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Research article

Jordan matrix algebras defined by generators and relations

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Abstract: In the present paper we describe Jordan matrix algebras over a field by generators and relations. We prove that the minimun number of generators of some special Jordan matrix algebras over a field is 2.

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1. Introduction

Let F be a field of characteristic not 2. Let A be an F-algebra. By $A^{(+)}$ we denote the Jordan algebra induced by A in the usual manner:

$$a \circ b = \frac{1}{2}(ab + ba).$$

We call $A^{(+)}$ the *special Jordan algebra*. There exist Jordan algebras that are not special, they are called *exceptional*. Let S be a subset of $A^{(+)}$. By $< S >$ we denote the subalgebra of $A^{(+)}$ generated by S. For more detailed introduction of Jordan algebras we refer the reader a book of Jacobson [7].

Let X be a nonempty set. The free algebra on X over F will be denoted by $F(X)$. Let $F(X)^{(+)}$ be the Jordan algebra induced by $F(X)$ in the usual manner. Let R be a nonempty subset of $F(X)^{(+)}$. By (R) we denote the ideal of $F(X)^{(+)}$ generated by R. This forms the Jordan factor algebra $F(X)^{(+)}/(R)$. By the universal property of $F(X)$, a function $f : X \to A$ can be extended to an algebra homomorphism $\bar{f}$ from $F(X)$ into A. It is clear that $\bar{f}$ induces an algebra homomorphism from $F(X)^{(+)}$ into $A^{(+)}$. Suppose that $\bar{f}(R) = 0$. We note that there exists an algebra homomorphism $\hat{f}$ from $F(X)^{(+)}/(R)$ into $A^{(+)}$.

Set $X = \{\xi_i \mid i \in I\}$ and $R = \{f_j = f_j(\xi_{i_1}, \ldots, \xi_{i_{n(j)}}) \mid j \in J\}$. Note that every element in R is a Jordan polynomial. For example,

$$2\xi_1 + \xi_2 \circ \xi_3 - \xi_3^4.$$

Denote the coset $\xi_i + (R)$ in $F(X)^{(+)}/(R)$ by x_i. Note that

$$f_j(x_{i_1}, \ldots, x_{i_{n(j)}}) = 0$$

for every $j \in J$. Following the case of algebras in [1, Section 6.2], we write $F(X)^{(+)}/(R)$ as $L(X;R)^{(+)}$. We say that this Jordan algebra is defined by the *generators* x_i and *relations* f_j. We always hope that the number of generators of $L(X,R)^{(+)}$ is the minimum. For more detailed introduction of generators and relations of algebras we refer the reader to [1, Section 6.2].

As we know, both matrix algebras and Jordan matrix algebras are important algebras that we often come across. We remand the reader to the papers [5–13] for a general theory of matrix Jordan algebras and the papers [2–4] that focus on polynomial identities of Jordan matrix algebras. It is easy to check that the minimun number of generators of matrix algebras over a field is 2 (see Proposition 2.1). However, it is not easy to determine the minimun number of generators of Jordan matrix algebras over a field.

In the present paper we shall describe Jordan matrix algebras over a field by generators and relations (see Theorem 2.1). We prove that the minimun number of generators of some special Jordan matrix algebras over a field is 2.

2. Jordan matrix algebras defined by generators and relations

Let $n \geq 2$ be an integer. By $M_n(F)$ we denote the algebra of all $n \times n$ matrices over F. By e_{ij} we denote the standard matrix unit of $M_n(F)$. By δ_{ij} we denote the symbol of Kronecker delta.

We begin with the following simple result.

Theorem 2.1. *Let F be a field. Set*

$$X = \{\xi, \eta, \rho\}.$$

Set

$$\xi_{11} = \xi, \quad \xi_{12} = 2\xi_{11} \circ \eta.$$

We set inductively

$$\xi_{1,i+1} = 2\xi_{1i} \circ \eta$$

for all $i = 2, \ldots, n-1$. Similarly, we set inductively

$$\xi_{i+1,1} = 2\xi_{i1} \circ \rho$$

for all $i = 1, \ldots, n-1$. Furthemore, we set

$$\xi_{ij} = 2\xi_{i1} \circ \xi_{1j} - \delta_{ij}\xi_{11}$$

for all $i, j = 2, \ldots, n$. Let R be the following subset of $L(X)^{(+)}$:

$$\xi - \xi_{11}; \ \ \eta - \sum_{i=1}^{n-1} \xi_{i,i+1}; \ \ \rho - \sum_{i=1}^{n-1} \xi_{i+1,i};$$

$$2\xi_{ij} \circ \xi_{st} - \delta_{js}\xi_{it} - \delta_{it}\xi_{sj}, \quad i, j, s, t = 1, \ldots, n;$$

$$\sum_{i=1}^{n} \xi_{ii} - 1.$$

We have that $L(X,R)^{(+)} \cong M_n(F)^{(+)}$.

