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Research article

On generalized fractional integral operator associated with generalized Bessel-Maitland function

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Abstract: In this paper, we describe generalized fractional integral operator and its inverse with generalized Bessel-Maitland function (BMF-V) as its kernel. We discuss its convergence, boundedness, its relation with other well known fractional operators (Saigo fractional integral operator , Riemann-Liouville fractional operator), and establish its integral transform. Moreover, we have given the relationship of BMF-V with Mittag-Leffler functions.

Keywords: extended Bessel-Maitland function; integral transform; Riemann-Liouville fractional integral operator
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1. Introduction

In recent years, many authors developed the class of integral formula involving a variety of special functions [1–10]. Fractional integral operator having special functions as their kernel is of great use in many research fields [11–14]. One of the most valuable special function is Bessel function [15–24]. The mathematician Daniel Bernoulli first introduced the Bessel function, which was later generalized by the German astronomer Friedrich Whilhelm Bessel. The theory of Bessel function correlate with linear differential equation which is further extended by the researchers due to its wide range of applications. For more details, reader may consult the work of Watson [25].

The Bessel-Maitland function (BMF-I) defined by the following series representation [26], as

$$\mathrm{J}_{\nu}^{\mu}(s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!\Gamma(\mu n + \nu + 1)} = \phi(\mu, \nu + 1; -s). \tag{1.1}$$

The generalization of Bessel-Maitland function (BMF-II) introduced by Singh et al. [27] as

$$\mathrm{J}_{\nu,q}^{\mu,\gamma}(s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-s)^n}{\Gamma(\mu n + \nu + 1)n!}, \tag{1.2}$$

where $\mu, \nu, \gamma \in \mathbb{C}$, $\Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0$ and $q \in (0, 1) \bigcup \mathbb{N}$.

The extended Bessel-Maitland function (BMF-III) investigated by Ghayasuddin and Khan [28], defined as

$$\mathrm{J}_{\nu,\gamma,\delta}^{\mu,q,p}(s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-s)^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}}, \tag{1.3}$$

where $\mu, \nu, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0, \Re(\nu) > -1, \Re(\gamma) > 0, \Re(\delta) \geq 0$; $p, q > 0$ and $q < \Re(\alpha) + p$.

The generalization of generalized Bessel-Maitland function (BMF-IV) is introduced by Ali [29]

$$\mathrm{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(s) = \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n}(\gamma)_{\sigma n}(-s)^n}{\Gamma(\mu n + \nu + 1)(\rho)_{mn}}, \tag{1.4}$$

where $\mu, \nu, \eta, \rho, \gamma \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$; $\xi, m, \sigma \geq 0$ and $m, \xi > \Re(\mu) + \sigma$.

Notation used in generalized (BMF-IV) is defined as

$${}_{\mu,\nu}^{\gamma,\sigma}\mathcal{Q}_{\rho,m;n}^{\eta,\xi} = \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n}(\gamma)_{\sigma n}(-1)^n}{\Gamma(\mu n + \nu + 1)(\rho)_{mn}}, \tag{1.5}$$

A new extension of Bessel-Maitland function (BMF-V) is introduced and investigated by Khan et al. [30] as

$$\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(s) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-s)^n}{\Gamma(\alpha n + \beta + 1)(\nu)_{\sigma n}(\delta)_{pn}}, \tag{1.6}$$

where $\alpha, \beta, \mu, \rho, \nu, \gamma, \sigma, \delta \in \mathbb{C}, \Re(\beta) > 0$, $\Re(\rho) > 0, \Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\alpha) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\sigma) > 0$; $p, q > 0$ and $q < \Re(\alpha) + p$.

Saigo fractional integral operators are defined by Saigo [31], for $s > 0$, $a, c, d \in \mathbb{C}$, and $\Re(a) > 0$

$$\begin{aligned}(\mathcal{F}_{0+}^{a,c,d}g)(s) &= \frac{s^{-a-c}}{\Gamma(a)} \int_0^s (s-\tau)^{a-1} \\ &\times {}_2F_1(a+c, -d; a; (1-\frac{\tau}{s}))g(\tau)d\tau,\end{aligned} \tag{1.7}$$

and

$$(\mathcal{F}_{0-}^{a,c,d}g)(s) = \frac{1}{\Gamma(a)} \int_s^{\infty} (\tau - s)^{a-1}\tau^{-a-c}$$

$$\times {}_2F_1(a+c,-d;a;(1-\frac{s}{\tau}))g(\tau)d\tau. \tag{1.8}$$

Samko et al. [32] defined the Riemann-Liouville fractional operators for $\Re(b) > 0$ and $n = [Re(b)] + 1$ as

$$(\mathcal{F}^b_{0+}g)(s) = \frac{1}{\Gamma(b)} \int_0^s (s-\tau)^{b-1} g(\tau)d\tau, \tag{1.9}$$

and

$$(\mathcal{D}^b_{0+}\phi)(s) = \frac{d^n}{ds^n}(\mathcal{F}^{n-b}_{0+}\phi)(s). \tag{1.10}$$

The Gauss Hypergeometric function can be considered as infinite series defined by Saigo [31], denoted by ${}_2F_1(a,-b;c;s)$ for all $a,b,c \in \mathbb{C}$ where a,b,c are parameters and s is a variable, $c \neq 0$ and $|s| < 1$.

$${}_2F_1(a,-b;c;s) = \sum_{n=0}^{\infty} \frac{(a)_n(-b)_n s^n}{(c)_n n!}, \tag{1.11}$$

where $(a)_n, (-b)_n, (c)_n$ are the Pochhammer's symbols.

The Pochhammer's symbols defined by Petojevic [33],

$$(z)_n = \begin{cases} z(z+1)(z+2)\cdots(z+n-1) & \text{for } n \geq 1 \\ 1 & \text{for } n = 0, z \neq 0, \end{cases} \tag{1.12}$$

where $z \in \mathbb{C}$ and $n \in \mathbb{N}$ and in gamma form it can be write as

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \tag{1.13}$$

The beta function is defined as [33,34] , for $\Re(x) > 0, \Re(y) > 0$ and also expressed in gamma form respectively

$$\beta(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du, \tag{1.14}$$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.15}$$

The gamma function is defined [33,34] for $\Re(x) > 0$ as

$$\Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u}du. \tag{1.16}$$

The generalized hypergeometric function is defined by Rainville [35]

$${}_kR_r(p_1,\cdots,p_k,q_1,\cdots,q_r;s) = \sum_{n=0}^{\infty} \frac{(p_1)_n,\cdots,(p_k)_n}{(q_1)_n,\cdots,(q_r)_n}\frac{s^n}{n!}, \tag{1.17}$$

where $p_i, q_j \in \mathbb{C}$, $q_j \neq 0,-1,\cdots,(i=-1,-2,\cdots,k; j=-1,-2,\cdots,r)$.

The generalized Fox-Wright function [36] is defined as follows,

$${}_r\psi_s(\tau) = {}_r\psi_s\left[\begin{array}{c} (p_i,q_i)_{1,r} \\ (x_j,y_j)_{1,s} \end{array}\middle|\tau\right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}\Gamma(p_i+q_in)}{\prod_{j=1}^{s}\Gamma(x_j+y_jn)}\frac{\tau^n}{n!}, \tag{1.18}$$

where $\tau \in \mathbb{C}$, $p_i, x_j \in \mathbb{C}$ and $q_j, y_j \in \mathbb{R}$ $(i = 1, 2, \cdots, r; j = 1, 2, \cdots, s)$.

