(a - x_1^2)}{bx_1}$$

and

$$F(-1) = 4 - 2x_1 + \frac{(1 - x_1)[4x_1^2 + d(a - x_1^2)]}{a + x_1^2}.$$

Noting $0 < x_1 < \sqrt{a}$, $F(1) > 0$ and $F(-1) > 0$ are always true.

(1) For $d < \frac{x_1(a-2x_1+3x_1^2)}{(1-x_1)(a-x_1^2)}$, one sees $C < 1$. By Lemma 4.2 (i.1), the two eigenvalues satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$. According to Definition 4.1(1), E_1 is a sink.

(2) For $d = \frac{x_1(a-2x_1+3x_1^2)}{(1-x_1)(a-x_1^2)}, a > x_1(2 - 3x_1)$, one has $C = 1$, and $-2 < B < 2$. Then Lemma 4.2 (i.5) tells us the two eigenvalues satisfy $|\lambda_1| = |\lambda_2| = 1$. Hence, from Definition 4.1(4), E_1 is non-hyperbolic.

(3) For $d > \frac{x_1(a-2x_1+3x_1^2)}{(1-x_1)(a-x_1^2)}$, one gets $C > 1$. According to Lemma 4.2 (i.4), the two eigenvalues verify $|\lambda_{1,2}| > 1$. So it follows from Definition 4.1(2) that E_1 is a source.

Theorem 2.5. *For $\frac{b}{d} - 1 < a < \frac{b^2}{4d^2} < 1$, the positive fixed point $E_2 = (x_2, y_2)$ of the system (1.9) occurs. Moreover, the following statements are valid about the positive fixed point E_2.*

(1) If $d > \frac{(2x_2-4)(a+x_2^2)-4x_2^2}{(1-x_2)(a-x_2^2)}$, E_2 is a source;

(2) if $d = \frac{(2x_2-4)(a+x_2^2)-4x_2^2}{(1-x_2)(a-x_2^2)}$, E_2 is non-hyperbolic;

(3) if $d < \frac{(2x_2-4)(a+x_2^2)-4x_2^2}{(1-x_2)(a-x_2^2)}$, E_2 is a saddle.

The proof is completely similar to the one for the fixed point E_1 and omitted here.

Theorem 2.6. *For $0 < a = \frac{b^2}{4d^2} < 1$ the positive fixed point $E_0 = (x_0, y_0)$ of the system (1.9) occurs. Moreover, the following results in the Table 2 are valid about the positive fixed point E_0.*

Table 2. Properties of the positive fixed point E_0.

Conditions	Eigenvalues $\lambda_1 = 2(1 - \sqrt{a}), \lambda_2 = 1$	Properties
$a \in (0, \frac{1}{4}) \bigcup (\frac{1}{4}, 1)$	$\lvert\lambda_1\rvert \neq 1, \lambda_2 = 1$	non-hyperbolic
$a = \frac{1}{4}$	$\lambda_1 = 1, \lambda_2 = 1$	non-hyperbolic

The proof is easy and omitted here.

3. Bifurcation analysis

In this section, we consider the local bifurcation problems of the system (1.9) at the fixed points $A(1,0)$ and $E_0(\sqrt{a}, 2a(1-\sqrt{a}))$. We first study the bifurcation problems of codimnesion one at the fixed points A, then the codimension two bifurcation problem at the fixed points E_0, which is most important and difficult, and is also our main result in this paper.

3.1. Codimension one bifurcation at the fixed point A

Theorem 3.1. *Assume the parameters* $(a,b,d) \in \Omega_1 = \{(a,b,d) \in R_+^3 | a \neq 1\}$, *and let* $b_0 = d(a+1)$, *then the system* (1.9) *undergoes a transcritical bifurcation at the fixed point* $A(1,0)$ *when the parameter* b *goes through the critical value* b_0.

Proof. Let $u_t = x_t - 1, v_t = y_t - 0$, which transforms $A = (1,0)$ to the origin $O(0,0)$, give a small perturbation b^* of the parameter b around b_0, namely, $b^* = b - b_0$, with $0 < |b^*| \ll 1$, and set $b^*_{t+1} = b^*_t = b^*$, then the system (1.9) may be written into

$$\begin{cases} u_{t+1} = (u_t+1)e^{-u_t - \frac{v_t}{a+(u_t+1)^2}} - 1, \\ v_{t+1} = v_t e^{-d+\frac{(u_t+1)(d(a+1)+b^*_t)}{a+(u_t+1)^2}}, \\ b^*_{t+1} = b^*_t. \end{cases} \tag{3.1}$$

