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Research article

Coefficient functionals for a class of bounded turning functions related to modified sigmoid function

Muhammad Ghaffar Khan[1], **Nak Eun Cho**[2,*], **Timilehin Gideon Shaba**[3], **Bakhtiar Ahmad**[4] **and Wali Khan Mashwani**[1]

[1] Institute of Numerical Sciences, Kohat university of science and technology, Kohat, Pakistan
[2] Department of Applied Mathematics, Pukyong National University Busan 48513, Korea
[3] Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, Nigeria
[4] Govt. Degree College Mardan, Mardan 23200, Pakistan

* **Correspondence:** Email: necho@pknu.ac.kr.

Abstract: The main objective of the present article is to define the class of bounded turning functions associated with modified sigmoid function. Also we investigate and determine sharp results for the estimates of four initial coefficients, Fekete-Szegö functional, the second-order Hankel determinant, Zalcman conjucture and Krushkal inequality. Furthermore, we evaluate bounds of the third and fourth-order Hankel determinants for the class and for the 2-fold and 3-fold symmetric functions.

Keywords: analytic functions; modified sigmoid function; subordination; Hankel determinant; 2-fold and 3-fold symmetric function; Zalcman conjucture; Krushkal inequality
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1. Introduction

Let $\mathcal{A}$ represent the collections of analytic functions defined in open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ whose normalization is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ comprising of functions of the form (1.1) which are also univalent in $\mathbb{D}$.

Let $\mathcal{P}$ represent the class of all functions p that are analytic in $\mathbb{D}$ with $\Re(p(z)) > 0$ and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \ (z \in \mathbb{D}). \tag{1.2}$$

Next we recall the definition of subordination. For two functions $h_1, h_2 \in \mathcal{A}$, we say that h_1 is subordinate to h_2 and is symbolically written as $h_1 \prec h_2$ if there exists an analytic function w with the property $|w(z)| \leq |z|$ and $w(0) = 0$ such that $h_1(z) = h_2(w(z))$ for $z \in \mathbb{D}$. Further, if $h_2 \in \mathcal{S}$, then the condition becomes

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \text{ and } h_1(\mathbb{D}) \subset h_2(\mathbb{D}).$$

Now we consider the following class $\mathcal{S}^*(\varphi)$ as follows:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \tag{1.3}$$

where φ is an analytic univalent function with positive real part in $\mathbb{D}$, $\varphi(\mathbb{U})$ is symmetric about the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class $\mathcal{S}^*(\varphi)$ was introduced by Ma and Minda [20]. In particular, if we take $\varphi(z) = (1+z)/(1-z)$, then the class $\mathcal{S}^*(\varphi)$ is the well-known class of starlike functions. If we vary the function φ on the right hand side of (1.3), then we obtain some several subclasses of $\mathcal{S}$ whose image domains have some interesting geometrical configurations as follows:

(1) The class $\mathcal{S}^*(\varphi)$ with $\varphi(z) = 1 + \sin z$ is introduced and studied by Cho et al. [6].
(2) The class $\mathcal{S}^*(\varphi)$ with $\varphi(z) = 1 + z - \frac{1}{3}z^3$, which is a nephroid shaped domain, was introduced and investigated by Wani and Swaminathan [39].
(3) The class $\mathcal{S}^*(\varphi)$ with $\varphi(z) = \sqrt{1+z}$, which is bounded by lemniscate of Bernoulli in right half plan, was developed by Sokól and Stankiewicz [30].
(4) The class $\mathcal{S}^*(\varphi)$ with $\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$ was introduced by Sharma et al. [29].
(5) The class $\mathcal{S}^*(\varphi)$ with $\varphi(z) = e^z$ was introduced and studied by Mendiratta et al. [21].
(6) The class $\mathcal{S}^*(\varphi)$ with $\varphi(z) = z + \sqrt{1+z^2}$, which maps $\mathbb{D}$ to crescent shaped region, was introduced by Raina and Sokól [26].

Also we note that lately many subclasses of starlike functions are introduced see [7, 9, 12] by choosing some particular functions such as functions associated with Bell numbers, shell-like curve connected with Fibonacci numbers, functions connected with conic domains and rational functions instead of φ in (1.3).