The Gauss hypergeometric function in gamma form can be written as

$$ {}_2R_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0. \tag{1.19}$$

The Laplace transform of function $f(z)$ is defined as [37]

$$\mathcal{L}[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)dt. \tag{1.20}$$

Dirichlet formula (Fubini's theorem) [32] is given by

$$\int_d^c dx \int_x^c u(x,t)dt = \int_d^c dt \int_d^t u(x,t)dx. \tag{1.21}$$

General form of Mittag-Leffler function [38] defined for $\Re(\acute{\delta}) > 0, \Re(\acute{\alpha}) > 0, \Re(\acute{\beta}) > 0$ as follows:

$$\mathfrak{E}^{\acute{\delta}}_{\acute{\alpha},\acute{\beta}}(z) = \sum_{n=0}^{\infty} \frac{(\acute{\delta})_n z^n}{n!\Gamma(\acute{\alpha}n + \acute{\beta})}. \tag{1.22}$$

2. Relations between Bessel-Maitland and Mittag-Leffler functions

In this section, we discuss generalized Bessel-Maitland function, and establish its relations with generalized Mittag-Leffler functions:

- On setting $p = 0$ in Eq (1.6), we get the following relation:

$$\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,0}(s) = \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma}(s), \tag{2.1}$$

  where $\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma}(s)$ is BMF-IV investigated by Ali in [29].
- on setting $\mu = \nu = \sigma = \rho = 1$ in Eq (1.6), we obtain the relation:

$$\mathrm{J}^{1,1,\gamma,q}_{\alpha,\beta,1,1,\delta,p}(s) = \mathrm{J}^{\alpha,q,p}_{\beta,\gamma,\delta}(s), \tag{2.2}$$

  where $\mathrm{J}^{\alpha,q,p}_{\alpha,\gamma,\delta,p}(s)$ is BMF-III introduced and investigated by Ghayasuddin and Khan [28].
- On replacing $\mu = \nu = \sigma = \rho = \delta = p = 1$ in Eq (1.6), we obtain the relation:

$$\mathrm{J}^{1,1,\gamma,q}_{\alpha,\beta,1,1,1,1}(s) = \mathrm{J}^{\alpha,q}_{\beta,\gamma}(s). \tag{2.3}$$

  where $\mathrm{J}^{\alpha,q}_{\beta,\gamma}(s)$ is BMF-II defined the Singh et al. [27].
- On setting $\mu = \nu = \sigma = \delta = \rho = p = q = 1$ in Eq (1.6).

$$\mathrm{J}^{1,1,1,1}_{\alpha,\beta,1,1,1,1}(s) = \mathrm{J}^{\alpha}_{\beta}(s) \tag{2.4}$$

  where $\mathrm{J}^{\alpha}_{\beta}(s)$ is BMF-I in [26].
- On replacing α by $\alpha - 1$ in Eq (1.6), we get the following interesting relation:

$$\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha-1,\beta,\nu,\sigma,\delta,p}(-s) = \mathrm{E}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(s), \tag{2.5}$$

  where $(\mathrm{E}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p})(s)$ is the Mittag-Leffler function introduced by Khan and Ahmad [39].

- On setting $\mu = \nu = \sigma = \rho = 1$ and replace α by $\alpha - 1$ in Eq (1.6), we get

$$\mathrm{J}^{1,1,\gamma,q}_{\alpha-1,\beta,1,1,\delta,p}(-s) = \mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(s), \tag{2.6}$$

  where $\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}$ is the Mittag-Leffler function defined by Salim and Faraz [40].
- On setting $\mu = \nu = \sigma = \rho = \delta = p = 1$ and replacing α by $\alpha - 1$ in Eq (1.6), we get

$$\mathrm{J}^{1,1,\gamma,q}_{\alpha-1,\beta,1,1,1,1}(-s) = \mathrm{E}^{\gamma,q}_{\alpha,\beta}(s) \tag{2.7}$$

  where $\mathrm{E}^{\gamma,q}_{\alpha,\beta}$ is the Mittag-Leffler function defined by Shukla and Prajapati [41].
- On setting $\mu = \nu = \sigma = \rho = \delta = p = q = 1$ and replacing α by $\alpha - 1$ in Eq (1.6)

$$(\mathrm{J}^{1,1,\gamma,1}_{\alpha-1,\beta,1,1,1,1})(-s) = \mathrm{E}^{\gamma}_{\alpha,\beta}(s), \tag{2.8}$$

  where $\mathrm{E}^{\gamma}_{\alpha,\beta}(s)$ is the Mittag-leffler function defined by Prabhakar [38]
- On setting $\mu = \nu = \sigma = \rho = \delta = \gamma = p = q = 1$ and replacing α by $\alpha - 1$ in Eq (1.6)

$$(\mathrm{J}^{1,1,1,1}_{\alpha-1,\beta,1,1,1,1})(-s) = \mathrm{E}_{\alpha,\beta}(s) \tag{2.9}$$

  where $\mathrm{E}_{\alpha,\beta}(s)$ is the Mittag-Leffler function defined by Wiman [42].
- On setting $\mu = \nu = \sigma = \rho = \delta = \gamma = p = q = 1, \alpha = 0$ and replacing α by $\alpha - 1$in Eq (1.6) and we obtain

$$\mathrm{J}^{1,1,1,1}_{0,\beta,1,1,1,1}(-s) = \mathrm{E}_{\beta}(s) \tag{2.10}$$

  where $\mathrm{E}_{\beta}(s)$ is the Magnus Gösta Mittag-Leffler function [43].

3. Generalized fractional integral operator

In this section, we discuss generalized fractional integral operator and its special cases.

Definition 3.1. *The generalized fractional integral operator involving the generalized Bessel-Maitland function (BMF-V) as its kernel is defined for* $\alpha, \beta, \mu, \rho, \nu, \gamma, \sigma, \delta, w \in \mathbb{C}, \Re(\beta) > 0$, $\Re(\rho) > 0, \Re(\mu) > 0$, $\Re(w) > 0$, $\Re(\nu) > 0$, $\Re(\alpha) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\sigma) > 0$; $p, q > 0$ *and* $q < \Re(\alpha) + p$.

$$(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi)(s) = \int_a^s (s-\tau)^{\alpha} \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}[w(s-\tau)^{\beta}]\phi(\tau)d\tau, \tag{3.1}$$

Remark 3.1. *On setting* $w = 0$ *and replacing* α *by* $\alpha - 1$, *it will become a left sided Riemann-Liouville fractional integral operator.*

Remark 3.2. *If we put* $p = 0$ *in* (3.1), *we obtain the generalized fractional integral operator defined by Ali et al. [29].*

Remark 3.3. *If we put* $\mu = \nu = \sigma = \rho = \delta = p = q = 1$ *in* (3.1), *we obtain the Srivastava fractional integral operator defined in [44].*

Definition 3.2. *We define the following notation which use in our results as*

$$^{\mu,\rho}_{\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n} = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^n}{\Gamma(\alpha n + \beta + 1)(\nu)_{\sigma n}(\delta)_{pn}}. \tag{3.2}$$

Definition 3.3. *The left inverse operator of integral operator* (3.1) *for* $\alpha, \beta, \mu, \rho, \nu, \gamma, \sigma, \delta, w \in \mathbb{C}, \Re(\beta) > 0$, $\Re(\rho) > 0, \Re(\mu) > 0$, $\Re(w) > 0$, $\Re(\nu) > 0$, $\Re(\alpha) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\sigma) > 0$; $p, q > 0$ *and* $q < \Re(\alpha) + p$ *and* $n = [\alpha]$ *as* $n - \alpha > 0$ *is defined as follows:*

$$\begin{aligned}(\mathrm{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi)(s) &= \frac{d^n}{ds^n}(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\beta,\alpha-n,\nu,\sigma,\delta,p,w,a^+}\phi)(s)\\ &= \frac{d^n}{ds^n}\int_a^s (s-\tau)^{n-\alpha}\mathrm{J}^{\mu,\rho,\gamma,q}_{\beta,n-\alpha,\nu,\sigma,\delta,p}[w(s-\tau)^{\beta}]\phi(\tau)d\tau. \end{aligned} \tag{3.3}$$

Remark 3.4. *On setting* $w = 0$ *and replacing* α *by* $\alpha + 1$, *then* (3.3) *becomes the Riemann-Liouville fractional differential operator. i.e*

$$\begin{aligned} &= \frac{d^n}{ds^n}\int_a^s (s-\tau)^{n-\alpha-1}\frac{1}{\Gamma(n-\alpha)}\phi(\tau)d\tau\\ &= \frac{d^n}{ds^n}\mathcal{F}^{n-\alpha}_{a+}\phi(\tau)d\tau. \end{aligned}$$

4. Convergence and boundedness of generalized fractional integral operator

In this section, we discuss the convergence and boundedness of generalized fractional integral operator involving BMF-V as its kernel in the form of theorem.