Taylor expanding (3.1) at (u_t, v_t, b^*_t)=(0,0,0) gets

$$\begin{pmatrix} u_t \\ v_t \\ b^*_t \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \frac{-1}{a+1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_t \\ v_t \\ b^*_t \end{pmatrix} + \begin{pmatrix} g_1(u_t, v_t, b^*_t) + o(\rho_1^2) \\ g_2(u_t, v_t, b^*_t) + o(\rho_1^2) \\ 0 \end{pmatrix}, \tag{3.2}$$

where $\rho_1 = \sqrt{u_t^2 + v_t^2 + (b^*_t)^2}$,

$$\begin{aligned} g_1(u_t, v_t, b^*_t) &= -\frac{1}{2}u_t^2 + \frac{2}{(a+1)^2}u_t v_t + \frac{1}{2(a+1)^2}v_t^2, \\ g_2(u_t, v_t, b^*_t) &= (d - \frac{2d}{a+1})u_t v_t + \frac{1}{a+1}v_t b^*_t. \end{aligned}$$

It is easy to derive the three eigenvalues of the matrix $A = \begin{pmatrix} 0 & \frac{-1}{a+1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to be $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$ with corresponding eigenvectors $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} \frac{-1}{a+1} \\ 1 \\ 0 \end{pmatrix}$ and $\xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Set $T = (\xi_1,\ \xi_2,\ \xi_3)$, namely,

$$T = \begin{pmatrix} 1 & \frac{-1}{a+1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } T^{-1} = \begin{pmatrix} 1 & \frac{1}{a+1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformation $\begin{pmatrix} u_t \\ v_t \\ b_t^* \end{pmatrix} = T \begin{pmatrix} l_t \\ m_t \\ \sigma_t \end{pmatrix}$ changes the system (3.2) into

$$\begin{pmatrix} l_{t+1} \\ m_{t+1} \\ \sigma_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_t \\ m_t \\ \sigma_t \end{pmatrix} + \begin{pmatrix} g_3(l_t, m_t, \sigma_t) + o(\rho_2^3) \\ g_4(l_t, m_t, \sigma_t) + o(\rho_2^3) \\ 0 \end{pmatrix}, \tag{3.3}$$

where $\rho_2 = \sqrt{l_t^2 + m_t^2 + \sigma_t^2}$,

$$\begin{aligned} g_3(l_t, m_t, \sigma_t) &= g_1(l_t - \frac{1}{a+1} m_t, m_t, \sigma_t) + \frac{1}{(a+1)} g_2(l_t - \frac{1}{a+1} m_t, m_t, \sigma_t), \\ g_4(l_t, m_t, \sigma_t) &= g_2(l_t - \frac{1}{a+1} m_t, m_t, \sigma_t). \end{aligned}$$

Assume on the center manifold

$$l_t = h(m_t, \sigma_t) = a_{20} m_t^2 + a_{11} m_t \sigma_t + a_{02} \sigma_t^2 + o(\rho_3^2),$$

where $\rho_3 = \sqrt{m_t^2 + \sigma_t^2}$, then, according to the following relations

$$\begin{aligned} l_{t+1} &= h(m_{t+1}, \sigma_{t+1}) = g_3(h(m_t, \sigma_t), m_t, \sigma_t) + o(\rho_3^2), \\ l_{t+1} &= a_{20} m_{t+1}^2 + a_{11} m_{t+1} \sigma_t + a_{02} \sigma_t^2 + o(\rho_3^2), \\ m_{t+1} &= m_t + g_4(h(m_t, \sigma_t), m_t, \sigma_t) + o(\rho_3^2), \end{aligned}$$

and by comparing the corresponding coefficients of terms with the same orders in the above equation, one derives that

$$a_{20} = \frac{d(1-a) - 2}{(a+1)^3}, a_{11} = \frac{1}{(a+1)^2}, a_{02} = 0.$$

So, the system (3.3) restricted to the center manifold is given by

$$m_{t+1} = f_1(m_t, \sigma_t) := m_t + \frac{1}{a+1} m_t \sigma_t + \frac{d(1-a)}{(a+1)^2} m_t^2 + o(\rho_3^2).$$

Therefore, the following results are derived:

$$\begin{aligned} & f_1(0,0) = 0, \frac{\partial f_1}{\partial m_t}|_{(0,0)} = 1, \frac{\partial f_1}{\partial \sigma_t}|_{(0,0)} = 0, \\ & \frac{\partial^2 f_1}{\partial m_t^2}|_{(0,0)} = \frac{2d(1-a)}{(a+1)^2} =: \beta, \frac{\partial^2 f_1}{\partial m_t \partial \sigma_t}|_{(0,0)} = \frac{1}{a+1} \neq 0. \end{aligned}$$

When $a \neq 1$, $\beta \neq 0$. According to (21.1.43)–(21.1.46) in [27, p507], all conditions hold for a transcritical bifurcation to occur, hence, the system (1.9) undergoes a transcritical bifurcation at the fixed point $A(1, 0)$.