Pommerenke [24, 25] introduced the Hankel determinant $H_{q,n}(f)$ for function $f \in \mathcal{S}$ of the form (1.1), where the parameters $q, n \in \mathbb{N} = \{1, 2, 3, \cdots\}$ as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}. \tag{1.4}$$

The Hankel determinants for different orders are obtained for different values of q and n. When $q = 2$ and $n = 1$, the determinant is

$$\left|H_{2,1}(f)\right| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left|a_3 - a_2^2\right|, \quad \text{where } a_1 = 1.$$

Note that $H_{2,1}(f) = a_3 - a_2^2$, is the classical Fekete-Szegö functional. For various subclasses of $\mathcal{A}$, the best possible value of the upper bound for $\left|H_{2,1}(f)\right|$ was investigated by different authors (see [13–15] for details). Furthermore, when $q = 2$ and $n = 2$, the second Hankel determinant is

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The upper bound of $\left|H_{2,2}(f)\right|$ has been studied by several authors in the last few decades. For instance, the readers may refer to the works of Hayman [11], the Noonan and Thomas [22], Ohran et al. [23] and Shi et al. [34]. Moreover, Babalola [3] studied the Hankel determinant $H_{3,1}(f)$ for some subclasses of analytic functions. For some recent works on third order Hankel determinant we may refer the interested reader to such more recent works as (for example) [28, 32, 38]. The bound of the fourth Hankel determinant for a class of analytic functions with bounded turning associated with cardoid domain was approximated by Srivastava et al. in [37]. It should be remarked that a wide variety of applications of Hankel systems arise in linear filtering theory, discrete inverse scattering, and discretization of certain integral equations arising in mathematical physics [40].

Evaluating these Hankel determinants for various new subclasses has been an attracting area lately. One such field of interest is the Quantum Calculus (q-calculus), which is a generalization of classical calculus by replacing the limit by a parameter q. For the basics and preliminaries, the readers are advised to see the works and expositions in [31, 35, 37]. It is important to mention here the work on a q-differential operator by Srivastava et al. [33], in which they determined the upper bound of second Hankel determinant for a subclass of bi-univalent functions in q-analogue. Recently, the upper bound estimate for q-analogue of a subclass of starlike functions in connection with exponential function were evaluated in [36].

Recently, a class of starlike functions associated with Modified sigmoid function was defined by Goel and Kumar [10], i.e,

$$\mathcal{S}^{*}_{\mathcal{S}_G} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-z}} \right\} \quad (z \in \mathbb{D}).$$

Motivated by all the works mentioned above and [4], in this article we introduce and investigate the class $\mathcal{R}_{\mathcal{S}_G}$, which is defined as follows:

$$\mathcal{R}_{\mathcal{S}_G} = \left\{ f \in \mathcal{S} : f'(z) \prec \frac{2}{1+e^{-z}} \right\} \quad (z \in \mathbb{D}). \tag{1.5}$$

We also establish some sharp results such as coefficient bounds, Fekete-Szegö inequality, second-order determinant, Zalcman conjecture and Krushkal inequality for functions belonging to the class $\mathcal{R}_{\mathcal{S}_G}$. Moreover, we estimate bounds of the third and forth-order Hankel determinants for this class $\mathcal{R}_{\mathcal{S}_G}$ and for the 2-fold and 3-fold symmetric functions.

2. A set of lemmas

For the proofs of our main findings, we need the following lemmas.

Lemma 1. *Let $p \in \mathcal{P}$ have the series expansion of the form* (1.2). *Then, for x and σ with $|x| \leq 1, |\sigma| \leq 1$, such that*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.1}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)lc_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\sigma. \tag{2.2}$$

We note that (2.1) and (2.2) are taken from [18].

Lemma 2. *If $p \in \mathcal{P}$ and has the series of the form* (1.2)*, then*

$$|c_{n+k} - \mu c_n c_k| \leq 2, \ 0 \leq \mu \leq 1, \tag{2.3}$$

$$|c_n| \leq 2 \text{ for } n \geq 1, \tag{2.4}$$

$$|c_2 - \zeta c_1^2| \leq 2 \max\{1, |2\zeta - 1|\}, \ \zeta \in \mathbb{C}. \tag{2.5}$$

We note that the inequalities (2.3), (2.4) and (2.6) in the above can be found in [2, 25] and (2.5) is given by [13].

Lemma 3. *[2] If $p \in \mathcal{P}$ and has the series of the form* (1.2)*, then*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|, \tag{2.6}$$

where J, K and L are real numbers.

Lemma 4. *[27] Let m, n, l and r satisfy the inequalities $0 < m < 1, \ 0 < r < 1$ and*

$$8r(1-r)\left[(mn - 2l)^2 + (m(r+m) - n)^2\right] + m(1-m)(n - 2rm)^2 \leq 4m^2(1-m)^2 r(1-r).$$

If $p \in \mathcal{P}$ and has power series (1.2)*, then*

$$\left| lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4 \right| \leq 2.$$