Theorem 4.1. *Let the operator* $(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi)(s)$ *is defined on* $L(a, c)$ *with* $\mu, \rho, \gamma, q, \alpha, \beta, \nu, \sigma, \delta, p, w \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$, $\Re(\sigma) > 0$, $\Re(\delta) > 0$, $\Re(\gamma) > 0$ $\Re(w) > 0$, $\Re(\rho) > 0, p, q > 0$ *and* $q < \Re(\alpha) + p$, *then*

$$||\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi||_c \leq B||\phi||_c, \tag{4.1}$$

where

$$B = (c-a)^{Re(\alpha)}\sum_{n=0}^{\infty}\frac{|(\mu)_{\rho n}||(\gamma)_{qn}|}{|(\delta)_{pn}||(\nu)_{\sigma n}|}\frac{|(-w(c-a)^{Re(\beta)})^n|}{|\Gamma(\beta n+\alpha+1)||Re(\beta)n+Re(\alpha)+1|}. \tag{4.2}$$

Proof. Let K_n be denote the nth term of (4.2), then

$$\begin{aligned}\left|\frac{K_{n+1}}{K_n}\right| &= \left|\frac{(\mu)_{\rho n+\rho}}{(\mu)_{\rho n}}\right|\left|\frac{(\gamma)_{qn+q}}{(\gamma)_{qn}}\right|\left|\frac{(\delta)_{pn}}{(\delta)_{pn+p}}\right|\left|\frac{(\nu)_{\sigma n}}{(\nu)_{\sigma n+\sigma}}\right|\left|\frac{\Gamma(\beta n+\alpha+1)}{\Gamma(\beta n+\beta+\alpha+1)}\right|\\ &\quad\left|\frac{Re(\beta)n+Re(\alpha)+1}{Re(\beta)(n+1)+Re(\alpha)+1}\right|\left|\frac{(-1)^{n+1}}{(-1)^n}\right||w(c-a)^{Re(\beta)}|\\ &\approx \frac{(\rho n)^{\rho}(qn)^{q}|w(c-a)^{Re(\beta)}|}{(pn)^{p}(\sigma n)^{\sigma})|n+1||(|\beta|n)^{Re(\beta)}|} \qquad as\ n\to\infty.\end{aligned}$$

Hence $|\frac{K_{n+1}}{K_n}| \to 0$ as $n \to \infty$ and $q < \Re(\alpha) + p$ which means that the right hand side of (4.2) is convergent and finite under the given condition. The condition of boundedness of the integral operator $(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi)(x)$ is discussed in space of Lebesgue measurable [45] $L(a, c)$ of continuous function on (a, c) where $c > a$,

$$L(a,c) = \{g(x) : ||g||_c = \int_a^c |g(x)|dx < \infty\}. \tag{4.3}$$

According to Eqs (1.6) and (3.2), we have

$$\|\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi\|_c = \int_a^c \left| \int_a^s (s-t)^\alpha |\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}[w(s-t)^\beta]|\phi(t)dt \right| ds$$
$$\leq \int_a^c \left[\int_t^c (s-t)^\alpha |\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}[w(s-t)^\beta]|ds \right] |\phi(t)|dt. \tag{4.4}$$

By putting these values $s-t=u \Rightarrow ds=du$, $s=c \Rightarrow u=c-t$ and $s=t \Rightarrow u=0$ in Eq (4.4), we have

$$\|\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi\|_c = \int_a^c \left[\int_0^{c-t} u^{Re(\alpha)} \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}[wu^\beta]du \right] |\phi(t)|dt$$
$$\leq \int_a^c \left[\int_0^{c-a} u^{Re(\alpha)} |\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(wu^\beta)|du \right] |\phi(t)|dt$$
$$B = \int_0^{c-a} u^{Re(\alpha)} |\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(wu^\beta)|du. \tag{4.5}$$

Let

$$B = \sum_{n=0}^{\infty} \frac{|(\mu)_{\rho n}||(\gamma)_{qn}||(-w)^n|}{|(\nu)_{\sigma n}||(\delta)_{pn}||\Gamma(\beta n+\alpha+1)} \int_0^{c-a} u^{Re(\beta)n+Re(\alpha)}du$$
$$= \sum_{n=0}^{\infty} \frac{(c-a)^{Re(\alpha)+1}|(\mu)_{\rho n}||(\gamma)_{qn}||(-w(c-a)^{Re(\beta)})^n|}{|(\nu)_{\sigma n}||(\delta)_{pn}||\Gamma(\beta n+\alpha+1)||Re(\beta)n+Re(\alpha)+1|}.$$

Hence

$$\|\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi\|_c \leq \int_a^c B|\phi(t)|dt \leq B\|\phi\|_c.$$
$$\Rightarrow \|\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}\phi\|_c \leq B\|\phi\|_c \quad .$$

□

5. Generalized Bessel-Maitland function with Saigo integral operators and Riemann-Liouville integral operators

In this section, we discuss the behavior of generalized fractional operators (Saigo and Riemann-Liouville) with BMF-V.

Theorem 5.1. *Let* $a, c, d, \alpha, \beta, \mu, \rho, \nu, \gamma, \sigma, \delta \in \mathbb{C}$ *with* $\Re(a) > 0$, $\rho > max[0, \Re(c-d)]$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\nu) > 0$, $\Re(\sigma) > 0$, $\Re(\delta) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > 0$, $\Re(w) > 0$, $p, q > 0$ *and* $q < \Re(\alpha) + p$ *, then the following relation holds,*

$$\mathcal{F}^{a,c,d}_{0+}[\tau^\rho \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(\tau^\delta)](s)$$
$$= \frac{s^{\rho-c}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} \times {}_5\psi_5 \left[\begin{matrix} (\mu,\rho)(\gamma,q)(\rho+1,\delta)(\rho+d-c+1,\delta)(1,1) \\ (\alpha+1,\beta)(\nu,\sigma)(\delta,p)(\rho-c+1,\delta)(a+\rho+d+1,\delta) \end{matrix} \middle| -s^\delta \right]. \tag{5.1}$$

Proof. Consider the left-sided Saigo fractional integral operator (1.7) in which using the power function with BMF-V (1.6), we get

$$\mathcal{F}^{a,c,d}_{0+}[\tau^\rho \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(\tau^\delta)](s) = \mathcal{F}^{a,c,d}_{0+}[\tau^\rho \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(\tau^\delta)](s).$$

$$=\mathcal{F}_{0+}^{a,c,d}[\tau^{\rho}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-\tau^{\delta})^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}d\tau](s).$$

$$=\mathcal{F}_{0+}^{a,c,d}[\tau^{\rho+\delta n}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}d\tau](s).$$

$$=\mathcal{F}_{0+}^{a,c,d}\tau^{\rho+\delta n}(s)\Big[\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\Big]d\tau. \tag{5.2}$$

$$I_{0,\lambda}^{\alpha,\beta,\gamma}z^{\lambda}=\frac{\Gamma(\lambda+1)\Gamma(\lambda-\beta+\gamma+1)}{\Gamma(\lambda-\beta+1)\Gamma(\lambda+\alpha+\gamma+1)}z^{\lambda-\beta}. \tag{5.3}$$

Using the above relation in (5.2).

$$\mathcal{F}_{0+}^{a,c,d}[\tau^{\rho}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\tau^{\delta})](s)$$
$$=\frac{\Gamma(\rho+\delta n+1)\Gamma(\rho+\delta n-c+d+1)}{\Gamma(\rho+\delta n-c+1)\Gamma(\rho+\delta n+a+d+1)}s^{\rho+\delta n-c}\Big[\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\Big]d\tau.$$
$$=\frac{\Gamma(\rho+\delta n+1)\Gamma(\rho+\delta n-c+d+1)s^{\rho-c}}{\Gamma(\rho+\delta n-c+1)\Gamma(\rho+\delta n+a+d+1)}\Big[\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-s^{\delta})^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\Big]d\tau. \tag{5.4}$$

