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Research article

Some new Caputo fractional derivative inequalities for exponentially $(\theta, h-m)$–convex functions

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Abstract: Firstly, we obtain some inequalities of Hadamard type for exponentially $(\theta, h-m)$–convex functions via Caputo k–fractional derivatives. Secondly, using integral identity including the $(n+1)$–order derivative of a given function via Caputo k-fractional derivatives we prove some of its related results. Some new results are given and known results are recaptured as special cases from our results.

Keywords: Hermite–Hadamard inequality; Caputo k-fractional derivatives; Hölder's inequality; exponentially $(\theta, h-m)$–convex functions
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction and preliminaries

Convex functions has applications in almost all branches of mathematics for example in mathematical analysis, optimization theory and mathematical statistics etc. A convex function can be expressed and visualized in many different ways which further provide the motivation and encouragement for defining new concepts. Let us recall its definition as follows:

A function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y), \tag{1.1}$$

holds for all $x, y \in J$ and $t \in [0, 1]$. Likewise f is concave if $(-f)$ is convex.

A convex function is generalized in different forms, one of the generalization is the exponentially $(\theta, h{-}m)$-convex function. Farid and Mahreen [11] introduced exponentially $(\theta, h{-}m)$-convex functions as follows:

Definition 1.1. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. For fix $t \in (0, 1)$, $(\theta, m) \in (0, 1]^2$ and $\eta \in \mathbb{R}$. A function $f : [0, b] \to \mathbb{R}$ is called exponentially $(\theta, h - m)$-convex function, if f is non-negative and for all $x, y \in [0, b]$ one has

$$f(tx + m(1 - t)y) \leq h(t^\theta)\frac{f(x)}{e^{\eta x}} + mh(1 - t^\theta)\frac{f(y)}{e^{\eta y}}. \tag{1.2}$$

Remark 1.2. By selecting suitable function h and particular values of parameter m and η, the above definition produces the functions comprise in the following remark:

(i) By setting $\eta = 0$, $(\theta, h - m)$-convex function [12] can be obtained.
(ii) By taking $\eta = 0$ and $\theta = 1$, $(h - m)$-convex function can be captured.
(iii) By choosing $\eta = 0$ and $h(t^\theta) = t^\theta$, (θ, m)-convex function can be obtained.
(iv) By setting $\eta = 0$, $\theta = 1$ and $m = 1$, h-convex function [32] can be captured.
(v) By taking $\eta = 0$, $\theta = 1$ and $h(t) = t$, m-convex function [31] can be obtained.
(vi) By choosing $\eta = 0$, $\theta = 1$, $m = 1$ and $h(t) = t$, convex function can be captured.
(vii) By setting $\eta = 0$, $m = 1$, $\theta = 1$ and $h(t) = 1$, P-function [6] can be obtained.
(viii) By taking $\theta = 1$ and $h(t) = t^s$, exponentially (s, m)-convex function [28] can be captured.
(ix) By choosing $\theta = 1$, $m = 1$ and $h(t) = t^s$, exponentially s-convex function [23] can be obtained.
(x) By setting $\theta = 1$, and $h(t) = t$, exponentially m-convex function [29] can be captured.
(xi) By taking $\theta = 1$, $m = 1$ and $h(t) = t$, exponentially convex function [3] can be obtained.
(xii) By choosing $\eta = 0$, $\theta = 1$ and $h(t) = t^s$, (s, m)-convex function [2] can be captured.
(xiii) By setting $\theta = 1$, $\eta = 0$, $m = 1$ and $h(t) = t^s$, s-convex function [23] can be obtained.
(xiv) By taking $\theta = 1$, $\eta = 0$, $m = 1$ and $h(t) = \frac{1}{t}$, Godunova-Levin function [14] can be captured.
(xv) By choosing $\theta = 1$, $\eta = 0$, $m = 1$ and $h(t) = \frac{1}{t^s}$, s-Godunova-Levin function of second kind can be obtained.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.3. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on J and $a, b \in J$ with $a < b$. Then the following double inequality holds:*

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a}\int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1.3}$$

Various extensions of this notion have been reported in the literature in recent years, see [1, 4, 7, 16–18, 21, 22, 26, 30].