3.2. Codimension-two bifurcation at the fixed points E_0

By the Theorem 2.1 in Section 2, when $a = \frac{b^2}{4d^2} < 1$, the system(1.9) has a unique positive fixed point E_0 and $J(E_0)$ has an eigenvalue 1 with multiplicity 2 if $b = b_0 := d$ (hence $a = a_0 := \frac{1}{4}$). Thus, the 1:1 strong resonance bifurcation may occur at the fixed point E_0.

In this subsection, by selecting the parameters a and b as bifurcation parameter and using the bifurcation theory established in [19,28], we analyze the 1:1 strong resonance bifurcation of the system (1.9) at the fixed point $E_0(\sqrt{a}, 2a(1-\sqrt{a}))$. The main result is as follows.

Theorem 3.2. *Consider the parameter vector* $(a, b, d) \in \{(a, b, d) \in R_+^3 | 0 < a = \frac{b^2}{4d^2} < 1, b > 0, 0 < d \neq 2, \frac{26 \pm 4\sqrt{34}}{3}\}$. *Let* $a_0 = \frac{1}{4}$ *and* $b_0 = d$, *then the system* (1.9) *undergoes the 1:1 strong resonance bifurcation at the fixed point* $E_0(\sqrt{a}, 2a(1-\sqrt{a}))$ *when the two parameters a and b vary in a sufficiently small neighborhood of* (a_0, b_0). *Moreover, for sufficiently small* $|\alpha|$, *where* $\alpha =: (a^*, b^*)$ *is a small perturbation of* $\alpha_0 =: (a_0, b_0)$, *i.e.,* $a^* = a - a_0, b^* = b - b_0$,

(i) There is a fold bifurcation which occurs on the curve

$$f_\pm : \eta_1(\alpha) = \frac{1}{4}\eta_2^2(\alpha) + O(|\alpha|^3);$$

(ii) Near the fixed point born at the fold bifurcation of (i) there is a Neimark-Sacker bifurcation which occurs on the curve

$$NS : \eta_1(\alpha) = O(|\alpha|^3), \quad \eta_2(\alpha) = O(|\alpha|^2) < 0;$$

The invariant circle created at the Neimark–Sacker bifurcation is stable;

(iii) There is a homoclinic bifurcation which occurs on two curves h_1 *and* h_2 *with the asymptotic forms*

$$\eta_1(\alpha) = -\frac{6}{25}\eta_2^2(\alpha) + O(|\alpha|^3), \quad \eta_2(\alpha) = O(|\alpha|^2) < 0.$$

The distance between the two homoclinic tangency bifurcation curves is exponentially small with respect to $\sqrt{|\alpha|}$.

These curves are illustrated in the following Figure 1.