3. Bounds of $\left|H_{3,1}(f)\right|$ for the class $\mathcal{R}_{S_G}$

Theorem 1. *Let $f \in \mathcal{R}_{S_G}$ and be of the form* (1.1)*. Then*

$$|a_2| \leq \frac{1}{4}, \tag{3.1}$$

$$|a_3| \leq \frac{1}{6}, \tag{3.2}$$

$$|a_4| \leq \frac{1}{8}, \tag{3.3}$$

$$|a_5| \leq \frac{1}{10}, \tag{3.4}$$

$$|a_6| \leq \frac{355}{288}, \tag{3.5}$$

$$|a_7| \leq \frac{381\,377}{282\,0}. \tag{3.6}$$

The first four inequalities are sharp for the functions defined below respectively

$$f_n(z) = \int_0^z \left(\frac{2}{1 + e^{-t^n}} \right) dt = z + \frac{1}{2(n+1)} z^{n+1} + \cdots, \quad \textit{where } n = 1, 2, 3, 4. \tag{3.7}$$

Proof. Let $f \in \mathcal{R}_{\mathcal{S}_G}$. Then, (1.5) can be put in the form of Schwarz function $w(z)$ as

$$f'(z) = \frac{2}{1 + e^{-w(z)}} \ (z \in \mathbb{D}). \tag{3.8}$$

Also, if $p \in \mathcal{P}$, then it may be written in terms of the Schwarz function w as

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 \cdots = \frac{1 + w(z)}{1 - w(z)},$$

or equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2 \right) z^2 + \left(\frac{1}{8} c_1^3 - \frac{1}{2} c_2 c_1 + \frac{1}{2} c_3 \right) z^3 + \cdots. \tag{3.9}$$

Now

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots, \tag{3.10}$$

By a simplification and using the series expansion (3.9), we have

$$\begin{aligned} \frac{2}{1 + e^{-w(z)}} &= 1 + \frac{c_1}{4} z + \left(\frac{c_2}{4} - \frac{c_1^2}{8} \right) z^2 + \left(\frac{11 c_1^3}{192} - \frac{c_2 c_1}{4} + \frac{c_3}{4} \right) z^3 \\ &\quad + \left(-\frac{3}{128} c_1^4 + \frac{11}{64} c_1^2 c_2 - \frac{1}{4} c_3 c_1 - \frac{1}{8} c_2^2 + \frac{1}{4} c_4 \right) z^4 + \cdots. \end{aligned} \tag{3.11}$$

Comparing (3.10) and (3.11), we get

$$a_2 = \frac{1}{8} c_1, \tag{3.12}$$

$$a_3 = \frac{1}{3} \left(\frac{1}{4} c_2 - \frac{1}{8} c_1^2 \right), \tag{3.13}$$

$$a_4 = \frac{1}{4} \left(\frac{11}{192} c_1^3 - \frac{1}{4} c_1 c_2 + \frac{1}{4} c_3 \right) \tag{3.14}$$

$$a_5 = -\frac{1}{20} \left(\frac{3}{32} c_1^4 - \frac{11}{16} c_1^2 c_2 + c_3 c_1 + \frac{1}{2} c_2^2 - c_4 \right). \tag{3.15}$$

$$a_6 = \frac{1}{18432} \left(\begin{array}{c} -5c_1^6 + 122 c_1^4 c_2 - 288 c_1^3 c_3 - 432 c_1^2 c_2^2 + 528 c_4 c_1^2 + 1056 c_1 c_2 c_3 \\ -768 c_5 c_1 + 176 c_2^3 - 768 c_4 c_2 - 384 c_3^2 + 768 c_6 \end{array} \right) \tag{3.16}$$

and

$$a_7 = \frac{1}{36\,126\,720} \left(\begin{array}{c} -2537 c_1^7 - 50\,400 c_1^5 c_2 + 204\,960 c_1^4 c_3 + 409\,920 c_1^3 c_2^2 - 483\,840 c_4 c_1^3 \\ -1451\,520 c_1^2 c_2 c_3 + 887\,040 c_5 c_1^2 - 483\,840 c_1 c_2^3 + 1774\,080 c_4 c_1 c_2 + 887\,040 c_1 c_3^2 \\ -1290\,240 c_6 c_1 + 887\,040 c_2^2 c_3 - 1290\,240 c_5 c_2 - 1290\,240 c_4 c_3 + 1290\,240 c_7 \end{array} \right). \tag{3.17}$$

For a_2, putting (2.4) in (3.12), we have

$$|a_2| \leq \frac{1}{4}.$$

For a_3, simplifying (3.13), we get

$$a_3 = \frac{1}{12}\left(c_2 - \frac{c_1^2}{2}\right)$$

and applying (2.3), we have

$$|a_3| \leq \frac{1}{6}.$$

For a_4, using (3.14), we obtain

$$|a_4| = \frac{1}{4}\left|\frac{11}{192}c_1^3 - \frac{1}{4}c_1c_2 + \frac{1}{4}c_3\right|. \tag{3.18}$$

By applying Lemma 3 to (3.18), we get

$$|a_4| \leq \frac{1}{4}\left[2\left|\frac{11}{192}\right| + 2\left|\frac{1}{4} - 2\left(\frac{11}{192}\right)\right| + \left|\frac{11}{192} - \frac{1}{4} + \frac{1}{4}\right|\right] = \frac{1}{8}.$$