The objective of this paper is to obtain inequalities of Hadamard type via Caputo k-fractional derivatives of exponentially $(\theta, h-m)$-convex functions. Study of integration or differentiation of fractional order is known as fractional calculus. Its history is as old as the history of calculus. A lot of work has been published since the day of Leibniz (1695) and since then has occupied great number of mathematicians of their time [15, 20, 25, 27].

Fractional integral inequalities are in the study of several researchers, see [5, 9, 13] and references therein. The classical Caputo fractional derivatives are defined as follows:

Definition 1.4. [20] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$ (the set of all functions f such that $f^{(n)}$ are absolutely continuous on $[a, b]$). The Caputo fractional derivatives of order α are defined by

$$ {}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a, \tag{1.4}$$

and

$$ {}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b. \tag{1.5}$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $({}^{C}D^{\alpha}_{a+}f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^{C}D^{\alpha}_{b-}f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular, we have

$$ ({}^{C}D^{0}_{a+}f)(x) = ({}^{C}D^{0}_{b-}f)(x) = f(x), \tag{1.6}$$

where $n = 1$ and $\alpha = 0$.

In [9], Farid et al. defined Caputo $k-$fractional derivatives as follows:

Definition 1.5. Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo k-fractional derivatives of order α are given as

$$ {}^{C}D^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \quad x > a, \tag{1.7}$$

and

$$ {}^{C}D^{\alpha,k}_{b-}f(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \quad x < b, \tag{1.8}$$

where $\Gamma_k(\alpha)$ is the k-gamma function defined as

$$ \Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt. $$

Also

$$ \Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha). $$

Motivated by above results and literatures, the paper is organized in the following manner:

In section 2, we present some inequalities of Hadamard type for exponentially $(\theta, h-m)-$convex functions via Caputo $k-$fractional derivatives. In section 3, we use the integral identity including the $(n+1)$-order derivative of f to establish interesting Hadamard type inequalities for exponentially $(\theta, h-m)-$convexity via Caputo k-fractional derivatives. In section 4, a briefly conclusion will be provided as well.

2. Main results

In this section, we give the Caputo k–fractional derivatives inequality of Hadamard type for a function whose n-th derivatives are exponentially $(\theta, h-m)$–convex.

Theorem 2.1. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha]+1$ and $[a, b] \subset [0, +\infty)$, $f : [0, +\infty) \to \mathbb{R}$ be a function such that $f \in AC^n[a, mb]$, where $a < mb$. Also, assume that $f^{(n)}$ be an exponentially $(\theta, h-m)$-convex function with $(\theta, m) \in (0, 1]^2$ and $\eta \in \mathbb{R}$. Then the following inequalities for Caputo k-fractional derivatives hold:*

$$\begin{aligned}
&\frac{1}{g(\eta)} f^{(n)}\left(\frac{bm+a}{2}\right) \leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}} \\
&\times\left(h\left(1-\frac{1}{2^\theta}\right) m^{n-\frac{\alpha}{k}+1}(-1)^n ({}^C D^{\alpha,k}_{b^-} f)\left(\frac{a}{m}\right) + h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{a^+} f)(mb)\right) \\
&\leq \frac{kn-\alpha}{k} \\
&\times\left\{\left(h\left(1-\frac{1}{2^\theta}\right) m^2 \frac{f^{(n)}(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}} + h\left(\frac{1}{2^\theta}\right) m \frac{f^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} h(1-t^\theta) dt \right. \\
&\left. + \left(h\left(1-\frac{1}{2^\theta}\right) m \frac{f^{(n)}(b)}{e^{\eta b}} + h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(a)}{e^{\eta a}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} h(t^\theta) dt\right\},
\end{aligned} \tag{2.1}$$

where $g(\eta) = \frac{1}{e^{\eta b}}$ for $\eta < 0$ and $g(\eta) = \frac{1}{e^{\frac{\eta a}{m}}}$ for $\eta \geq 0$.