Proof. Set

$$x = \xi + (R);$$
$$y = \eta + (R);$$
$$z = \rho + (R);$$
$$x_{ij} = \xi_{ij} + (R), \quad i, j = 1, \ldots, n.$$

It follows from the elements in R that

$$x = x_{11}; \ \ y = \sum_{i=1}^{n-1} x_{i,i+1}; \ \ z = \sum_{i=1}^{n-1} x_{i+1,i};$$
$$2x_{ij} \circ x_{st} = \delta_{js}x_{it} + \delta_{it}x_{sj}, i, j, s, t = 1, \ldots, n;$$
$$\sum_{i=1}^{n} x_{ii} = 1.$$

It follows from the relation above that every element of $L(X; R)^{(+)}$ is a linear combination of the following set

$$T = \{x_{ij} \mid i, j = 1, 2, \ldots, n\}.$$

We claim that T is an independent subset of $L(X, R)^{(+)}$. Suppose that

$$\sum_{i,j=1}^{n} \lambda_{ij}x_{ij} = 0 \tag{2.1}$$

for some $\lambda_{ij} \in F$. We define a function $f : X \to M_n(F)^{(+)}$ as follows:

$$f(\xi) = e_{11}, \quad f(\eta) = \sum_{i=1}^{n-1} e_{i,i+1}, \quad f(\rho) = \sum_{j=1}^{n-1} e_{j+1,j}.$$

By the universal property of $F(X)$ we have that there exists an algebra homomorphism $\bar{f} : F(X)^{(+)} \to M_n(F)^{(+)}$ such that

$$\bar{f}(\xi) = e_{11}, \quad \bar{f}(\eta) = \sum_{i=1}^{n-1} e_{i,i+1}, \quad \bar{f}(\rho) = \sum_{j=1}^{n-1} e_{j+1,j}$$

and

$$\bar{f}(\xi_{ij}) = 2\bar{f}(\xi_{i1} \circ \xi_{1j}) - \delta_{ij}\bar{f}(\xi_{11} = 2\bar{f}(\xi_{i1}) \circ \bar{f}(\xi_{1j}) - \delta_{ij}\bar{f}(\xi_{11}) = 2e_{i1} \circ e_{1j} - \delta_{ij}e_{11} = e_{ij}$$

for all $i, j = 1, \ldots, n$. Note that $\{e_{ij} \mid i, j = 1, \ldots, n\}$ is a basis of $M_n(F)^{(+)}$. This implies that $\bar{f}$ is surjective. It is easy to check that $\bar{f}(R) = 0$. Hence there exists a surjective algebra homomorphism $\hat{f} : L(X; R)^{(+)} \to M_n(F)^{(+)}$ such that

$$\hat{f}(x) = e_{11}, \quad \hat{f}(y) = \sum_{i=1}^{n-1} e_{i,i+1}, \quad \hat{f}(z) = \sum_{j=1}^{n-1} e_{j+1,j}$$

and

$$\hat{f}(x_{ij}) = e_{ij}$$

for all $i, j = 1, \ldots, n$. We get from (2.1) that

$$\sum_{i=1}^{n} \lambda_{ij} e_{ij} = 0.$$

It implies that $\lambda_{ij} = 0$ for all $i, j = 1, \ldots, n$. Consequently, T is a basis of $L(X, R)^{(+)}$. In view of the above relations we get that the Jordan operation table of T is the same as that of $\{e_{ij} \mid i, j = 1, \ldots, n\}$, a standard basis of $M_n(F)^{(+)}$. Therefore $L(X, R)^{(+)} \cong M_n(F)^{(+)}$. The proof of the result is complete. □

We remark that Theorem 2.1 implies the following result:

Corollary 2.1. *Let F be a field. We have that*

$$M_n(F)^{(+)} = \left\langle e_{11}, \sum_{i=1}^{n-1} e_{i,i+1}, \sum_{j=1}^{n-1} e_{j+1,j} \right\rangle.$$

In view of Corollary 2.1 we see that every Jordan matrix algebra over a field can be generated by three elements.

We remark that the minimun number of generators of matrix algebras over a field is 2. For example, we easily prove the following result. We give its proof for completeness.