By using Eq (1.13) in Eq (5.4), we get

$$\mathcal{F}_{0+}^{a,c,d}[\tau^{\rho}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\tau^{\delta})](s)$$
$$=\sum_{n=0}^{\infty}\frac{s^{\rho-c}\Gamma(\nu)\Gamma(\delta)\Gamma(\mu+\rho n)\Gamma(\gamma+qn)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\beta n+\alpha+1)}\frac{\Gamma(\rho+\delta n+1)\Gamma(\rho+\delta n-c+d+1)(-s^{\delta})^{n}}{\Gamma(\rho+\delta n-c+1)\Gamma(\rho+\delta n+a+d+1)}.$$

Hence, we attain require result

$$(\mathcal{F}_{0+}^{a,c,d}[\tau^{\rho}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\tau^{\delta})])(s)$$
$$=\frac{s^{\rho-c}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}\times{}_{5}\psi_{5}\left[\begin{array}{c}(\mu,\rho)(\gamma,q)(\rho+1,\delta)(\rho+d-c+1,\delta)(1,1)\\(\alpha+1,\beta)(\nu,\sigma)(\delta,p)(\rho-c+1,\delta)(a+\rho+d+1,\delta)\end{array}\Bigg|-s^{\delta}\right]. \tag{5.5}$$

□

Theorem 5.2. *Let* $a,c,d,\alpha,\beta,\mu,\rho,\nu,\gamma,\sigma,\delta\in\mathbb{C}$ *with* $\Re(a)>0$, $\rho>max[0,\Re(c-d)]$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\sigma)>0$, $\Re(\delta)>0$, $\Re(\gamma)>0$, $\Re(w)>0$, $\Re(\rho)>0$, $p,q>0$ *and* $q<\Re(\alpha)+p$ *, then the following relation holds,*

$$\mathcal{F}_{0-}^{a,c,d}[\tau^{-\rho}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\tau^{-\delta})](s)$$
$$=\frac{s^{-c-\rho}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}\times{}_{5}\psi_{5}\left[\begin{array}{c}(\mu,\rho)(\gamma,q)(d+\rho,\delta)(c+\rho,\delta)(1,1)\\(\alpha+1,\beta)(\nu,\sigma)(\delta,p)(\rho,\delta)(a+c+d+\rho,\delta)\end{array}\Bigg|-s^{-\delta}\right]. \tag{5.6}$$

Proof. Consider the right-sided Sagio fractional integral operator (1.8) in which using the power function with BMF-V (1.6), then we get

$$\mathcal{F}_{0-}^{a,c,d}[\tau^{-\rho}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\tau^{-\delta})](s)=\mathcal{F}_{0-}^{a,c,d}[\tau^{-\rho}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(\tau^{-\delta})](s).$$

$$=\mathcal{F}^{a,c,d}_{0-}[\tau^{-\rho}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-\tau^{-\delta})^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}d\tau](s).$$
$$=\mathcal{F}^{a,c,d}_{0-}[\tau^{-\rho-\delta n}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}d\tau](s).$$
$$=\mathcal{F}^{a,c,d}_{0-}\tau^{-\rho-\delta n}(s)\Big[\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\Big]d\tau. \tag{5.7}$$

By using the important relation defined in [46] in Eq (5.7), we have

$$\mathcal{F}^{a,c,d}_{0-}[\tau^{-\rho}\mathbf{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(\tau^{-\delta})](s)$$
$$=\frac{\Gamma(c+\rho+\delta n)\Gamma(d+\rho-\delta n)}{\Gamma(\rho+\delta n)\Gamma(a+c+d+\rho+\delta n)}s^{-\rho-\delta n-c}\Big[\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-1)^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\Big]d\tau.$$
$$=\frac{\Gamma(c+\rho+\delta n)\Gamma(d+\rho+\delta n)s^{-\rho-c}}{\Gamma(\rho+\delta n)\Gamma(a+c+d+\rho+\delta n)}\Big[\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-s^{-\delta})^{n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\Big]d\tau. \tag{5.8}$$

By using Eq (1.13) in Eq (5.8), we have result

$$\mathcal{F}^{a,c,d}_{0-}[\tau^{-\rho}\mathbf{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(\tau^{-\delta})](s)$$
$$=\sum_{n=0}^{\infty}\frac{s^{-c-\rho}\Gamma(\mu+\rho n)}{\Gamma(\beta n+\alpha+1)}\frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\gamma+qn)\Gamma(c+\rho+\delta n)\Gamma(d+\rho+\delta n)(-s^{-\delta})^{n}}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\nu+\sigma n)\Gamma(\rho+\delta n)\Gamma(a+c+d+\rho+\delta n)}$$
$$=\frac{s^{-c-\rho}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}\times{}_5\psi_5\left[\begin{array}{c}(\mu,\rho)(\gamma,q)(d+\rho,\delta)(c+\rho,\delta)(1,1)\\(\alpha+1,\beta)(\nu,\sigma)(\delta,p)(\rho,\delta)(a+c+d+\rho,\delta)\end{array}\Bigg|-s^{-\delta}\right]. \tag{5.9}$$

□

Theorem 5.3. *Let $\lambda,\mu,\rho,\gamma,\alpha,\beta,\nu,\sigma,\delta,w\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\nu)>0$, $\Re(\sigma)>0$, $\Re(\delta)>0$, $\Re(\rho)>0$, $\Re(\lambda)$, $\Re(\gamma)>0$, $\Re(w)>0$, $p,q>0$ and $q<\Re(\alpha)+p$, then the following relation holds,*

$$\mathcal{F}^{\lambda}_{a+}\Big[\frac{\mathbf{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(\tau-a)^{\beta})}{(\tau-a)^{-\alpha}}\Big](s-a)=\frac{\mathbf{J}^{\mu,\rho,\gamma,q}_{\alpha+\lambda,\beta,\nu,\sigma,\delta,p}(w(s-a)^{\beta})}{(s-a)^{-\lambda-\alpha}}. \tag{5.10}$$

Proof. Consider the left-sided Riemann-Liouville fractional integral operator (1.9) in which using the power function with BMF-V (1.6), we get

$$\mathcal{F}^{\lambda}_{a+}[(\tau-a)^{\alpha}\mathbf{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(\tau-a)^{\beta})](s-a)$$
$$=[{}^{\mu,\rho}_{\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}]\int_{a}^{s}\frac{(\tau-a)^{\alpha}(w(\tau-a)^{\beta})^{n}}{\Gamma(\lambda)(s-\tau)^{-\lambda+1}}d\tau$$
$$=\frac{[{}^{\mu,\rho}_{\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}]}{\Gamma(\lambda)(w)^{-n}}\int_{a}^{s}(s-\tau)^{\lambda-1}(\tau-a)^{\alpha+\beta n}d\tau\ . \tag{5.11}$$

By putting these values $\frac{\tau-a}{s-a}=y\Rightarrow d\tau=(s-a)dy$, $\tau=s\Rightarrow y=1$ and $\tau=a\Rightarrow y=0$ in Eq (5.11), we obtain

$$\mathcal{F}_{a+}^{\lambda}[(\tau-a)^{\alpha}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(w(\tau-a)^{\beta})](s-a)$$
$$=\frac{[{}_{\alpha,\beta}^{\mu,\rho}Q_{\nu,\sigma,\delta,p;n}^{\gamma,q}](w)^{n}}{\Gamma(\lambda)}\int_{0}^{1}\frac{(s-(s-a)y-a)^{\lambda-1}}{((s-a)y)^{-\beta n-\alpha}(s-a)^{-1}}dy$$
$$=\frac{(s-a)^{\lambda+\alpha}}{(w)^{-n}\Gamma(\lambda)}[{}_{\alpha,\beta}^{\mu,\rho}Q_{\nu,\sigma,\delta,p;n}^{\gamma,p}(s-a)^{\beta n}]\int_{0}^{1}(1-y)^{\lambda-1}y^{\beta n+\alpha}dy. \tag{5.12}$$

By using Eqs (1.14) and (1.15) in Eq (5.12), we have

$$\mathcal{F}_{a+}^{\lambda}[(\tau-a)^{\alpha}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(w(\tau-a)^{\beta})](s-a)$$
$$=(s-a)^{\lambda+\alpha}[{}_{\alpha,\beta}^{\mu,\rho}Q_{\nu,\sigma,\delta,p;n}^{\gamma,q}(w)^{n}(s-a)^{\beta n}]\frac{\Gamma(\beta n+\alpha+1)}{\Gamma(\lambda+\beta n+\alpha+1)}. \tag{5.13}$$

Now, by using Eq (3.2) in Eq (5.13), then obtain the required result

$$\mathcal{F}_{a+}^{\lambda}[(\tau-a)^{\alpha}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(w(\tau-a)^{\beta})](s-a)$$
$$=(s-a)^{\lambda+\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-w(s-a)^{\beta})^{n}}{\Gamma(\lambda+\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}$$
$$=(s-a)^{\lambda+\alpha}\mathrm{J}_{\alpha+\lambda,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(w(s-a)^{\beta}). \tag{5.14}$$

□

6. Composition of Riemann-Liouville fractional operators with new operator

In this section, we discuss the Riemann-Liouville fractional integral and differential operators with fractional integral operator, and results can be seen some other generalized fractional integral operator, in the form of theorems.