Proof. Since $f^{(n)}$ is an exponentially $(\theta, h-m)$-convex function on $[a, b]$, then

$$f^{(n)}\left(\frac{um+v}{2}\right) \leq h\left(1-\frac{1}{2^\theta}\right)\frac{mf^{(n)}(u)}{e^{\eta u}} + h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(v)}{e^{\eta v}}, \quad u, v \in [a, b].$$

By setting $u = (1-t)\frac{a}{m} + tb \leq b$ and $v = m(1-t)b + ta \geq a$ in the above inequality for $t \in [0, 1]$, then by integrating with respect to t over $[0, 1]$ after multiplying with $t^{n-\frac{\alpha}{k}-1}$, we have

$$\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} dt \\
\leq & h\left(1-\frac{1}{2^\theta}\right)\left(\int_0^1 t^{n-\frac{\alpha}{k}-1}\frac{mf^{(n)}\left((1-t)\frac{a}{m}+tb\right)}{e^{\eta\left((1-t)\frac{a}{m}+tb\right)}} dt\right. \\
&\left. + h\left(\frac{1}{2^\theta}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(m(1-t)b+ta)}{e^{\eta(m(1-t)b+ta)}} dt\right).
\end{aligned}$$

Now, if we let $w = (1-t)\frac{a}{m} + tb$ and $z = m(1-t)b + ta$ in right hand side of above inequality, we get

$$\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right)\frac{1}{n-\frac{\alpha}{k}} \leq h\left(1-\frac{1}{2^\theta}\right)\left(\int_{\frac{a}{m}}^b \left(\frac{w-\frac{a}{m}}{b-\frac{a}{m}}\right)^{n-\frac{\alpha}{k}-1}\frac{mf^{(n)}(w)dw}{e^{\eta w}\left(b-\frac{a}{m}\right)}\right. \\
&\left. + h\left(\frac{1}{2^\theta}\right)\int_a^{mb}\left(\frac{mb-z}{mb-a}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(z)dz}{e^{\eta z}(mb-a)}\right).
\end{aligned}$$

Hence

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq \left\{\frac{g(\eta)k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\right.$$
$$\left.\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{a^+}f)(mb)\right)\right\}. \tag{2.2}$$

On the other hand by using exponentially $(\theta, h-m)$-convexity of $f^{(n)}$, we have

$$mf^{(n)}\left((1-t)\frac{a}{m}+tb\right) \leq m^2 h(1-t^\theta)\frac{f^{(n)}(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}}+mh(t^\theta)\frac{f^{(n)}(b)}{e^{\eta b}}.$$

By multiplying both sides of above inequality with $(n-\frac{\alpha}{k})h\left(1-\frac{1}{2^\theta}\right)t^{n-\frac{\alpha}{k}-1}$ and integrating with respect to t over $[0,1]$, after some calculations we get

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)\right)$$
$$\leq h\left(1-\frac{1}{2^\theta}\right)\left(n-\frac{\alpha}{k}\right)\left\{m^2\frac{f^{(n)}(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt\right. \tag{2.3}$$
$$\left.+m\frac{f^{(n)}(b)}{e^{\eta b}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\right\}.$$

Similarly,

$$f^{(n)}(m(1-t)b+ta) \leq mh(1-t^\theta)\frac{f^{(n)}(b)}{e^{\eta b}}+h(t^\theta)\frac{f^{(n)}(a)}{e^{\eta a}}.$$

By multiplying both sides of above inequality with $(n-\frac{\alpha}{k})h\left(\frac{1}{2^\theta}\right)t^{n-\frac{\alpha}{k}-1}$ and integrating with respect to t over $[0,1]$, after some calculations we get

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left(h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{a^+}f)(mb)\right)$$
$$\leq h\left(\frac{1}{2^\theta}\right)\left(n-\frac{\alpha}{k}\right)\left\{m\frac{f^{(n)}(b)}{e^{\eta b}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt\right. \tag{2.4}$$
$$\left.+\frac{f^{(n)}(a)}{e^{\eta a}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\right\}.$$

By adding (2.3) and (2.4), we obtain

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{a^+}f)(mb)\right)$$
$$\leq\left(n-\frac{\alpha}{k}\right)\left\{\left(h\left(1-\frac{1}{2^\theta}\right)m^2\frac{f^{(n)}(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}}+h\left(\frac{1}{2^\theta}\right)m\frac{f^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt\right.$$
$$\left.+\left(h\left(1-\frac{1}{2^\theta}\right)m\frac{f^{(n)}(b)}{e^{\eta b}}+h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(a)}{e^{\eta a}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\right\}.$$