For a_5, applying Lemma 4 to (3.15), we get

$$|a_5| \leq \frac{1}{10}.$$

For a_6, re-arranging (3.16) and applying the triangle inequality, we get

$$|a_6| \leq \frac{1}{18432}\left[\begin{array}{c} 122\,|c_1|^4\left|c_2 - \frac{5}{122}c_1^2\right| + 1056\,|c_1|\,|c_3|\left|c_2 - \frac{3}{11}c_1^2\right| + 528\,|c_1|^2\left|c_4 - \frac{9}{11}c_2^2\right| \\ +768\,|c_6 - c_1c_5| + 768\,|c_2|\left|c_4 - \frac{88}{89}c_2^2\right| + 384\,|c_3|^2 \end{array}\right].$$

By applying (2.3) and (2.4) to the above, we get

$$|a_6| \leq \frac{355}{288}.$$

For a_7, re-arranging (3.17) and applying the triangle inequality, we get

$$|a_7| \leq \frac{1}{36\,126\,720}\left[\begin{array}{c} 204960\,|c_1|^4\left|c_3 - \frac{105}{427}c_1c_2\right| + 483840\,|c_1|^3\left|c_4 - \frac{61}{72}c_1^2\right| + 1290240\,|c_1|\left|c_6 - \frac{11}{16}c_1c_5\right| \\ +1774080\,|c_1|\,|c_2|\left|c_4 - \frac{9}{11}c_1c_3\right| + 887040\,|c_2|^2\left|c_3 - \frac{6}{11}c_1c_2\right| + 1290240\,|c_7 - c_2c_5| \\ +1290240\,|c_3|\left|c_4 - \frac{11}{16}c_1c_3\right| + 2537\,|c_1|^7 \end{array}\right].$$

Also by using (2.3) and (2.4) to the above, we obtain

$$|a_7| \leq \frac{381\,377}{282\,240}.$$

□

Next, we consider the Fekete-Szegö problem and the Hankel determinants for the class $\mathcal{R}_{S_G}$.

Theorem 2. *If f of the form* (1.1) *belongs to $\mathcal{R}_{\mathcal{S}_G}$, then*

$$\left|a_3 - \zeta a_2^2\right| \leq \frac{1}{6} \max\left\{1, \frac{3|\zeta|}{8}\right\} \quad (\zeta \in \mathbb{C}). \tag{3.19}$$

The result is sharp for the function f_2 defined by (3.7) *for $|\zeta| \leq 8/3$ and the functiom f_1 defined by* (3.7) *for $|\zeta| \geq 8/3$.*

Proof. Using (3.12) and (3.13), we can write

$$|a_3 - \zeta a_2^2| = \left|\frac{c_2}{12} - \frac{c_1^2}{24} - \zeta\frac{c_1^2}{64}\right|.$$

By rearranging we have

$$|a_3 - \zeta a_2^2| = \frac{1}{12}\left|c_2 - \left(\frac{3\zeta + 8}{16}\right)c_1^2\right|.$$

Applying (2.5) we get

$$\left|a_3 - \zeta a_2^2\right| \leq \frac{1}{12} \max\left\{2, 2\left|2\left(\frac{3\zeta + 8}{16}\right) - 1\right|\right\}.$$

Then with simple calculations, we obtain

$$\left|a_3 - \zeta a_2^2\right| \leq \frac{1}{6} \max\left\{1, \frac{3|\zeta|}{8}\right\}.$$

For the sharpness consider the function

$$f_2(z) = z + \frac{1}{6}z^3 - \frac{1}{168}z^7 + \cdots, \tag{3.20}$$

which gives equality in (3.19) when $|\zeta| \leq \frac{8}{3}$, namely

$$\left|a_3 - \zeta a_2^2\right| = |a_3| = \frac{1}{6} = \frac{1}{6} \max\left\{1, \frac{3|\zeta|}{8}\right\}.$$

For the case $|\zeta| \geq \frac{8}{3}$ consider

$$f_1(z) = z + \frac{1}{4}z^2 - \frac{1}{96}z^4 + \cdots,$$

which gives

$$\left|a_3 - \zeta a_2^2\right| = |\zeta a_2^2| = \frac{|\zeta|}{16}.$$

□

If we put $\zeta = 1$, then the above result becomes:

Corollary 1. *If f of the form* (1.1) *belongs to $\mathcal{R}_{\mathcal{S}_G}$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6}. \tag{3.21}$$

Theorem 3. *If f of the form* (1.1) *belongs to $\mathcal{R}_{S_G}$, then*

$$|a_2a_3 - a_4| \leq \frac{1}{8}. \tag{3.22}$$

The result is sharp for the function f_3 defined by (3.7).