Theorem 6.1. *Let* $\lambda,\alpha,\beta,\mu,\rho,\gamma,\nu,\sigma,\delta,w\in\mathbb{C},\Re(\lambda)>0,\Re(w)>0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\nu)>0,\Re(\sigma)>0,\Re(\delta)>0,\Re(\gamma)>0,\Re(\rho)>0,p,q>0$ *and* $q<\Re(\alpha)+p$ *, then the following relation holds:*

$$(\mathcal{F}_{0^{+}}^{\lambda}\mathcal{Z}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^{+}}^{\mu,\rho,\gamma,q}\phi)(s)=(\mathcal{Z}_{\alpha+\lambda,\beta,\nu,\sigma,\delta,p,w,0^{+}}^{\mu,\rho,\gamma,q}\phi)(s). \tag{6.1}$$

Proof. Consider the left sided Riemann-Liouville integral operator (1.9) involving new fractional integral operator (3.1), as

$$(\mathcal{F}_{0^{+}}^{\lambda}\mathcal{Z}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^{+}}^{\mu,\rho,\gamma,q}\phi)(s)$$
$$=\frac{1}{\Gamma(\lambda)}\int_{0}^{s}(s-y)^{\lambda-1}\int_{0}^{y}(y-\tau)^{\alpha}\mathrm{J}_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(w(y-\tau)^{\beta}))\phi(\tau)\,d\tau dy. \tag{6.2}$$

By using Eq (1.21) in Eq (6.2), we have

$$(\mathcal{F}_{0^{+}}^{\lambda}\mathcal{Z}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^{+}}^{\mu,\rho,\gamma,q}\phi)(s)$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^s \int_\tau^s (s-y)^{\lambda-1} (y-\tau)^{\alpha} \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(y-\tau)^{\beta}) dy\, \phi(\tau)\, d\tau. \tag{6.3}$$

By putting these values $t = y - \tau \Rightarrow dt = dy$, $y = s \Rightarrow t = s - \tau$ and $y = \tau \Rightarrow t = 0$ in Eq (6.3), we get

$$\begin{aligned}
&(\mathcal{F}^{\lambda}_{0^+} \mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s) \\
&= \frac{1}{\Gamma(\lambda)} \int_0^s \int_0^{s-\tau} (t)^{\alpha} \frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(t)^{\beta})}{(s-\tau-t)^{1-\lambda}} dt\, \phi(\tau)\, d\tau \\
&= \int_0^s \frac{1}{\Gamma(\lambda)} \int_0^{s-\tau} (t)^{\alpha} \frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(t)^{\beta})}{(s-\tau-t)^{1-\lambda}} dt\, \phi(\tau)\, d\tau.
\end{aligned} \tag{6.4}$$

By using Eq (1.9) in Eq (6.4), we have

$$(\mathcal{F}^{\lambda}_{0^+} \mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s) = \int_0^s [\mathcal{F}^{\lambda}_{0^+}(t)^{\alpha} \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(t)^{\beta})](s-\tau)\phi(\tau) d\tau. \tag{6.5}$$

By using Eq (5.11) in Eq (6.5), we obtain

$$\begin{aligned}
(\mathcal{F}^{\lambda}_{0^+} \mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s) &= \int_0^s \frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha+\lambda,\beta,\nu,\sigma,\delta,p}(w(s-\tau)^{\beta})}{(s-\tau)^{-\alpha-\lambda}} \phi(\tau) d\tau \\
&= (\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha+\lambda,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s).
\end{aligned} \tag{6.6}$$

□

Theorem 6.2. *Let* $\lambda, \mu, \rho, \gamma, \alpha, \beta, \nu, \sigma, \delta, w \in \mathbb{C}, \Re(\lambda) > 0, \Re(w) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\nu) > 0, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(\gamma) > 0, \Re(\rho) > 0, p, q > 0$ *and* $q < \Re(\alpha) + p$ *, then the following relation holds:*

$$(\mathcal{D}^{\lambda}_{0^+} \mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s) = (\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha-\lambda,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s).$$

Proof. Consider the left sided Riemann-Liouville differential operator (1.10) involving new fractional integral operator (3.1), then

$$\begin{aligned}
&(\mathcal{D}^{\lambda}_{0^+} \mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s) \\
&= \frac{1}{\Gamma(m-\lambda)} (\frac{d}{ds})^m \int_0^s (s-y)^{m-\lambda-1} \int_0^y \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(y-\tau)^{\beta})(y-\tau)^{\alpha}\phi(\tau)\, d\tau\, dy.
\end{aligned} \tag{6.7}$$

By using Eq (1.21) in Eq (6.7), we have

$$\begin{aligned}
&(\mathcal{D}^{\lambda}_{0^+} \mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+,}\phi)(s) \\
&= \frac{1}{\Gamma(m-\lambda)} (\frac{d}{ds})^m \int_0^s \int_\tau^s (y-\tau)^{\alpha} \frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(y-\tau)^{\beta})}{(s-y)^{\lambda-m+1}} dy\, \phi(\tau)\, d\tau.
\end{aligned} \tag{6.8}$$

By putting these values $t = y - \tau \Rightarrow dt = dy$, $y = s \Rightarrow t = s - \tau$ and $y = \tau \Rightarrow t = 0$ in Eq (6.8), we get

$$
\begin{aligned}
&(\mathcal{D}^{\lambda}_{0^+}\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s)\\
&=\frac{1}{\Gamma(m-\lambda)}(\frac{d}{ds})^m\int_0^s\int_0^{s-\tau}\frac{(t)^{\alpha}\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(t)^{\beta})}{(s-\tau-t)^{-m+\lambda+1}}dt\,\phi(\tau)\,d\tau\\
&=\int_0^s(\frac{d}{ds})^m\frac{1}{\Gamma(m-\lambda)}\int_0^{s-\tau}\frac{(t)^{\alpha}\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(t)^{\beta})}{(s-\tau-t)^{-m+\lambda+1}}dt\,\phi(\tau)\,d\tau.
\end{aligned}
\tag{6.9}
$$

Now, by using the Eq (1.9) in Eq (6.9), we have

$$
(\mathcal{D}^{\lambda}_{0^+}\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s)=\int_0^s(\frac{d}{ds})^m\mathcal{F}^{m-\lambda}_{0^+}[(t)^{\alpha}\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(t)^{\beta})](s-\tau)\phi(\tau)d\tau. \tag{6.10}
$$

By using Eq (5.11) in Eq (6.10), we obtain

$$
(\mathcal{D}^{\lambda}_{0^+}\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s)=\int_0^s(\frac{d}{ds})^m\frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(w(s-\tau)^{\beta})}{(s-\tau)^{\lambda-m-\alpha}}\phi(\tau)d\tau. \tag{6.11}
$$