Combining above with (2.2), we get required result. □

Corollary 2.2. *By setting $k = 1$ in inequality (2.1), the following inequalities hold for exponentially $(\theta, h-m)$-convex functions via Caputo fractional derivatives:*

$$\begin{aligned}
&\frac{1}{g(\eta)} f^{(n)}\left(\frac{bm+a}{2}\right) \\
&\leq \frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left(h\left(1-\frac{1}{2^{\theta}}\right)m^{n-\alpha+1}(-1)^{n}({}^{C}D^{\alpha}_{b^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^{\theta}}\right)({}^{C}D^{\alpha}_{a^+}f)(mb)\right) \\
&\leq (n-\alpha)\left\{\left(h\left(1-\frac{1}{2^{\theta}}\right)m^{2}\frac{f^{(n)}(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}}+h\left(\frac{1}{2^{\theta}}\right)m\frac{f^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\alpha-1}h(1-t^{\theta})dt\right. \\
&\left.+\left(h\left(1-\frac{1}{2^{\theta}}\right)m\frac{f^{(n)}(b)}{e^{\eta b}}+h\left(\frac{1}{2^{\theta}}\right)\frac{f^{(n)}(a)}{e^{\eta a}}\right)\int_0^1 t^{n-\alpha-1}h(t^{\theta})dt\right\}.
\end{aligned}$$

Corollary 2.3. *Taking $\eta = 0$ in (2.1), the following inequalities hold for $(\theta, h-m)$-convex functions via Caputo k-fractional derivatives:*

$$\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right) \\
&\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^{\theta}}\right)m^{n-\frac{\alpha}{k}+1}(-1)^{n}({}^{C}D^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^{\theta}}\right)({}^{C}D^{\alpha,k}_{a^+}f)(mb)\right) \\
&\leq \left(\frac{kn-\alpha}{k}\right)\left\{\left(h\left(1-\frac{1}{2^{\theta}}\right)m^{2}f^{(n)}\left(\frac{a}{m^2}\right)+h\left(\frac{1}{2^{\theta}}\right)mf^{(n)}(b)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^{\theta})dt\right. \\
&\left.+\left(h\left(1-\frac{1}{2^{\theta}}\right)mf^{(n)}(b)+h\left(\frac{1}{2^{\theta}}\right)f^{(n)}(a)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^{\theta})dt\right\}.
\end{aligned}$$

Corollary 2.4. *Choosing $\eta = 0$ and $\theta = 1$ in (2.1), the following inequalities hold for $(h-m)$-convex functions via Caputo k-fractional derivatives defined in [[24], Theorem 2.1]:*

$$\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right) \leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(m^{n-\frac{\alpha}{k}+1}(-1)^{n}({}^{C}D^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+({}^{C}D^{\alpha,k}_{a^+}f)(mb)\right) \\
&\leq \frac{kn-\alpha}{2k}\left\{\left(m^{2}f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t)dt\right. \\
&\left.+\left(mf^{(n)}(b)+f^{(n)}(a)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt\right\}.
\end{aligned}$$

Corollary 2.5. *By setting $\eta = 0$, $\theta = 1$ and $k = 1$ in (2.1), the following inequalities hold for $(h-m)$-convex functions via Caputo fractional derivatives defined in [[24], Corollary 2.2]:*

$$\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right) \leq \frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(m^{n-\alpha+1}(-1)^{n}({}^{C}D^{\alpha}_{b^-}f)\left(\frac{a}{m}\right)+({}^{C}D^{\alpha}_{a^+}f)(mb)\right) \\
&\leq \frac{n-\alpha}{2}\left\{\left(m^{2}f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\int_0^1 t^{n-\alpha-1}h(1-t)dt\right. \\
&\left.+\left(mf^{(n)}(b)+f^{(n)}(a)\right)\int_0^1 t^{n-\alpha-1}h(t)dt\right\}.
\end{aligned}$$