Proof. From (3.12)–(3.14), we get

$$|a_2a_3 - a_4| = \frac{1}{16}\left|\frac{5}{16}c_1^3 - \frac{7}{6}c_2c_1 + c_3\right|.$$

Using Lemma 3, we get the required result. □

Theorem 4. *If f of the form* (1.1) *belongs to $\mathcal{R}_{S_G}$, then*

$$|H_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| \leq \frac{1}{36}. \tag{3.23}$$

The result is sharp for the function f_2 defined by (3.7).

Proof. From (3.12)–(3.14), we have

$$H_{2,2}(f) = \frac{1}{18432}c_1^4 - \frac{1}{1152}c_1^2c_2 + \frac{1}{128}c_1c_3 - \frac{1}{144}c_2^2.$$

Applying (2.1) and (2.2) to express c_2 and c_3 in terms of $c_1 = c$, with $0 \leq c \leq 2$, we get

$$H_{2,2}(f) = -\tfrac{1}{6144}c^4 - \tfrac{1}{512}c^2(4-c^2)x^2 - \tfrac{1}{576}(4-c^2)^2x^2 + \tfrac{1}{256}c(4-c^2)(1-|x|^2)\sigma.$$

With the aid of the triangle inequality and replacing $|\sigma| \leq 1$, $|x| = b$, with $b \leq 1$, we obtain

$$\left|H_{2,2}(f)\right| \leq \frac{1}{6144}c^4 + \frac{1}{512}c^2(4-c^2)b^2 + \frac{1}{576}(4-c^2)^2b^2 + \frac{1}{256}c(4-c^2)(1-b^2) := \phi(c,b).$$

It is a simple calculation to show that $\frac{\partial\phi(c,b)}{\partial b} \geq 0$ on $[0,1]$, so that $\phi(c,b) \leq \phi(c,1)$. Putting $b = 1$ gives

$$\left|H_{2,2}(f)\right| \leq \frac{1}{6144}c^4 + \frac{1}{512}c^2(4-c^2) + \frac{1}{576}(4-c^2)^2 := \phi(c,1).$$

Also, $\phi'(c,1) = 0$, has only root $c = 0 \in [0,2]$ and so $\phi''(0,1) < 0$. Thus, the maximum value at $c = 0$ is

$$\left|H_{2,2}(f)\right| \leq \frac{1}{36}.$$

□

Theorem 5. *If $f \in \mathcal{A}$ belongs to $\mathcal{R}_{S_G}$, then*

$$|a_2a_5 - a_3a_4| \leq \frac{217}{2880}. \tag{3.24}$$

Proof. From (3.12)–(3.15), we have

$$|a_2a_5 - a_3a_4| = \left|\frac{1}{92\,160}c_1^5 + \frac{23}{46\,080}c_1^3c_2 - \frac{7}{1920}c_3c_1^2 + \frac{1}{480}c_1c_2^2 + \frac{1}{160}c_4c_1 - \frac{1}{192}c_3c_2\right|.$$

Rearranging the above term, we get

$$\begin{aligned}|a_2a_5 - a_3a_4| &= \left|\frac{1}{92\,160}c_1^5 - \frac{7}{1920}c_1^2\left(c_3 - \frac{23}{168}c_1c_2\right) - \frac{1}{192}c_2\left(c_3 - \frac{2}{5}c_1c_2\right) + \frac{1}{160}c_1c_4\right| \\ &\leq \frac{1}{92\,160}|c_1|^5 + \frac{7}{1920}|c_1|^2\left|c_3 - \frac{23}{168}c_1c_2\right| + \frac{1}{192}|c_2|\left|c_3 - \frac{2}{5}c_1c_2\right| + \frac{1}{160}|c_1||c_4|.\end{aligned}$$

Using (2.3) and (2.4), we get the required result. □

Theorem 6. *If $f \in \mathcal{A}$ belongs to $\mathcal{R}_{S_G}$, then*

$$|a_5 - a_2a_4| \leq \frac{1}{10}. \tag{3.25}$$

The result is sharp for function f_4 defined by (3.7).

Proof. From (3.12)–(3.15), we have

$$|a_5 - a_2a_4| = \frac{1}{20}\left|\frac{199}{1536}c_1^4 - \frac{27}{32}c_1^2c_2 + \frac{37}{32}c_3c_1 + \frac{1}{2}c_2^2 - c_4\right|.$$