By using Eq (1.6) in Eq (6.11), and taking one time derivative then

$$
\begin{aligned}
&(\mathcal{D}^{\lambda}_{0^+}\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s)\\
&=\sum_{n=0}^{\infty}\frac{(-w)^n(\beta n+m+\alpha-\lambda)}{\Gamma(\beta n+m+\alpha-\lambda+1)}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{(\nu)_{\sigma n}(\delta)_(pn)}(\frac{d}{ds})^{m-1}\int_0^s(s-\tau)^{-\lambda+\beta n+m+\alpha-1}\phi(\tau)d\tau\\
&=(\frac{d}{ds})^{m-1}\int_0^s\frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha+m-\lambda-1,\beta,\nu,\sigma,\delta,p}(w(s-\tau)^{\beta})}{(s-\tau)^{-m-\alpha+\lambda+1}}\phi(\tau)d\tau.
\end{aligned}
\tag{6.12}
$$

Now, taking the *(m-1)*, derivative of Eq (6.12), then get

$$
\begin{aligned}
(\mathcal{D}^{\lambda}_{0^+}\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s)&=\int_0^s\frac{\mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha-\lambda,\beta,\nu,\sigma,\delta,p}(w(s-\tau)^{\beta})}{(s-\tau)^{\lambda-\alpha}}\phi(\tau)d\tau\\
&=(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha-\lambda,\beta,\nu,\sigma,\delta,p,w,0^+}\phi)(s).
\end{aligned}
\tag{6.13}
$$

□

Theorem 6.3. *Let* $\alpha,\beta,\mu,\rho,\nu,\gamma,\sigma,\delta,w\in\mathbb{C},\Re(w)>0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\nu)>0,\Re(\sigma)>0,\Re(\delta)>0,\Re(\gamma)>0,\Re(\rho)>0,p,q>0$ *and* $q<\Re(\alpha)+p$ *, then the following relation holds:*

$$
\mathcal{L}[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi]=\frac{s^{-\alpha-1}\Gamma(\nu)\Gamma(\delta)\phi(s)}{\Gamma(\mu)\Gamma(\gamma)}\times{}_4\psi_3\left[\begin{array}{c}(\mu,\rho)(\gamma,q)(1,1)\\(\nu,\sigma)(\delta,p)\end{array}\middle|-(\frac{w}{s})^{\beta}\right]. \tag{6.14}
$$

Proof. Consider the new fractional integral operator (3.1), then

$$
\mathcal{L}[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi]=\int_0^{\infty}e^{-st}\Big[\int_0^t(t-y)^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n(t-y)^{\beta n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}}\phi(y)dy\Big]dt. \tag{6.15}
$$

Now, after changing the order of integration, then we obtain

$$\mathcal{L}[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi] = \int_0^\infty \int_y^\infty \frac{(t-y)^\alpha}{e^{st}} \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n(t-y)^{\beta n}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} dt\phi(y)dy$$
$$= \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} \int_0^\infty \int_y^\infty \frac{(t-y)^{\beta n+\alpha}}{e^{st}} dt\phi(y)dy. \quad (6.16)$$

By putting $t - y = \tau$, then

$$\mathcal{L}[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\phi]$$
$$= \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} \int_0^\infty \frac{\phi(y)}{e^{sy}} \int_0^\infty e^{-s\tau}\tau^{\beta n+\alpha} d\tau dy$$
$$= \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} \frac{\Gamma(\beta n+\alpha+1)}{s^{\beta n+\alpha}}\phi(s)$$
$$= \frac{s^{-\alpha-1}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} \sum_{n=0}^\infty \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)(-ws^{-\beta})^n\phi(s)}{\Gamma(\delta+pn)\Gamma(\nu+\sigma n)}$$
$$= \frac{s^{-\alpha-1}\Gamma(\nu)\Gamma(\delta)\phi(s)}{\Gamma(\mu)\Gamma(\gamma)} \times {}_4\psi_3\left[\begin{array}{c} (\mu,\rho)(\gamma,q)(1,1) \\ (\nu,\sigma)(\delta,p) \end{array}\middle| -(\frac{w}{s})^\beta\right]. \quad (6.17)$$

□

Theorem 6.4. *Let $\chi,\mu,\rho,\gamma,\alpha,\beta,\nu,\sigma,\delta,w \in \mathbb{C}, \Re(w) > 0, \Re(\chi) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\nu) > 0, \Re(\sigma) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\rho) > 0, p, q > 0$ and $q < \Re(\alpha) + p$, then the following relation holds:*

$$(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\tau^\chi)(s)$$
$$= \frac{s^{\alpha+\chi}\Gamma(\nu)\Gamma(\delta)\Gamma(\chi+1)}{\Gamma(\mu)\Gamma(\gamma)} \times {}_3\psi_3\left[\begin{array}{c} (\mu,\rho)(\gamma,q)(1,1) \\ (\alpha+\chi+2,\beta)(\nu,\sigma)(\delta,p) \end{array}\middle| -ws^\beta\right]. \quad (6.18)$$

Proof. Consider the new fractional integral operator (3.1),

$$(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\tau^\chi(s)$$
$$= \int_0^s (s-\tau)^\alpha \mathrm{J}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}[w(s-\tau)^\beta]\tau^\chi d\tau$$
$$= \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n s^{\beta n+\alpha}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} \int_0^s (1-\frac{\tau}{s})^{\beta n+\alpha}\tau^\chi d\tau. \quad (6.19)$$

By putting $\frac{\tau}{s} = y$, then we obtain required result

$$(\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}\tau^\chi(s)$$
$$= \sum_{n=0}^\infty \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n s^{\beta n+\alpha}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} \int_0^1 (1-y)^{\beta n+\alpha}(sy)^\chi s dy$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n s^{\beta n+\alpha+\chi}}{\Gamma(\beta n+\alpha+1)(\nu)_{\sigma n}(\delta)_{pn}} \frac{\Gamma(\beta n+\alpha+1)\Gamma(\chi+1)}{\Gamma(\beta n+\alpha+\chi+2)}$$
$$= \frac{s^{\alpha+\chi}\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)(-ws^{\beta})^n\Gamma(\chi+1)}{\Gamma(\beta n+\alpha+\chi+2)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}$$
$$= \frac{s^{\alpha+\chi}\Gamma(\nu)\Gamma(\delta)\Gamma(\chi+1)}{\Gamma(\mu)\Gamma(\gamma)} \times {}_3\psi_3\left[\begin{matrix}(\mu,\rho)(\gamma,q)(1,1)\\(\alpha+\chi+2,\beta)(\nu,\sigma)(\delta,p)\end{matrix}\middle|(-ws)^{\beta}\right].$$

□

7. Applications of inverse operator

In this section, we will discuss some applications of the inverse fractional operator. We derive some results of the inverse fractional operator with the Mittag-leffler function and Bessel-Maitland function.

Theorem 7.1. *Let* $\lambda,\mu,\rho,\gamma,\alpha,\beta,\nu,\sigma,\delta,w \in \mathbb{C}, \Re(\lambda) > 0, \Re(w) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\nu) > 0, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(\gamma) > 0, \Re(\rho) > 0, p, q > 0$ *and* $q < \Re(\alpha) + p$ *, then the following relation holds:*

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau-a)^{\rho-1}\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(\tau-a)^{\lambda}](s)$$
$$= \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+qm)(s-a)^{\lambda m+\rho-\alpha}}{\Gamma(\alpha m+\beta)} \frac{\Gamma(\rho+\lambda m)}{\Gamma(\delta+pm)}$$
$$\times {}_3\psi_3\left[\begin{matrix}(\mu,\rho)(\gamma,q)(1,1)\\(\delta,p)(\nu,\sigma)(\lambda m+\rho-\alpha+1,\beta)\end{matrix}\middle|-w(s-a)^{\beta}\right]. \tag{7.1}$$

Proof. If we consider the new fractional integral operator (3.3),

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau-a)^{\rho-1}\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(\tau-a)^{\lambda}](s)$$
$$= (\frac{d}{ds})^p \int_a^s (s-\tau)^{p-\alpha} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w(s-\tau)^{\beta})^n(\tau-a)^{\rho-1}}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+p-\alpha+1)} \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}(\tau-a)^{\lambda m}}{\Gamma(\alpha m+\beta)(\delta)_{pm}} d\tau$$
$$= (\frac{d^p}{ds^p})^{\mu,\rho}_{p-\alpha,\beta} Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}(w)^n \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}}{\Gamma(\alpha m+\beta)(\delta)_{pm}} \int_a^s (s-\tau)^{p+\beta n-\alpha}(\tau-a)^{\rho+\lambda m-1} d\tau. \tag{7.2}$$