Corollary 2.6. *Taking $\theta = 1$ and $h(t) = t^s$ in (2.1), the following inequalities hold for exponentially (s, m)-convex functions via Caputo k-fractional derivatives:*

$$\begin{aligned}
&\frac{1}{g(\eta)} f^{(n)}\left(\frac{bm+a}{2}\right) \\
&\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2^s(mb-a)^{n-\frac{\alpha}{k}}}\left(m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-} f)\left(\frac{a}{m}\right) + ({}^C D^{\alpha,k}_{a^+} f)(mb)\right) \\
&\leq \frac{kn-\alpha}{k2^s}\left\{\left(\frac{m^2 f^{(n)}(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}} + \frac{mf^{(n)}(b)}{e^{\eta b}}\right)\beta\left(\frac{kn-\alpha}{k}, s+1\right)\right. \\
&\left. + \left(\frac{mf^{(n)}(b)}{e^{\eta b}} + \frac{f^{(n)}(a)}{e^{\eta a}}\right)\frac{k}{kn-\alpha+ks}\right\},
\end{aligned}$$

where $\beta(\cdot, \cdot)$ is well–known beta function.

Corollary 2.7. *Choosing $\eta = 0$, $\theta = 1$, $m = 1$ and $h(t) = t$ in (2.1), the following inequalities hold for convex functions via Caputo k-fractional derivatives defined in [[8], Theorem 2.2]:*

$$\begin{aligned}
f^{(n)}\left(\frac{a+b}{2}\right) &\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left((-1)^n({}^C D^{\alpha,k}_{b^-} f)(a) + ({}^C D^{\alpha,k}_{a^+} f)(b)\right) \\
&\leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}.
\end{aligned}$$

Corollary 2.8. *By setting $\eta = 0$, $\theta = 1$, $m = 1$, $h(t) = t$ and $k = 1$ in (2.1), the following inequalities hold for convex functions via Caputo fractional derivatives:*

$$\begin{aligned}
f^{(n)}\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left((-1)^n({}^C D^{\alpha}_{b^-} f)(a) + ({}^C D^{\alpha}_{a^+} f)(b)\right) \\
&\leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}.
\end{aligned}$$

In the following we generalize the fractional Hadamard type inequalities for exponentially $(\theta, h-m)-$convex function via Caputo k-fractional derivatives.

Theorem 2.9. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$ and $[a, b] \subset [0, +\infty)$, $f : [0, +\infty) \to \mathbb{R}$ be a function such that $f \in AC^n[a, mb]$, where $a < mb$. Also, assume that $f^{(n)}$ be an exponentially $(\theta, h-m)$-convex function with $(\theta, m) \in (0, 1]^2$ and $\eta \in \mathbb{R}$. Then the following inequalities for Caputo k-fractional derivatives hold:*

$$\begin{aligned}
&\frac{1}{g(\eta)} f^{(n)}\left(\frac{a+bm}{2}\right) \leq \frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}\right. \\
&\left.({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-} f)\left(\frac{a}{m}\right) + h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+} f)(mb)\right) \leq \left(\frac{kn-\alpha}{k}\right)\left\{\left(h\left(1-\frac{1}{2^\theta}\right)\right.\right. \\
&\left.\frac{m^2 f^{(n)}\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}} + h\left(\frac{1}{2^\theta}\right)\frac{mf^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} h\left(1-\left(\frac{t}{2}\right)^\theta\right)dt + \left(h\left(1-\frac{1}{2^\theta}\right)\right. \\
&\left.\left.\frac{mf^{(n)}(b)}{e^{\eta b}} + h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(a)}{e^{\eta a}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} h\left(\left(\frac{t}{2}\right)^\theta\right)dt\right\},
\end{aligned} \tag{2.5}$$

where $g(\eta)=\frac{1}{e^{\eta b}}$ *for* $\eta<0$ *and* $g(\eta)=\frac{1}{e^{\frac{\eta a}{m}}}$ *for* $\eta\geq 0$.