Proposition 2.1. *Let F be a field. We have that*

$$M_n(F) = \left\langle \sum_{i=1}^{n-1} e_{i,i+1}, \sum_{j=1}^{n-1} e_{j+1,j} \right\rangle.$$

Proof. Set

$$E = \sum_{i=1}^{n-1} e_{i,i+1}, \quad Q = \sum_{j=1}^{n-1} e_{j+1,j}.$$

We have that

$$E^{n-1}Q^{n-1} = e_{11} \in < E, Q > .$$

We get that $< e_{11}, E, Q > = < E, Q >$. In view of Corollary 2.1 we note that $M_n(F)^{(+)} = < e_{11}, E, Q >$. This implies that

$$M_n(F) = < e_{11}, E, Q > = < E, Q > .$$

This proves the result. □

We now give the main result of the paper, which shows that some special Jordan matrix algebras over a field can be generated by two elements.

Theorem 2.2. *Let F be a field. Let $n \geq 3$ be an integer. Suppose that $char(F) = 0$ or $char(F) > [\frac{3(n-1)}{2}]$, the integer part of $\frac{3(n-1)}{2}$. Set*

$$X = \{\xi, \eta\}.$$

We set

$$\begin{aligned} a_i &= i, \quad i = 1, \ldots, n-2; \\ a_{n-1} &= [\frac{3(n-1)}{2}]; \\ a_n &= n-1 \end{aligned}$$

and

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} a_1, a_1^2, \ldots, a_1^n \\ a_2, a_2^2, \ldots, a_2^n \\ \vdots \\ a_n, a_n^2, \ldots, a_n^n \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.2}$$

Set

$$\xi_{11} = \lambda_1 \xi \circ \eta + \lambda_2 (\xi \circ \eta)^2 + \cdots + \lambda_n (\xi \circ \eta)^n.$$

We set

$$\xi_{12} = \xi_{11} \circ \xi.$$

Furthermore, we set inductively

$$\xi_{1,i+1} = [\frac{i+1}{2}]^{-1} \xi_{1i} \circ \xi$$

for all $i = 1, \ldots, n-2$ and

$$\xi_{1n} = (n-1)^{-1} \xi_{1,n-1} \circ \xi.$$

Similarly, we set

$$\xi_{21} = 2\xi_{11} \circ \eta.$$

We set inductively

$$\xi_{i+1,1} = 2\xi_{i1} \circ \eta$$

for all $i = 1, \ldots, n-1$. Moreover, we set

$$\xi_{ij} = 2(\xi_{i1} \circ \xi_{1j}) - \delta_{ij} \xi_{11}$$

for all $i, j = 2, \ldots, n$. Let R be the following subset of $L(X)^{(+)}$:

$$\begin{aligned} &\xi - \sum_{i=1}^{n-2} 2[\frac{i+1}{2}] \xi_{i,i+1} - 2(n-1)\xi_{n-1,n}; \\ &\eta - \sum_{i=1}^{n-1} \xi_{i+1,i}; \\ &2\xi_{ij} \circ \xi_{st} - \delta_{js} \xi_{it} - \delta_{it} \xi_{sj}, \quad i, j, s, t = 1, \ldots, n; \\ &\sum_{i=1}^{n} \xi_{ii} - 1. \end{aligned}$$

We have that $L(X, R)^{(+)} \cong M_n(F)^{(+)}$. Moreover, the minimun number of generators of $M_n(F)^{(+)}$ is 2.

Proof. Set

$$x = \xi + (R);$$
$$y = \eta + (R);$$
$$x_{ij} = \xi_{ij} + (R), \quad i, j = 1, \dots, n.$$

It follows from the elements in R that

$$x = \sum_{i=1}^{n-2} 2[\frac{i+1}{2}]x_{i,i+1} + 2(n-1)x_{n-1,n};$$
$$y = \sum_{i=1}^{n-1} x_{i+1,i};$$
$$2x_{ij} \circ x_{st} = \delta_{js}x_{it} + \delta_{it}x_{sj}, i, j, s, t = 1, \dots, n;$$
$$\sum_{i=1}^{n} x_{ii} = 1.$$

It follows from the relation above that every element of $L(X;R)^{(+)}$ is a linear combination of the following set

$$T = \{x_{ij} \mid i, j = 1, 2, \dots, n\}.$$

We claim that T is an independent subset of $L(X,R)^{(+)}$. Suppose that

$$\sum_{i,j=1}^{n} \lambda_{ij}x_{ij} = 0 \tag{2.3}$$

for some $\lambda_{ij} \in F$. We define a function $f : X \to M_n(F)^{(+)}$ as follows:

$$f(\xi) = 2\left(\sum_{i=1}^{n-2}[\frac{i+1}{2}]e_{i,i+1} + (n-1)e_{n-1,n}\right), \quad f(\eta) = \sum_{j=1}^{n-1} e_{j+1,j}.$$