Substituting $y = (\frac{s-\tau}{s-a})$ in Eq (7.2), we get

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau-a)^{\rho-1}\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(\tau-a)^{\lambda}](s)$$
$$= (\frac{d^p}{ds^p})^{\mu,\rho}_{p-\alpha,\beta} Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}(w)^n \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}(s-a)^{p-\alpha+\beta n+\lambda m+\rho}}{\Gamma(\alpha m+\beta)(\delta)_{pm}} \int_0^1 y^{p+\beta n-\alpha}(1-y)^{\rho+\lambda m-1} dy. \tag{7.3}$$

By using Eqs (1.14) and (1.15), we get

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau-a)^{\rho-1}\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(\tau-a)^{\lambda}](s)$$

$$= (\frac{d^p}{ds^p})^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}(w)^n \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}(s-a)^{p-\alpha+\beta n+\lambda m+\rho}}{\Gamma(\alpha m+\beta)(\delta)_{pm}} \frac{\Gamma(\beta n+p-\alpha+1)\Gamma(\rho+\lambda m)}{\Gamma(\beta n+\rho-\alpha+\lambda m+1)} d\tau. \tag{7.4}$$

Now, back substituting ${}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}$ in Eq (7.4), we have

$$\begin{aligned}
&[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau-a)^{\rho-1}\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(\tau-a)^{\lambda}](s) \\
&= \sum_{m,n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n(\frac{d^p}{ds^p})(s-a)^{p-\alpha+\beta n+\lambda m+\rho}}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+\rho+\lambda m+p-\alpha+1)} \frac{(\gamma_{qm})\Gamma(\rho+\lambda m)}{\Gamma(\alpha m+\beta)(\delta)_{pm}} \\
&= \sum_{m,n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n(s-a)^{-\alpha+\beta n+\lambda m+\rho}}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+\rho+\lambda m-\alpha+1)} \frac{(\gamma_{qm})\Gamma(\rho+\lambda m)}{\Gamma(\alpha m+\beta)(\delta)_{pm}} \\
&= \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+qm)(s-a)^{\lambda m+\rho-\alpha}}{\Gamma(\alpha m+\beta)} \frac{\Gamma(\rho+\lambda m)}{\Gamma(\delta+pm)} \\
&\times {}_3\psi_3\left[\begin{matrix}(\mu,\rho)(\gamma,q)(1,1)\\ (\delta,p)(\nu,\sigma)(\lambda m+\rho-\alpha+1,\beta)\end{matrix}\,\middle|-w(s-a)^{\beta}\right].
\end{aligned} \tag{7.5}$$

□

Corollary 7.1. *On setting $\alpha = -\alpha$ in theorem 9, we obtain*

$$\begin{aligned}
&[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau-a)^{\rho-1}\mathrm{E}^{\gamma,\delta,q}_{\alpha,\beta,p}(\tau-a)^{\lambda}](s) \\
&= \frac{\Gamma(\nu)\Gamma(\delta)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+qm)(s-a)^{\lambda m+\rho+\alpha}}{\Gamma(\alpha m+\beta)} \frac{\Gamma(\rho+\lambda m)}{\Gamma(\delta+pm)} \\
&\times {}_3\psi_3\left[\begin{matrix}(\mu,\rho)(\gamma,q)(1,1)\\ (\delta,p)(\nu,\sigma)(\lambda m+p+\alpha+1,\beta)\end{matrix}\,\middle|-w(s-a)^{\beta}\right].
\end{aligned} \tag{7.6}$$

Theorem 7.2. *Let $\lambda,\mu,\rho,\gamma,\alpha,\beta,\nu,\sigma,\delta,w \in \mathbb{C}, \Re(\lambda) > 0, \Re(w) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\nu) > 0, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(\gamma) > 0, \Re(\rho) > 0, p, q > 0$ and $q < \Re(\alpha) + p$, then the following relation holds:*

$$\begin{aligned}
&[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau)^{-1}\mathrm{J}^{\alpha}_{\beta}(\tau)\mathrm{J}^{\alpha,\gamma}_{\beta,q}(\tau)^{-\lambda}](s) \\
&= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\gamma)} \sum_{m,n=0}^{\infty} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)}{\Gamma(\nu+\sigma n)\Gamma(\delta+pn)} \frac{(-ws^{\beta})^n(-s)^m(s)^{-\alpha}}{\Gamma(\alpha m+\beta+1)m!} \\
&\times {}_2\psi_2\left[\begin{matrix}(\gamma,q)(m,-\lambda)\\ (\beta+1,\alpha)(\beta n+m-\alpha+1,-\lambda)\end{matrix}\,\middle|-s^{-\lambda}\right].
\end{aligned} \tag{7.7}$$

Proof. Consider the new fractional integral operator (3.3),

$$\begin{aligned}
&[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{-1}\mathrm{J}^{\alpha,\beta}(\tau)\mathrm{J}^{\alpha,\nu}_{\beta,q}(\tau)^{-\lambda}](s) \\
&= (\frac{d}{ds})^p \int_a^s (s-\tau)^{p-\alpha} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w(s-\tau)^{\beta})^n}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+p-\alpha+1)}
\end{aligned}$$

$$\times \sum_{m=0}^{\infty} \frac{(\tau)^{-1}(-\tau)^m}{m!\Gamma(\alpha m+\beta+1)} \sum_{w=0}^{\infty} \frac{(\gamma)_{qw}(-1)^w(\tau)^{-\lambda w}}{w!\Gamma(\alpha w+\beta+1)} d\tau$$

$$= (\frac{d^p}{ds^p}) {}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}(w)^n \sum_{m,w=0}^{\infty} \frac{(-1)^{m+w}(s)^{p-\alpha+\beta n}}{m!\Gamma(\alpha m+\beta+1)}$$

$$\times \frac{(\gamma)_{qw}(w)^n}{w!\Gamma(\alpha w+\beta+1)} \int_0^s (1-\frac{\tau}{s})^{p-\alpha+\beta n}\tau^{-1+m-\lambda w} d\tau. \tag{7.8}$$

By substituting $y = (\frac{\tau}{s})$ in Eq (7.8)

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau)^{-1}\mathrm{J}^{\alpha}_{\beta}(\tau)\mathrm{J}^{\alpha,\nu}_{\beta,q}(\tau)^{-\lambda}](s)$$

$$= (\frac{d^p}{ds^p}) {}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n} \sum_{m,w=0}^{\infty} \frac{(w)^n(-1)^{w+m}(\gamma)_{qw}(s)^{\beta n}(s)^{m-\lambda w+p-\alpha}}{m!\Gamma(\alpha m+\beta+1)w!\Gamma(\alpha w+\beta+1)} \int_0^1 (1-y)^{p-\alpha+\beta n}y^{-1+m-\lambda w} dy. \tag{7.9}$$

By using Eqs (1.14) and (1.15),we get

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau)^{-1}\mathrm{J}^{\alpha}_{\beta}(\tau)\mathrm{J}^{\alpha,\gamma}_{\beta,q}(\tau)^{-\lambda}](s)$$

$$= {}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n} \sum_{m,w=0}^{\infty} \frac{(w)^n(-1)^{w+m}(\gamma)_{qw}s^{\beta n}s^{p-\alpha+m-\lambda w}\Gamma(p-\alpha+\beta n+1)\Gamma(m-\lambda w)}{m!\Gamma(\alpha m+\beta+1)w!\Gamma(\alpha w+\beta+1)\Gamma(p-\alpha+\beta n+m-\lambda w+1)}. \tag{7.10}$$

Now, back substituting ${}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p;n}$ in Eq (7.10), we have