Proof. From exponentially $(\theta, h-m)$−convexity of $f^{(n)}$ one can have

$$f^{(n)}\left(\frac{um+v}{2}\right)\leq h\left(1-\frac{1}{2^\theta}\right)\frac{mf^{(n)}(u)}{e^{\eta u}}+h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(v)}{e^{\eta v}}.$$

Putting $u=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $u=\frac{t}{2}a+m\frac{(2-t)}{2}b$ in the above inequality where $t\in[0,1]$, and multiplying with $t^{n-\frac{\alpha}{k}-1}$, then integrating with respect to t over $[0,1]$ one can have

$$\begin{aligned}&f^{(n)}\left(\frac{a+bm}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt\\&\leq h\left(1-\frac{1}{2^\theta}\right)\left(\int_0^1 t^{n-\frac{\alpha}{k}-1}\frac{mf^{(n)}\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)}{e^{\eta\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)}}dt\right.\\&\left.+h\left(\frac{1}{2^\theta}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)}{e^{\eta\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)}}dt\right).\end{aligned}$$

By change of variables, we get

$$\begin{aligned}&\frac{1}{g(\eta)}f^{(n)}\left(\frac{a+bm}{2}\right)\leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}\\&\times\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\theta}\right)({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)\right).\end{aligned}\tag{2.6}$$

Now, using the exponentially $(\theta, h-m)$−convexity of $f^{(n)}$, we can write

$$f^{(n)}\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)\leq h\left(\left(\frac{t}{2}\right)^\theta\right)\frac{f^{(n)}(a)}{e^{\eta a}}+mh\left(1-\left(\frac{t}{2}\right)^\theta\right)\frac{f^{(n)}(b)}{e^{\eta b}}.$$

Multiplying both sides of above inequality with $(n-\frac{\alpha}{k})h\left(\frac{1}{2^\theta}\right)t^{n-\frac{\alpha}{k}-1}$ and integrating with respect to t over $[0,1]$, then by change of variables, we have

$$\begin{aligned}&h\left(\frac{1}{2^\theta}\right)\frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)\right)\\&\leq\left(n-\frac{\alpha}{k}\right)h\left(\frac{1}{2^\theta}\right)\left\{\frac{mf^{(n)}(b)}{e^{\eta b}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(1-\left(\frac{t}{2}\right)^\theta\right)dt\right.\\&\left.+\frac{f^{(n)}(a)}{e^{\eta a}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\left(\frac{t}{2}\right)^\theta\right)dt\right\}.\end{aligned}\tag{2.7}$$

Again by using the exponentially $(\theta, h-m)$−convexity of $f^{(n)}$, we can write

$$mf^{(n)}\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)\leq mh\left(\frac{t}{2}\right)^\theta\frac{f^{(n)}(b)}{e^{\eta b}}+m^2h\left(1-\left(\frac{t}{2}\right)^\theta\right)\frac{f^{(n)}\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}}.$$

Multiplying both sides of above inequality with $(n-\frac{\alpha}{k})h\left(1-\frac{1}{2^\theta}\right)t^{n-\frac{\alpha}{k}-1}$ and integrating with respect to t over $[0,1]$, then by change of variables, we have

$$\begin{aligned}
&h\left(1-\frac{1}{2^\theta}\right)\frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}\left(m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right)\\
&\leq\left(n-\frac{\alpha}{k}\right)h\left(1-\frac{1}{2^\theta}\right)\left\{\frac{m^2f^{(n)}\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(1-\left(\frac{t}{2}\right)^\theta\right)dt\right.\\
&\left.+\frac{mf^{(n)}(b)}{e^{\eta b}}\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\left(\frac{t}{2}\right)^\theta\right)dt\right\}.
\end{aligned}\tag{2.8}$$

Adding (2.7) and (2.8), we get

$$\begin{aligned}
&\frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right.\\
&\left.+h\left(\frac{1}{2^\theta}\right)({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)\right)\leq\left(n-\frac{\alpha}{k}\right)\left\{\left(h\left(1-\frac{1}{2^\theta}\right)\frac{m^2f^{(n)}\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}}\right.\right.\\
&\left.+h\left(\frac{1}{2^\theta}\right)\frac{mf^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(1-\left(\frac{t}{2}\right)^\theta\right)dt+\left(h\left(1-\frac{1}{2^\theta}\right)\frac{mf^{(n)}(b)}{e^{\eta b}}\right.\\
&\left.\left.+h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(a)}{e^{\eta a}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\left(\frac{t}{2}\right)^\theta\right)dt\right\}.
\end{aligned}$$