By the universal property of $F(X)$ we have that there exists an algebra homomorphism $\bar{f} : F(X)^{(+)} \to M_n(F)^{(+)}$ such that

$$\bar{f}(\xi) = 2\left(\sum_{i=1}^{n-2}[\frac{i+1}{2}]e_{i,i+1} + (n-1)e_{n-1,n}\right), \quad \bar{f}(\eta) = \sum_{j=1}^{n-1} e_{j+1,j}$$

and

$$\begin{aligned}
\bar{f}(\xi \circ \eta) &= \bar{f}(\xi) \circ \bar{f}(\eta) \\
&= 2\left(\sum_{i=1}^{n-2}[\frac{i+1}{2}]e_{i,i+1} + (n-1)e_{n-1,n}\right) \circ \left(\sum_{j=1}^{n-1} e_{j+1,j}\right) \\
&= \sum_{i=1}^{n-1} ie_{ii} + [\frac{3(n-1)}{2}]e_{n-1,n-1} + (n-1)e_{nn} \\
&= \sum_{i=1}^{n} a_ie_{ii}.
\end{aligned}$$

We get from (2.2) that

$$\begin{aligned}\bar{f}(\xi_{11}) &= \bar{f}(\lambda_1\xi \circ \eta + \lambda_2(\xi \circ \eta)^2 + \cdots + \lambda_n(\xi \circ \eta)^n) \\ &= \lambda_1 \bar{f}(\xi \circ \eta) + \lambda_2 \bar{f}(\xi \circ \eta)^2 + \cdots + \lambda_n \bar{f}(\xi \circ \eta)^n \\ &= \lambda_1 \left(\sum_{i=1}^{n} a_i e_{ii}\right) + \lambda_2 \left(\sum_{i=1}^{n} a_i e_{ii}\right)^2 \\ &\quad + \cdots + \lambda_n \left(\sum_{i=1}^{n} a_i e_{ii}\right)^n \\ &= (\lambda_1 a_1 + \lambda_2 a_1^2 + \cdots + \lambda_n a_1^n) e_{11} + \cdots + \\ &\quad (\lambda_1 a_n + \lambda_2 a_n^2 + \cdots + \lambda_n a_n^n) e_{nn} \\ &= e_{11}.\end{aligned}$$

We have that

$$\begin{aligned}\bar{f}(\xi_{12}) &= \bar{f}(\xi_{11} \circ \xi) \\ &= \bar{f}(\xi_{11}) \circ \bar{f}(\xi) \\ &= e_{11} \circ \left(\sum_{i=1}^{n-2} 2[\frac{i+1}{2}] e_{i,i+1} + 2(n-1) e_{n-1,n}\right) \\ &= e_{12}\end{aligned}$$

and

$$\bar{f}(\xi_{1,i+1}) = [\frac{i+1}{2}]^{-1} \bar{f}(\xi_{1i} \circ \xi) = e_{1,i+1}$$

for all $i = 1, \ldots, n-2$. Moreover, we have that

$$\bar{f}(\xi_{1n}) = (n-1)^{-1} \bar{f}(\xi_{1,n-1} \circ \xi) = e_{1n}.$$

Similarly, we have that

$$\bar{f}(\xi_{i1}) = e_{i1}$$

for all $i = 2, \ldots, n-1$. Moreover, we have that

$$\bar{f}(\xi_{ij}) = 2\bar{f}(\xi_{i1} \circ \xi_{1j}) - \delta_{ij} \bar{f}(\xi_{11}) = 2e_{i1} \circ e_{1j} - \delta_{ij} e_{11} = e_{ij}$$

for all $i, j = 2, \ldots, n$. Note that $\{e_{ij} \mid i, j = 1, \ldots, n\}$ is a basis of $M_n(F)^{(+)}$. This implies that $\bar{f}$ is surjective. It is easy to check that $\bar{f}(R) = 0$. Hence there exists a surjective algebra homomorphism $\hat{f} : L(X; R)^{(+)} \to M_n(F)^{(+)}$ such that

$$\hat{f}(x) = \sum_{i=1}^{n-2} 2[\frac{i+1}{2}] e_{i,i+1} + 2(n-1) e_{n-1,n}, \quad \hat{f}(y) = \sum_{j=1}^{n-1} e_{j+1,j}.$$