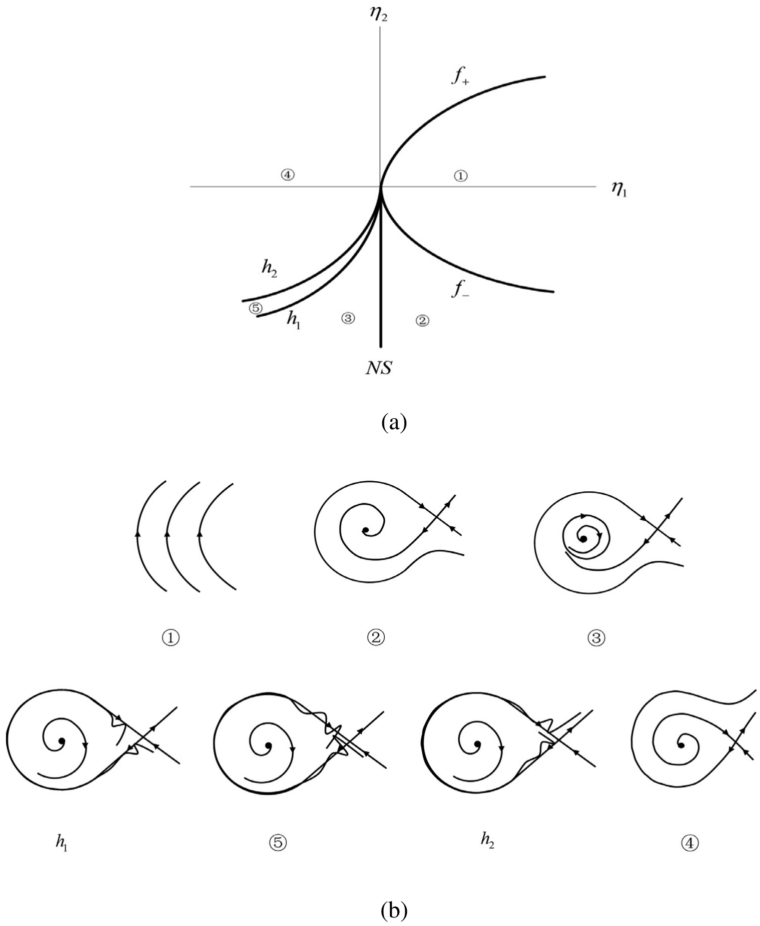

Figure 1. Bifurcation diagram and Phase portraits of the system (1.9). (a) Bifurcation diagram of the system (1.9) near $(l_t, m_t) = (0, 0)$ and $\eta = (0, 0)$, where $f_{\pm}, NS$ and $h_j(j = 1, 2)$ represent the bifurcation curves for the fold, Neimark-Sacker and homoclinic bifurcation to occur respectively. (b) Phase portraits of the system (1.9) under different cases. For the detailed cites, see[28, pp.323–325].

Proof. In order to transfer the fixed point $E_0(\sqrt{a}, 2a(1-\sqrt{a}))$ to the origin point $(0, 0)$, let $u_t = x_t - \sqrt{a}$, $v_t = y_t - 2a(1-\sqrt{a})$. Under the small perturbation α^* of α_0, the system (1.9) can be rewritten as

$$\begin{cases} u_{t+1} = & (u_t + \sqrt{a_0 + a^*})e^{1-(u_t+\sqrt{a_0+a^*})-\frac{v_t+2(a_0+a^*)(1-\sqrt{a_0+a^*})}{(a_0+a^*)+(u_t+\sqrt{a_0+a^*})^2}} \\ & -\sqrt{a_0 + a^*}, \\ v_{t+1} = & [v_t + 2(a_0 + a^*)(1 - \sqrt{a_0 + a^*})]e^{-d+\frac{(b_0+b^*)(u_t+\sqrt{a_0+a^*})}{(a_0+a^*)+(u_t+\sqrt{a_0+a^*})^2}} \\ & -2(a_0 + a^*)(1 - \sqrt{a_0 + a^*}). \end{cases} \tag{3.4}$$

Taylor expanding (3.4) at $(u_t, v_t) = (0,0)$ with $(a_0, b_0) = (\frac{1}{4}, d)$ produces

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \begin{pmatrix} 2(1-\sqrt{a^*+\frac{1}{4}})u_t - \frac{1}{2\sqrt{a^*+\frac{1}{4}}}v_t + (2\sqrt{a^*+\frac{1}{4}} + \frac{1}{\sqrt{a^*+\frac{1}{4}}})u_t^2 \\ +\frac{1}{\sqrt{a^*+\frac{1}{4}}}u_t v_t + \frac{1}{8}(a^*+\frac{1}{4})^{-\frac{3}{2}}v_t^2 + O(r_1^3) \\ 2(1-e^{\frac{b^*+d}{2\sqrt{a^*+\frac{1}{4}}}-d})(a^*+\frac{1}{4})(\sqrt{a^*+\frac{1}{4}}-1) + e^{\frac{b^*+d}{2\sqrt{a^*+\frac{1}{4}}}-d}v_t \\ +e^{\frac{b^*+d}{2\sqrt{a^*+\frac{1}{4}}}-d}(\frac{1}{2} - \frac{1}{2(a^*+\frac{1}{4})})(b^*+d)u_t^2 + O(r_1^3) \end{pmatrix}, \tag{3.5}$$

where $r_1 = \sqrt{u_t^2 + v_t^2}$.