By combining above with (2.6), we get required result. □

Corollary 2.10. *By setting $k=1$ in inequality (2.5), the following inequalities hold for exponentially $(\theta,h-m)$-convex functions via Caputo fractional derivatives:*

$$\begin{aligned}
&\frac{1}{g(\eta)}f^{(n)}\left(\frac{a+bm}{2}\right)\leq\frac{2^{(n-\alpha)}\Gamma(n-\alpha+1)}{(bm-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\alpha+1}(-1)^{(n)}\right.\\
&\left.({}^CD^{\alpha}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\theta}\right)({}^CD^{\alpha}_{(\frac{a+bm}{2})^+}f)(mb)\right)\leq(n-\alpha)\left\{\left(h\left(1-\frac{1}{2^\theta}\right)\right.\right.\\
&\left.\frac{m^2f^{(n)}\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}}+h\left(\frac{1}{2^\theta}\right)\frac{mf^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\alpha-1}h\left(1-\left(\frac{t}{2}\right)^\theta\right)dt+\left(h\left(1-\frac{1}{2^\theta}\right)\right.\\
&\left.\left.\frac{mf^{(n)}(b)}{e^{\eta b}}+h\left(\frac{1}{2^\theta}\right)\frac{f^{(n)}(a)}{e^{\eta a}}\right)\int_0^1 t^{n-\alpha-1}h\left(\left(\frac{t}{2}\right)^\theta\right)dt\right\}.
\end{aligned}$$

Corollary 2.11. *Taking $\eta=0$ in (2.5), the following inequalities hold for $(\theta,h-m)$-convex functions* *via Caputo k-fractional derivatives:*

$$f^{(n)}\left(\frac{a+bm}{2}\right) \leq \frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}\right.$$
$$\left.({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)+h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)\right) \leq \left(\frac{kn-\alpha}{k}\right)\left\{\left(h\left(1-\frac{1}{2^\theta}\right)\right.\right.$$
$$\left.m^2 f^{(n)}\left(\frac{a}{m^2}\right)+h\left(\frac{1}{2^\theta}\right)mf^{(n)}(b)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(1-\left(\frac{t}{2}\right)^\theta\right)dt+\left(h\left(1-\frac{1}{2^\theta}\right)\right.$$
$$\left.\left.mf^{(n)}(b)+h\left(\frac{1}{2^\theta}\right)f^{(n)}(a)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\left(\frac{t}{2}\right)^\theta\right)dt\right\}.$$

Corollary 2.12. *Choosing $\eta = 0$ and $\theta = 1$ in (2.5), the following inequalities hold for $(h-m)$-convex functions via Caputo k-fractional derivatives defined in [[24], Theorem 2.4]:*

$$f^{(n)}\left(\frac{a+bm}{2}\right) \leq \frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}\right.$$
$$\left.({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)+({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)\right) \leq \frac{kn-\alpha}{k}h\left(\frac{1}{2}\right)$$
$$\times\left\{\left(m^2 f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{2-t}{2}\right)dt\right.$$
$$\left.+\left(mf^{(n)}(b)+f^{(n)}(a)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{t}{2}\right)dt\right\}.$$

Corollary 2.13. *By setting $\eta = 0$, $\theta = 1$ and $k = 1$ in (2.5), the following inequalities hold for $(h-m)$-convex functions via Caputo fractional derivatives defined in [[24], Corollary 2.5]:*

$$f^{(n)}\left(\frac{a+bm}{2}\right) \leq \frac{2^{(n-\alpha)}\Gamma(n-\alpha+1)}{(bm-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(m^{n-\alpha+1}(-1)^{(n)}\right.$$
$$\left.({}^C D^{\alpha}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)+({}^C D^{\alpha}_{(\frac{a+bm}{2})^+}f)(mb)\right) \leq (n-\alpha)h\left(\frac{1}{2}\right)$$
$$\times\left\{\left(m^2 f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\int_0^1 t^{n-\alpha-1}h\left(\frac{2-t}{2}\right)dt\right.$$
$$\left.+\left(mf^{(n)}(b)+f^{(n)}(a)\right)\int_0^1 t^{n-\alpha-1}h\left(\frac{t}{2}\right)dt\right\}.$$