By applying of Lemma 4, we get the desired result. □

Theorem 7. *If $f \in \mathcal{A}$ belongs to $\mathcal{R}_{S_G}$, then*

$$\left|a_3a_5 - a_4^2\right| \leq \frac{146\,831}{3087\,360}. \tag{3.26}$$

Proof. From (3.13)–(3.15), we have

$$\begin{aligned}\left|a_3a_5 - a_4^2\right| &= \left|-\frac{29}{2949\,120}c_1^6 - \frac{1}{30\,720}c_1^4c_2 + \frac{3}{10\,240}c_1^3c_3 - \frac{1}{480}c_4c_1^2\right. \\ &\quad \left.+\frac{7}{1920}c_1c_2c_3 - \frac{1}{480}c_2^3 + \frac{1}{240}c_4c_2 - \frac{1}{256}c_3^2\right| \\ &\leq \frac{29}{2949\,120}|c_1|^6 + \frac{3}{10240}|c_1|^3\left|c_3 - \frac{1}{9}c_1c_2\right| + \frac{1}{240}|c_4|\left|c_2 - \frac{1}{2}c_1^2\right| \\ &\quad + \frac{1}{256}|c_3|\left|c_3 - \frac{14}{15}c_1c_2\right| + \frac{1}{480}|c_2|^3.\end{aligned}$$

Now using (2.3)–(2.5), we get the required result. □

We will now determine the bound of the third Hankel determinant $H_{3,1}(f)$ for $f \in \mathcal{R}_{S_G}$.

Theorem 8. *If $f \in \mathcal{A}$ belongs to $\mathcal{R}_{S_G}$, then*

$$\left|H_{3,1}(f)\right| \leq \frac{319}{8640}. \tag{3.27}$$

Proof. We have the third Hankel determinant form as follows:

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \tag{3.28}$$

where $a_1 = 1$. This yields

$$\left|H_{3,1}(f)\right| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{3.29}$$

By using (3.2)–(3.4) and (3.21)–(3.23), we obtain the desired result. □

4. Bounds of $|H_{4,1}(f)|$ for the class $\mathcal{R}_{\mathcal{S}_G}$

From (1.4), we can write $H_{4,1}(f)$ as

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3, \tag{4.1}$$

where

$$\delta_1 = a_3\,(a_2a_5 - a_3a_4) - a_4\,(a_5 - a_2a_4) + a_6(a_3 - a_2^2), \tag{4.2}$$

$$\delta_2 = a_3\left(a_3a_5 - a_4^2\right) - a_5\,(a_5 - a_2a_4) + a_6(a_4 - a_2a_3), \tag{4.3}$$

$$\delta_3 = a_4\left(a_3a_5 - a_4^2\right) - a_5\,(a_2a_5 - a_3a_4) + a_6(a_2a_4 - a_3^2). \tag{4.4}$$

In recent years, researchers start to find the fourth-order Hankel determinant for different subclasses of analytic functions. The trend of finding fourth-order Hankel determinant in geometric function theory started in 2018, when Arif et al. [1] studied and successfully obtained the upper bound for the class of bounded turning functions. Recently Khan et al. [16] obtained the third and fourth-order Hankel determinant for the class of bounded turning functions associated with sine function. Also, Zhang and Tang [41] studied the fouth-order Hankel determinat for class of starlike functions connected with sine function. Inspired from the above works, we discuss here the fourth-order Hankel determiant for the class $\mathcal{R}_{\mathcal{S}_G}$.

Theorem 9. *If $f \in \mathcal{A}$ belongs to $\mathcal{R}_{\mathcal{S}_G}$, then*

$$\left|H_{4,1}(f)\right| \leq \frac{2334\,260\,186\,533}{6535\,323\,648\,000} \simeq 0.357\,1.$$

Proof. From (4.1), we have

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3,$$

where δ_1, δ_2 and δ_3 are defined as in (4.2)–(4.4). Now by using the triangle inequality, we have

$$\left|H_{4,1}(f)\right| \leq |a_7|\left|H_{3,1}(f)\right| + |a_6|\,|\delta_1| + |a_5|\,|\delta_2| + |a_4|\,|\delta_3|\,, \tag{4.5}$$

Since

$$\begin{aligned}|\delta_1| &= \left|a_3\left(a_2a_5-a_3a_4\right)-a_4\left(a_5-a_2a_4\right)+a_6(a_3-a_2^2)\right| \\ &\leq |a_3|\,|a_2a_5-a_3a_4|+|a_4|\,|a_5-a_2a_4|+|a_6|\left|a_3-a_2^2\right|,\end{aligned}$$

by using (3.2), (3.3), (3.5), (3.21), (3.24) and (3.25), we get

$$|\delta_1| \leq \frac{3983}{17\,280}. \tag{4.6}$$

Since

$$\begin{aligned}|\delta_2| &= \left|a_3\left(a_3a_5-a_4^2\right)-a_5\left(a_5-a_2a_4\right)+a_6(a_4-a_2a_3)\right| \\ &\leq |a_3|\left|a_3a_5-a_4^2\right|+|a_5|\,|a_5-a_2a_4|+|a_6|\,|a_4-a_2a_3|,\end{aligned}$$