Moreover, we have that

$$\hat{f}(x_{1j}) = e_{1j}, \quad \hat{f}(x_{i1}) = e_{i1}$$

for all $i, j = 1, \ldots, n$ and so

$$\hat{f}(x_{ij}) = e_{ij}$$

for all $i, j = 1, \ldots, n$. It follows from (2.3) that

$$\sum_{i,j=1}^{n} \lambda_{ij} e_{ij} = 0.$$

This implies that $\lambda_{ij} = 0$ for all $i, j = 1, \ldots, n$. Consequently, T is a basis of $L(X, R)^{(+)}$. In view of the above relations we see that the Jordan operation table of T is the same as that of $\{e_{ij} \mid i, j = 1, 2, \ldots, n\}$, a standard basis of $M_n(F)^{(+)}$. Therefore $L(X, R)^{(+)} \cong M_n(F)^{(+)}$.

Suppose that $M_n(F)^{(+)} =< A >$ for some $A \in M_n(F)^{(+)}$. It is clear that $M_n(F) =< A >$. This implies that $M_n(F)$ is commutative, a contradiction. We get that the minimun number of generators of $M_n(F)^{(+)}$ is 2. The proof of the result is complete. □

We remark that Theorem 2.2 implies the following result:

Corollary 2.2. *Let F be a field. Let $n \geq 3$ be an integer. Suppose that $char(F) = 0$ or $char(F) > [\frac{3(n-1)}{2}]$. We have that*

$$M_n(F)^{(+)} = \left\langle \sum_{i=1}^{n-2} 2[\frac{i+1}{2}] e_{i,i+1} + 2(n-1)e_{n-1,n}, \sum_{j=1}^{n-1} e_{j+1,j} \right\rangle.$$

3. Conclusions

The main conclusion of the paper is to show that the minimun number of generators of $M_n(F)^{(+)}$ (where $n \geq 3$ and $char(F) = 0$ or $char(F) > [\frac{3(n-1)}{2}]$) is 2.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. M. Brešar, *Introduction to nocommutative algebras*, Springer, 2014. doi: 10.1007/978-3-319-08693-4.
2. L. Centrone, F. Martino, A note on cocharacter sequence of Jordan upper triangular matrix algebra, *Commun. Algebra*, **45** (2017), 1687–1695. doi: 10.1080/00927872.2016.1222408.
3. L. Centrone, F. Martino, M. da Silva Souza, Specht property for some varieties of Jordan algebras of almost polynomial growth, *J. Algebra*, **521** (2019), 137–165. doi: 10.1016/j.jalgebra.2018.11.017.

4. A. Cirrito, F. Martino, Ordinary and graded cocharacter of the Jordan algebra of 2×2 upper triangular matrices, *Linear Algebra Appl.*, **451** (2004), 246–259. doi: 10.1016/j.laa.2014.03.011.
5. C. Costara, Nonlinear invertibility preserving maps on matrix algebras, *Linear Algebra Appl.*, **602** (2020), 216–222. doi: 10.1016/j.laa.2020.05.010.
6. M. Grašič, Zero product determined Jordan algebra, *Linear Multilinear Algebra*, **59** (2011), 671–685. doi: 10.1080/03081087.2010.485199.
7. N. Jacobson, *Structure and representations of Jordan algebras*, Colloquium Publications, Vol. 39, American Mathematical Society, 1968.
8. M. Kochetov, F. Y. Yasumura, Group grading on the Lie and Jordan algebras of block-triangular matrices, *J. Algebra*, **537** (2019), 147–172. doi: 10.1016/j.jalgebra.2019.07.020.
9. X. B. Ma, G. H. Ding, L. Wang, Square-zero determined matrix algebras, *Linear Multilinear Algebra*, **59** (2011), 1311–1317. doi: 10.1080/03081087.2010.539606.
10. K. McCrimmon, *A taste of Jordan algebras*, Springer, 2004. doi: 10.1007/b97489.
11. H. P. Pertersson, The isotopy problem for Jordan matrix algebras, *Trans. Amer. Math. Soc.*, **244** (1978), 185–197. doi: 10.1090/S0002-9947-1978-0506615-7.
12. C. C. Xi, J. B. Zhang, Structure of centralizer matrix algebras, *Linear Algebra Appl.*, **622** (2021), 215–249. doi: 10.1016/j.laa.2021.03.034.
13. D. Y. Wang, Q. Hu C. G. Xia, Jordan derivation of certain Jordan matrix algebras, *Linear Multilinear Algebra*, **56** (2008), 581–588. doi: 10.1080/03081080701492165.