Denote

$$A(\alpha) = \begin{pmatrix} 2(1-\sqrt{a_0+a^*}) & \frac{-1}{2\sqrt{a_0+a^*}} \\ 0 & e^{\frac{b_0+b^*}{2\sqrt{a_0+a^*}}-d} \end{pmatrix},$$

where $\alpha = (a^*, b^*)$, $a_0 = \frac{1}{4}, b_0 = d$, and

$$A_0(\alpha) = \begin{pmatrix} 2(1-\sqrt{\frac{1}{4}+a^*}) & \frac{-1}{2\sqrt{\frac{1}{4}+a^*}} \\ 0 & e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}}-d} \end{pmatrix}.$$

Then the eigenvector and the generalized eigenvector of A_0 corresponding to the eigenvalue 1 are $q_0 = \begin{pmatrix} \frac{1}{2\sqrt{\frac{1}{4}+a^*}} \\ 1-e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}}-d} \end{pmatrix}$ and $q_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. At the same time, the eigenvector and the generalized eigenvector of A_0^T corresponding to the eigenvalue 1 are $p_1 = \begin{pmatrix} 1-2\sqrt{\frac{1}{4}+a^*} \\ \frac{-1}{2\sqrt{\frac{1}{4}+a^*}} \end{pmatrix}$ and $p_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. These four vectors with $\alpha = 0$ satisfy the following equality:

$$A_0 q_0 = q_0, \quad A_0 q_1 = q_1 + q_0, \quad A_0^T p_1 = p_1, \quad A_0^T p_0 = p_0 + p_1,$$
$$\langle q_0, p_0 \rangle = \langle q_1, p_1 \rangle = 1, \quad \langle q_1, p_0 \rangle = \langle q_0, p_1 \rangle = 0,$$

where $\langle . \rangle$ stands for the standard scalar product in $\mathbb{R}^2$. Make an invertible linear transformation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = l_t q_0 + m_t q_1 = \begin{pmatrix} \frac{1}{2\sqrt{\frac{1}{4}+a^*}} & 1 - e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}}-d} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} l_t \\ m_t \end{pmatrix}, \tag{3.6}$$

and denote $x = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$, then the new coordinates (l_t, m_t) can be computed explicitly by

$$\begin{cases} l_t = \langle p_0, x \rangle = u_t, \\ m_t = \langle p_1, x \rangle = (1 - 2\sqrt{\frac{1}{4} + a^*})u_t - \frac{1}{2\sqrt{\frac{1}{4}+a^*}}v_t. \end{cases} \tag{3.7}$$

In the coordinates (l_t, m_t), the system (3.5) takes the form:

$$\begin{pmatrix} l_{t+1} \\ m_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l_t \\ m_t \end{pmatrix} + \begin{pmatrix} f_1(l_t, m_t, \alpha) + O((|l_t| + |m_t|)^3) \\ f_2(l_t, m_t, \alpha) + O((|l_t| + |m_t|)^3) \end{pmatrix}, \tag{3.8}$$

where

$$
\begin{aligned}
f_1(l_t, m_t, \alpha) &= \langle p_0, g(l_t q_0 + m_t q_1)\rangle = a_{00}(\alpha) + a_{10}(\alpha) l_t + a_{01}(\alpha) m_t \\
&\quad + \frac{1}{2} a_{20}(\alpha) l_t^2 + a_{11}(\alpha) l_t m_t + \frac{1}{2} a_{02}(\alpha) m_t^2, \\
f_2(l_t, m_t, \alpha) &= \langle p_1, g(l_t q_0 + m_t q_1)\rangle = b_{00}(\alpha) + b_{10}(\alpha) l_t + b_{01}(\alpha) m_t \\
&\quad + \frac{1}{2} b_{20}(\alpha) l_t^2 + b_{11}(\alpha) l_t m_t + \frac{1}{2} b_{02}(\alpha) m_t^2,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
a_{00}(\alpha) =& 0, a_{10}(\alpha) = 0, a_{01}(\alpha) = 0, a_{20}(\alpha) = \frac{1}{2\sqrt{\frac{1}{4} + a^*}}(1 + \frac{1}{2(\frac{1}{4} + a^*)}), \\
a_{11}(\alpha) =& 2 + \frac{1}{2(\frac{1}{4} + a^*)} - (2 + \frac{1}{\frac{1}{4} + a^*}) e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d}, \\
a_{02}(\alpha) =& (1 - e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d})[(2\sqrt{\frac{1}{4} + a^*} + \frac{1}{\sqrt{\frac{1}{4} + a^*}})(1 - e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d}) \\
& - \frac{1}{\sqrt{\frac{1}{4} + a^*}}] + \frac{1}{8}(a^* + \frac{1}{4})^{-\frac{3}{2}},
\end{aligned}
$$

$$
\begin{aligned}
b_{00}(\alpha) =& 2(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d} - 1)(a^* + \frac{1}{4})(\sqrt{a^* + \frac{1}{4}} - 1), b_{10}(\alpha) = 0, \\
b_{01}(\alpha) =& (e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d} - 1), b_{20}(\alpha) = -\frac{e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d}}{1 + 4a^*}[(\frac{1}{2} - \frac{1}{2a^* + \frac{1}{2}})(b^* + d)], \\
b_{11}(\alpha) =& e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d} \frac{e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d} - 1}{\sqrt{\frac{1}{4} + a^*}}[(\frac{1}{2} - \frac{1}{2a^* + \frac{1}{2}})(b^* + d)], \\
b_{02}(\alpha) =& -(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d} - 1)^2 e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}} - d}[(\frac{1}{2} - \frac{1}{2a^* + \frac{1}{2}})(b^* + d)], \\
a_{00}(0) =& a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0.
\end{aligned}
$$