by using (3.2), (3.4), (3.5), (3.22), (3.25) and (3.26), we get

$$|\delta_2| \leq \frac{15\,931\,363}{92\,620\,800}. \tag{4.7}$$

Also, since

$$\begin{aligned}|\delta_3| &= \left|a_4\left(a_3a_5-a_4^2\right)-a_5\left(a_2a_5-a_3a_4\right)+a_6(a_2a_4-a_3^2)\right| \\ &\leq |a_4|\left|a_3a_5-a_4^2\right|+|a_5|\,|a_2a_5-a_3a_4|+|a_6|\left|a_2a_4-a_3^2\right|,\end{aligned}$$

by using (3.3)–(3.5), (3.23), (3.24) and (3.26), we get

$$|\delta_3| \leq \frac{53\,037\,859}{1111\,449\,600}. \tag{4.8}$$

Now by using the values of (4.6)–(4.8) along with (3.3)–(3.6) and (3.27) to (4.5), we get the desired estimate. □

5. Bounds of $|H_{4,1}(f)|$ for the 2-fold and 3-fold symmetric functions

A function f is said to be m-fold symmetric if the following condition holds true for $\varepsilon=\exp\left(\frac{2\pi i}{m}\right)$,

$$f(\varepsilon z)=\varepsilon f(z) \quad (z\in\mathbb{D}).$$

The set of all m-fold symmetric functions belonging to the familiar class $\mathcal{S}$ of univalent functions is denoted by $\mathcal{S}^{(m)}$, represented by the following series expansion

$$f(z)=z+\sum_{n=1}^{\infty}a_{mn+1}z^{mn+1} \quad (z\in\mathbb{D}). \tag{5.1}$$

An analytic function f of the form (5.1) belongs to the class $\mathcal{R}_{\mathcal{S}_G}^{(m)}$ if and only if

$$f'(z)=\frac{2}{1+e^{-\left(\frac{p(z)-1}{p(z)+1}\right)}} \quad (z\in\mathbb{D}), \tag{5.2}$$

where $p(z)$ belong to the class $\mathcal{P}^{(m)}$ which is defined as follows:

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{n=1}^{\infty} c_{mn} z^{mn} \right\}. \tag{5.3}$$

If a function f belongs to $\mathcal{S}^{(2)}$, then its series representation is

$$f(z) = z + a_3 z^3 + a_5 z^5 + \cdots,$$

and

$$H_{4,1}(f) = a_3 a_5 a_7 - a_3^3 a_7 + a_3^2 a_5^2 - a_5^3. \tag{5.4}$$

Further, if a function f belongs to $\mathcal{S}^{(3)}$, then its series representation is

$$f(z) = z + a_4 z^4 + a_7 z^7 + \cdots,$$

and

$$H_{4,1}(f) = a_4^2 \left(a_4^2 - a_7 \right). \tag{5.5}$$

Theorem 10. *If $f \in \mathcal{R}_{\mathcal{S}_G}^{(2)}$, then*

$$\left| H_{4,1}(f) \right| \leq \frac{299}{108000}.$$

Proof. Let $f \in \mathcal{R}_{\mathcal{S}_G}^{(2)}$. Then by the series (5.1)–(5.3) for $m = 2$, we have

$$f'(z) = 1 + 3a_3 z^2 + 5a_5 z^4 + 7a_7 z^6 + \cdots,$$

$$\frac{2}{1 + e^{-\left(\frac{c_2 z^2 + c_4 z^4 + \cdots}{2 + c_2 z^2 + c_4 z^4 + \cdots}\right)}} = 1 + \frac{1}{4} c_2 z^2 + \left(\frac{1}{4} c_4 - \frac{1}{8} c_2^2 \right) z^4 + \left(\frac{11}{192} c_2^3 - \frac{1}{4} c_4 c_2 + \frac{1}{4} c_6 \right) z^6 + \cdots.$$

After comparing, we get

$$\begin{aligned} a_3 &= \frac{1}{12} c_2, \\ a_5 &= \frac{1}{20} \left(c_4 - \frac{1}{2} c_2^2 \right), \\ a_7 &= \frac{11}{1344} c_2^3 - \frac{1}{28} c_2 c_4 + \frac{1}{28} c_6. \end{aligned}$$

Then by substituting the above values to (5.4), we get

$$\begin{aligned} H_{4,1}(f) = &- \frac{529}{290\,304\,000} c_2^6 + \frac{437}{192\,000} c_2^4 c_4 - \frac{23}{241\,920} c_6 c_2^3 \\ &+ \frac{113}{2016\,000} c_2^2 c_4^2 + \frac{1}{6720} c_6 c_2 c_4 - \frac{1}{8000} c_4^3 \end{aligned}$$

and after rearranging, we get

$$H_{4,1}(f) = \frac{437}{24\,192\,000} c_2^4 \left(c_4 - \frac{23}{228} c_2^2 \right) + \frac{1}{6720} c_2 c_6 \left(c_4 - \frac{23}{36} c_2^2 \right) - \frac{1}{8000} c_4^2 \left(c_4 - \frac{113}{252} c_2^2 \right).$$