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,pw,a^+}(\tau)^{-1}\mathrm{J}^{\alpha}_{\beta}(\tau)\mathrm{J}^{\alpha,\gamma}_{\beta,q}(\tau)^{-\lambda}](s)$$

$$= \sum_{m,n,w=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n(\frac{d^p}{ds^p})(s)^{p-\alpha+\beta n+m-\lambda w}}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+p-\alpha+\rho-\lambda w+1)} \frac{(\gamma)_{qw}\Gamma(m-\lambda w)(w)^n(-1)^{m+w}}{m!\Gamma(\alpha m+\beta+1)w!\Gamma(\alpha w+\beta+1)}$$

$$= \sum_{m,n,w=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n(s)^{-\alpha+\beta n+m-\lambda w}}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+m-\alpha-\lambda w+1)} \frac{(\gamma)_{qw}\Gamma(m-\lambda w)(w)^n(-1)^{m+w}}{m!\Gamma(\alpha m+\beta+1)w!\Gamma(\alpha w+\beta+1)}$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\gamma)} \sum_{m,n=0}^{\infty} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)(-ws^{\beta})^n(-s)^m(s)^{-\alpha}}{\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\alpha m+\beta+1)m!}$$

$$\times {}_2\psi_2\left[\begin{array}{c}(\gamma,q)(m,-\lambda)\\(\beta+1,\alpha)(\beta n+m-\alpha+1,-\lambda)\end{array}\middle| -s^{-\lambda}\right].$$

□

Corollary 7.2. *On setting $\alpha = -\alpha$ in theorem 10 ,we obtain the result,*

$$[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,a^+}(\tau)^{-1}\mathrm{J}^{\alpha}_{\beta}(\tau)\mathrm{J}^{\alpha,\gamma}_{\beta,q}(\tau)^{-\lambda}](s)$$

$$= \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)\Gamma(\gamma)} \sum_{m,n=0}^{\infty} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)(-ws^{\beta})^n(-s)^m(s)^{\alpha}}{\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\alpha m+\beta+1)m}$$

$$\times {}_2\psi_2\left[\begin{array}{c}(\gamma,q)(m,-\lambda)\\(\beta+1,\alpha)(\beta n+m+\alpha+1,-\lambda)\end{array}\middle| -s^{-\lambda}\right]. \tag{7.11}$$

Theorem 7.3. *Let $\lambda, \eta, \mu, \rho, \gamma, \alpha, \beta, \nu, \sigma, \delta, w \in \mathbb{C}, \Re(\lambda) > 0, \Re(w) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\nu) > 0, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(\gamma) > 0, \Re(\eta) > 0, \Re(\rho) > 0, p, q > 0$ and $q < \Re(\alpha) + p$, then the following relation holds:*

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{\frac{\alpha}{\beta}-1}{}_2R_1(\frac{\alpha}{\beta}+u,-\eta:\frac{\alpha}{\beta}:\tau)](s)$$
$$= \sum_{m,n=0}^{\infty} \frac{((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m}{m!(-\alpha+\beta n+(\frac{\alpha}{\beta})+1)_m} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)s^{-\alpha+m+(\frac{\alpha}{\beta})}}{\Gamma(-\alpha+\beta n+(\frac{\alpha}{\beta})+1)\Gamma(\delta+pn)} \frac{(-ws^{\beta})^n\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}. \tag{7.12}$$

Proof. Consider fractional integral operator (3.1) with Gauss hypergeometric function, then the following results hold:

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{\frac{\alpha}{\beta}-1}{}_2R_1(\frac{\alpha}{\beta}+u,-\eta:\frac{\alpha}{\beta}:\tau)](s)$$
$$= (\frac{d^p}{ds^p})\int_0^s (s-\tau)^{p-\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-w(s-\tau)^{\beta})^n}{(\nu)_{\sigma n}(\delta)_{pn}\Gamma(\beta n+p-\alpha+1)}\tau^{(\frac{\alpha}{\beta})-1}\sum_{m=0}^{\infty}\frac{((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m}{(\frac{\alpha}{\beta})_m m!}(\tau)^m d\tau$$
$$= (\frac{d^p}{ds^p}){}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p}(w)^n\sum_{m=0}^{\infty}\frac{((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m s^{p-\alpha+\beta n}}{(\frac{\alpha}{\beta})_m m!}\int_0^s(\frac{1-\tau}{s})^{p-\alpha+\beta n}\tau^{m+(\frac{\alpha}{\beta})-1}d\tau. \tag{7.13}$$

putting $y=(\frac{\tau}{s})$ in Eq (7.13), we get

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{\frac{\alpha}{\beta}-1}{}_2R_1(\frac{\alpha}{\beta}+u,-\eta:\frac{\alpha}{\beta}:\tau)](s)$$
$$= (\frac{d^p}{ds^p}){}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p}(w)^n\sum_{m=0}^{\infty}\frac{((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m s^{p-\alpha+\beta n}}{(\frac{\alpha}{\beta})_m m! s^{-m-(\frac{\alpha}{\beta})}}\int_0^1(1-y)^{p-\alpha+\beta n}y^{m+(\frac{\alpha}{\beta})-1}dy. \tag{7.14}$$

Now using Eqs (1.14) and (1.15) , we have

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{\frac{\alpha}{\beta}-1}{}_2R_1(\frac{\alpha}{\beta}+u,-\eta:\frac{\alpha}{\beta}:\tau)](s)$$
$$= (\frac{d^p}{ds^p}){}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p}(w)^n\sum_{m=0}^{\infty}\frac{((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m s^{p-\alpha+\beta n}}{(\frac{\alpha}{\beta})_m m! s^{-m-(\frac{\alpha}{\beta})}}\frac{\Gamma(p-\alpha+\beta n+1)\Gamma(m+(\frac{\alpha}{\beta}))}{\Gamma(p-\alpha+\beta n+m+(\frac{\alpha}{\beta})+1)}. \tag{7.15}$$

By substituting ${}^{\mu,\rho}_{p-\alpha,\beta}Q^{\gamma,q}_{\nu,\sigma,\delta,p}$ in Eq (7.15) , we get the required result:

$$[\mathcal{D}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{\frac{\alpha}{\beta}-1}{}_2R_1(\frac{\alpha}{\beta}+u,-\eta:\frac{\alpha}{\beta}:\tau)](s)$$
$$= \sum_{m,n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}(-w)^n((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m\Gamma(m+(\frac{\alpha}{\beta}))}{(\nu)_{\sigma n}(\delta)_{pn}(\frac{\alpha}{\beta})_m m!\Gamma(p-\alpha+\beta n+m+(\frac{\alpha}{\beta})+1)}\frac{d^p}{ds^p}s^{p-\alpha+\beta n+m+(\frac{\alpha}{\beta})}$$
$$= \sum_{m,n=0}^{\infty}\frac{((\frac{\alpha}{\beta})+\mu)_m(-\eta)_m}{m!(-\alpha+\beta n+(\frac{\alpha}{\beta})+1)_m}\frac{(-ws^{\beta})^n\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}\frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)}{\Gamma(\nu+\sigma n)\Gamma(\delta+pn)}\frac{s^{-\alpha+m+(\frac{\alpha}{\beta})}(-ws^{\beta})^n}{\Gamma(-\alpha+\beta n+(\frac{\alpha}{\beta})+1)}. \tag{7.16}$$

Corollary 7.3. *By putting $\alpha=-\alpha$ in Theorem 11, we get*

$$[\mathcal{Z}^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p,w,0^+}(\tau)^{\frac{\alpha}{\beta}-1}{}_2R_1(\frac{\alpha}{\beta}+u,-\eta:\frac{\alpha}{\beta}:\tau)](s)$$

$$= \sum_{n,m=0}^{\infty} \frac{((\frac{\alpha}{\beta}) + \mu)_m(-\eta)_m}{m!(\alpha + \beta n + (\frac{\alpha}{\beta}) + 1)_m} \frac{(-ws^{\beta})^n\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} \frac{\gamma(\mu + \rho n)\Gamma(\gamma + qn)}{\Gamma(\nu + \sigma n)\Gamma(\delta + pn)} \frac{s^{\alpha+m+(\frac{\alpha}{\beta})}(-ws^{\beta})^n}{\Gamma(\alpha + \beta n + (\frac{\alpha}{\beta}) + 1)}. \quad (7.17)$$

□

8. Conclusions

We have described new generalized fractional integral operator and its inverse operator with generalized Bessel-Maitland function (BMF-V) as its kernel. We examined the behavior of new operator's with some fractional operators (Riemann-Liouville and Saigo). We discussed the generalized fractional integral operator with some other special functions.

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Conflict of interest

The authors declare that they have no competing interests.

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