For sufficiently small $|\alpha|$, by Lemma 9.6 in [28] or Lemma 3.1 in [29], the map (3.8) can be expressed as

$$
\begin{pmatrix} l_{t+1} \\ m_{t+1} \end{pmatrix} \mapsto \Phi_\alpha^1(l_t, m_t) + O((|l_t| + |m_t|)^3),
\tag{3.10}
$$

where $\Phi_\alpha^1(l_t, m_t)$ is the time-one flow of the following planar system

$$
\begin{pmatrix} \dot{l_{t+1}} \\ \dot{m_{t+1}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} l_t \\ m_t \end{pmatrix} + \begin{pmatrix} g_1(l_t, m_t, \alpha) \\ g_2(l_t, m_t, \alpha) \end{pmatrix},
\tag{3.11}
$$

where

$$
\begin{aligned}
g_1(l_t, m_t, \alpha) =& c_{00}(\alpha) + c_{10}(\alpha)l_t + c_{01}(\alpha)m_t + \frac{1}{2}c_{20}(\alpha)l_t^2 \\
&+ c_{11}(\alpha)l_t m_t + \frac{1}{2}c_{02}(\alpha)m_t^2, \\
g_2(l_t, m_t, \alpha) =& d_{00}(\alpha) + d_{10}(\alpha)l_t + d_{01}(\alpha)m_t + \frac{1}{2}d_{20}(\alpha)l_t^2 \\
&+ d_{11}(\alpha)l_t m_t + \frac{1}{2}d_{02}(\alpha)m_t^2,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
c_{00}(\alpha) =& a_{00}(\alpha) - (\frac{1}{2}a_{10}(\alpha) - \frac{1}{3}b_{10}(\alpha))a_{00}(\alpha) \\
&- (\frac{1}{2} - \frac{1}{3}a_{10}(\alpha) + \frac{1}{2}a_{01}(\alpha) + \frac{1}{4}b_{10}(\alpha) - \frac{1}{3}b_{01}(\alpha))b_{00}(\alpha), \\
c_{10}(\alpha) =& a_{10}(\alpha) - \frac{1}{2}b_{10}(\alpha), \\
c_{01}(\alpha) =& a_{01}(\alpha) - \frac{1}{2}a_{10}(\alpha) + \frac{1}{3}b_{10}(\alpha) - \frac{1}{2}b_{01}(\alpha), \\
c_{20}(\alpha) =& a_{20}(\alpha) - \frac{1}{2}b_{20}(\alpha), \\
c_{11}(\alpha) =& a_{11}(\alpha) - \frac{1}{2}a_{20}(\alpha) + \frac{1}{3}b_{20}(\alpha) - \frac{1}{2}b_{11}(\alpha), \\
c_{02}(\alpha) =& a_{02}(\alpha) + \frac{1}{6}a_{20}(\alpha) - a_{11}(\alpha) - \frac{1}{6}b_{20}(\alpha) + \frac{2}{3}b_{11}(\alpha) - \frac{1}{2}b_{02}(\alpha), \\
d_{00}(\alpha) =& b_{00}(\alpha) - \frac{1}{2}b_{10}(\alpha)a_{00}(\alpha) + (\frac{1}{3}b_{10}(\alpha) - \frac{1}{2}b_{01}(\alpha))b_{00}(\alpha), \\
d_{10}(\alpha) =& b_{10}(\alpha), \\
d_{01}(\alpha) =& b_{01}(\alpha) - \frac{1}{2}b_{10}(\alpha), \\
d_{20}(\alpha) =& b_{20}(\alpha), \\
d_{11}(\alpha) =& b_{11}(\alpha) - \frac{1}{2}b_{20}(\alpha), \\
d_{02}(\alpha) =& b_{02}(\alpha) + \frac{1}{6}b_{20}(\alpha) - b_{11}(\alpha).
\end{aligned}
$$