Now by using the triangle inequality along with (2.3) and (2.4), we get the required result. □

Theorem 11. *If $f \in \mathcal{R}^{(3)}_{S_G}$, then*

$$\left|H_{4,1}(f)\right| \leq \frac{1}{896}.$$

Proof. Let $f \in \mathcal{R}^{(3)}_{S_G}$. Then by (5.1)–(5.3) for $m = 3$, we have

$$f'(z) = 1 + 4a_4z^3 + 7a_7z^6 + \cdots \tag{5.6}$$

$$\frac{2}{1 + e^{-\left(\frac{c_3z^3+c_6z^6+\cdots}{2+c_3z^3+c_6z^6+\cdots}\right)}} = 1 + \frac{1}{4}c_3z^3 + \left(\frac{1}{4}c_6 - \frac{1}{8}c_3^2\right)z^6 + \cdots . \tag{5.7}$$

After comparing (5.6) and (5.7), we get

$$a_4 = \frac{1}{16}c_3,$$
$$a_7 = \frac{1}{28}\left(c_6 - \frac{1}{2}c_3^2\right).$$

Then by substituting the above values to (5.5), we get

$$H_{4,1}(f) = -\frac{1}{7168}c_3^2\left(c_6 - \frac{39}{64}c_3^2\right).$$

Now by using the (2.3) and (2.4), we get the required result. □

6. Zalcman functional

One of the main conjectures in Geometric function theory, suggested by Lawrence Zalcman in 1960, is that the coefficients of class $\mathcal{S}$ satisfy the inequality,

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2. \tag{6.1}$$

Only the well-known Koebe function $k(z) = \frac{z}{(1-z)^2}$ and its rotations have equality in the above form. For the popular Fekete-Szego inequality, when $n = 2$, the equality holds. Many researchers have researched Zalcman functional in the literature [5, 8, 19].

Theorem 12. *Let $f \in \mathcal{A}$ belong to $\mathcal{R}_{S_G}$. Then*

$$|a_3^2 - a_5| \leq \frac{1}{10}. \tag{6.2}$$

The result is sharp for the function f_4 defined by (3.7).

Proof. We use the Eqs (3.13) and (3.15) to get the Zalcman functional, and then we get

$$\left|a_3^2 - a_5\right| = \frac{1}{20}\left|\frac{37}{288}c_1^4 - \frac{119}{144}c_1^2c_2 + c_3c_1 + \frac{23}{36}c_2^2 - c_4\right|.$$

Using Lemma 4, we can get the necessary result for the last expression. □

7. Krushkal inequality for the class $\mathcal{R}_{S_G}$

In this section we will give a direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \leq 2^{p(n-1)} - n^p$$

over the class $\mathcal{R}_{S_G}$ for the choice of $n = 4$, $p = 1$, and for $n = 5$, $p = 1$. Krushkal introduced and proved this inequality for the whole class of univalent functions in [17].

Theorem 13. *Let $f \in \mathcal{A}$ belong to $\mathcal{R}_{S_G}$. Then*

$$\left|a_4 - a_2^3\right| \leq \frac{1}{8}.$$

The result is sharp for the function f_3 defined by (3.7).

Proof. From Eqs (3.12) and (3.14), we get

$$\left|a_4 - a_2^3\right| = \left|\frac{19}{1536}c_1^3 - \frac{1}{16}c_2c_1 + \frac{1}{16}c_3\right|.$$

By applying (2.6) to the above,we get the required result. □

Theorem 14. *Let $f \in \mathcal{A}$ belong to $\mathcal{R}_{S_G}$. Then*

$$\left|a_5 - a_2^4\right| \leq \frac{1}{10}.$$

The result is sharp for the function f_4 defined by (3.7).

Proof. From Eqs (3.12) and (3.14), we get

$$\left|a_5 - a_2^4\right| = \frac{1}{20}\left|\frac{101}{1024}c_1^4 - \frac{11}{16}c_1^2c_2 + c_3c_1 + \frac{1}{2}c_2^2 - c_4\right|.$$

By using Lemma 4, we can get the necessary result for the last expression. □

8. Conclusions

In the present study, we have defined the class of bounded turning functions associated with modified sigmoid function. Also we have determined the sharp results for some coefficient functionals which play a very important role in the study of the geometric function theory. Furthermore, we have evaluated bounds of the third and fourth-order Hankel determinants for the 2-fold and 3-fold symmetric functions.

Recently, the usages of the quantum (or q-) calculus happens to provide another popular direction for researches in geometric function theory of complex analysis. This is evidenced by the recently-published survey-cum-expository review article by Srivastava [31]. Therefore the quantum (or q-) extensions of the results, which we have presented in this paper, are worthy of investigation.

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Conflict of interest

The authors declare that they have no competing interests.

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