In particular, $c_{00}(0) = c_{10}(0) = c_{01}(0) = d_{00}(0) = d_{10}(0) = d_{01}(0) = 0$. When $d \neq 2$, the following nondegeneracy conditions hold:

$$
\begin{aligned}
&d_{20}(0) = b_{20}(0) = \frac{3d}{2} \neq 0, \\
&c_{20}(0) + d_{11}(0) = a_{20}(0) + b_{11}(0) - b_{20}(0) = 3 - \frac{3}{2}d \neq 0.
\end{aligned}
\tag{3.13}
$$

It follows from Lemma 3.2 in [29] that under analytic changes of coordinates and rescaling of time system (3.11) becomes

$$
\begin{cases}
\dot{v}_1 = v_2, \\
\dot{v}_1 = \eta_1(\alpha) + \eta_2(\alpha)v_1 + v_1^2 + s v_1 v_2,
\end{cases}
\tag{3.14}
$$

where

$$s = sign[b_{20}(a_{20} + b_{11} - b_{20})](0) = \pm 1.$$

Then $\eta := (\eta_1, \eta_2)$ can be expressed by the coefficients a_{ij} and b_{ij} (and in turn by c_{ij} and d_{ij}) as follows:

$$\eta_1(\alpha) = \frac{8\beta_0^4}{b_{20}^3(0)}\beta_1(\alpha) - \frac{8\beta_0^3}{b_{20}^3(0)}\beta_2(\alpha)\beta_3(\alpha) + \frac{4\beta_0^2}{b_{20}^2(0)}\beta_2(\alpha),$$
$$\eta_2(\alpha) = \frac{4\beta_0^2}{b_{20}^2(0)}\beta_4(\alpha) - \frac{4\beta_0}{b_{20}(0)}\beta_2(\alpha),$$

in which

$$\begin{aligned}
\beta_0 &= a_{20}(0) + b_{11}(0) - b_{20}(0),\\
\beta_1(\alpha) &= b_{00}(\alpha) + \frac{1}{2}(\frac{1}{6}b_{20}(0) - b_{11}(0) + b_{02}(0))a_{00}^2(\alpha)\\
&\quad - (\frac{1}{6}a_{20}(0) - a_{11}(0) + a_{02}(0) - \frac{1}{12}b_{20}(0) + \frac{1}{6}b_{11}(0))a_{00}(\alpha)b_{00}(\alpha)\\
&\quad + \frac{1}{2}(\frac{1}{6}a_{20}(0) - a_{11}(0) + a_{02}(0) - \frac{1}{8}b_{20}(0) + \frac{5}{12}b_{11}(0) - \frac{1}{4}b_{02}(0))b_{00}^2(\alpha)\\
&\quad - a_{00}(\alpha)b_{01}(\alpha) - \frac{1}{2}a_{10}(\alpha)b_{00}(\alpha) + a_{01}(\alpha)b_{00}(\alpha) + \frac{5}{12}b_{00}(\alpha)b_{10}(\alpha)\\
&\quad - \frac{1}{2}b_{00}(\alpha)b_{01}(\alpha),\\
\beta_2(\alpha) &= a_{10}(\alpha) - b_{10}(\alpha) + (\frac{1}{2}a_{20}(0) - a_{11}(0) - \frac{1}{2}b_{20}(0)) + \frac{3}{2}b_{11}(0) - b_{02}(0))a_{00}(\alpha)\\
&\quad - (\frac{1}{12}a_{20}(0) + \frac{1}{2}a_{11}(0) - a_{02}(0) - \frac{1}{12}b_{20}(0) + \frac{1}{12}b_{11}(0))b_{00}(\alpha) + b_{01}(\alpha),\\
\beta_3(\alpha) &= b_{10}(\alpha) + (\frac{1}{2}b_{20}(0) - b_{11}(0))a_{00}(\alpha) - (\frac{1}{2}a_{20}(0) - a_{11}(0) - \frac{1}{12}b_{20}(0))b_{00}(\alpha),\\
\beta_4(\alpha) &= b_{10}(\alpha) + (\frac{1}{2}b_{20}(0) - b_{11}(0))a_{00}(\alpha)\\
&\quad + (\frac{1}{2}a_{20}(0) - a_{11}(0) - \frac{3}{4}b_{20}(0) + 2b_{11}(0) - b_{02}(0))b_{00}(\alpha).
\end{aligned}$$

Then

$$\begin{aligned}
\eta_1(\alpha) =&(\frac{2-d}{d})^3(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}-d}} - 1)(a^* + \frac{1}{4})(\sqrt{a^* + \frac{1}{4}} - 1)[(\frac{3}{2}d^2 - 26d + 54)\\
&+ (18 + d)(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}-d}} - 1) + (\frac{5}{8}d^2 - \frac{33}{4}d + 54)(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}-d}} - 1)\\
&(a^* + \frac{1}{4})(\sqrt{a^* + \frac{1}{4}} - 1)] + (4e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}-d}} + 2 - 3d)(\frac{2-d}{d})^3,\\
\eta_2(\alpha) =&\frac{4(d-2)}{d}(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}-d}} - 1) + \frac{4(2-d)(2d^2 - 9d + 6)}{d^2}[(e^{\frac{d+b^*}{2\sqrt{\frac{1}{4}+a^*}-d}} - 1)\\
&(a^* + \frac{1}{4})(\sqrt{a^* + \frac{1}{4}} - 1)] - \frac{3(2-d)^2}{d}.
\end{aligned}$$

The transversality condition requires

$$detD_{\alpha}\eta(0) \neq 0, \tag{3.15}$$

where

$$detD_{\alpha}\eta(0) = \frac{(d-2)^4(3d^2 - 52d + 44)}{4d^3}.$$

This equivalently requires that the parameter $d \neq 2$ and $d \neq \frac{26\pm4\sqrt{34}}{3}$. At this time, the system (3.14) is the versal unfolding of the Bogdanov-Takens singularity of codimension two. Applying those results in [28, pp.424–434], we obtain the existence of 1:1 strong resonance bifurcation of the system (1.9) and corresponding conclusions.

4. Conclusions and discussion

In this paper, we revisit a discrete predator-prey model with Holling-IV functional response considered by Huang et.al. in [26]. The main differences between our work and the known one lie in three aspects. First, we make use of mathematical skills to reduce the known continuous model into a simpler equivalent model, leading to the original six parameters to become into three ones. By using semidiscretization method instead of the forward Euler method, we derive a new discrete version without assuming D=1 carried out in [26] and avoid isolating the accurate requirement. Second, we comprehensively consider the stability of all the fixed points $O(0,0), A(1,0)$, E_1, E_2 and E_0. Third, we explore the bifurcation problems of the system (1.9), which have not been studied in any other literature. Especially, we find the existence of 1:1 strong resonance bifurcation of codim 2 at the fixed point E_0 of the system (1.9), showing that this system possesses complicate dynamics, namely, under different parameter cases, the fold bifurcation, Neimark-Sacker bifurcation and homoclinic bifurcation may occur.

Our results clearly demonstrate that deep investigations into the same system may dig out more newer and deeper dynamical properties.

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Conflict of interest

The authors declare that they have no competing interests.

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Appendix

Definition 4.1. *Let $E(x, y)$ be a fixed point of the system* (1.9) *with multipliers λ_1 and λ_2.*

(1) *If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, the fixed point $E(x, y)$ is called sink, so a sink is locally asymptotically stable.*

(2) *If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the fixed point $E(x, y)$ is called source, so a source is locally asymptotically unstable.*

(3) *If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), the fixed point*

$E(x, y)$ *is called saddle.*

(4) *If either* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$, *the fixed point* $E(x, y)$ *is called to be non-hyperbolic.*

Lemma 4.2. *Let* $F(\lambda) = \lambda^2 + B\lambda + C$, *where* B *and* C *are two real constants. Suppose* λ_1 *and* λ_2 *are two roots of* $F(\lambda) = 0$. *Then the following statements hold.*

(i) *If* $F(1) > 0$, *then*

(i.1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;

(i.2) $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *if and only if* $F(-1) = 0$ *and* $B \neq 2$;

(i.3) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) < 0$;

(i.4) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$;

(i.5) λ_1 *and* λ_2 *are a pair of conjugate complex roots and,* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $-2 < B < 2$ *and* $C = 1$;

(i.6) $\lambda_1 = \lambda_2 = -1$ *if and only if* $F(-1) = 0$ *and* $B = 2$.

(ii) *If* $F(1) = 0$, *namely,* 1 *is one root of* $F(\lambda) = 0$, *then the other root* λ *satisfies* $|\lambda| = (<, >)1$ *if and only if* $|C| = (<, >)1$.

(iii) *If* $F(1) < 0$, *then* $F(\lambda) = 0$ *has one root lying in* $(1, \infty)$. *Moreover,*

(iii.1) *the other root* λ *satisfies* $\lambda < (=) - 1$ *if and only if* $F(-1) < (=)0$;

(iii.2) *the other root* $-1 < \lambda < 1$ *if and only if* $F(-1) > 0$.