Corollary 2.14. *Taking $\theta = 1$ and $h(t) = t^s$ in (2.5), the following inequalities hold for exponentially (s,m)-convex functions via Caputo k-fractional derivatives:*

$$\frac{1}{g(\eta)}f^{(n)}\left(\frac{a+bm}{2}\right)\leq\frac{2^{(n-\frac{\alpha}{k}-s)}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{k}}}$$
$$\times\left(m^{n-\frac{\alpha}{k}+1}(-1)^{(n)}({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)+({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)\right).$$
$$\leq\frac{1}{2^{2s}}\left\{\left(\frac{m^2f^{(n)}\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}}+\frac{mf^{(n)}(b)}{e^{\eta b}}\right)k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)\right.$$
$$\left.+\left(\frac{mf^{(n)}(b)}{e^{\eta b}}+\frac{f^{(n)}(a)}{e^{\eta a}}\right)\frac{kn-\alpha}{kn-\alpha+ks}\right\}.$$

Corollary 2.15. *Choosing $\eta=0$, $\theta=1$, $m=1$ and $h(t)=t$ in (2.5), the following inequalities hold for convex functions via Caputo k-fractional derivatives defined in [[10], Theorem 6]:*

$$f^{(n)}\left(\frac{a+b}{2}\right)\leq\frac{2^{(n-\frac{\alpha}{k})}k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}$$
$$\times\left((-1)^{(n)}({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^-}f)(a)+({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^+}f)(b)\right)\leq\frac{f^{(n)}(a)+f^{(n)}(b)}{2}.$$

Corollary 2.16. *By setting $\eta=0$, $\theta=1$, $m=1$, $h(t)=t$ and $k=1$ in (2.5), the following inequalities hold for convex functions via Caputo fractional derivatives defined in [[19] Theorem 2.2]:*

$$f^{(n)}\left(\frac{a+b}{2}\right)\leq\frac{2^{(n-\alpha)}\Gamma(n-\alpha+1)}{2(b-a)^{n-\frac{\alpha}{k}}}$$
$$\times\left((-1)^{(n)}({}^{C}D^{\alpha}_{(\frac{a+b}{2})^-}f)(a)+({}^{C}D^{\alpha}_{(\frac{a+b}{2})^+}f)(b)\right)\leq\frac{f^{(n)}(a)+f^{(n)}(b)}{2}.$$

In the next, some other inequalities of Hadamard type for exponentially $(\theta,h-m)$−convex function via Caputo k−fractional derivatives are given.

Theorem 2.17. *Let $\alpha>0$, $k\geq 1$ and $\alpha\notin\{1,2,3,\ldots\}$, $n=[\alpha]+1$ and $f:[0,+\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,mb]$, where $a<mb$. Also, assume that $f^{(n)}$ be an exponentially $(\theta,h-m)$-convex function with $(\theta,m)\in(0,1]^2$ and $\eta\in\mathbb{R}$. Then the following inequalities for Caputo k-fractional derivatives hold:*

$$\begin{aligned}&\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}(({}^{C}D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^{C}D^{\alpha,k}_{b^-}f)(a))\leq\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)\\&\times\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt\\&\leq\frac{1}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t^\theta))^q dt\right)^{\frac{1}{q}}\\&\times\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right),\end{aligned}\tag{2.9}$$

where $p^{-1}+q^{-1}=1$ and $p>1$.

Proof. Since $f^{(n)}$ is exponentially $(\theta, h-m)$–convex on $[a,b]$, then for $(\theta,m)\in(0,1]^2$ and $t\in[0,1]$, we have

$$\begin{aligned}&f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)\\&\leq h(t^\theta)\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)+mh(1-t^\theta)\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right).\end{aligned}$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{n-\frac{\alpha}{k}-1}(f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb))dt\\&\leq\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\\&+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt.\end{aligned}$$

If we set $x=ta+(1-t)b$ in the left side of above inequality, we get the following inequality

$$\begin{aligned}&\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}((^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n(^C D^{\alpha,k}_{b^-}f)(a))\\&\leq\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\\&+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt.\end{aligned}\tag{2.10}$$

We get the first inequality of (2.9). The second inequality of (2.9) follows by using the Hölder's inequality

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\leq\frac{1}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t^\theta))^q dt\right)^{\frac{1}{q}}.$$

Combining it with (2.10) we get (2.9). □

Corollary 2.18. *By setting $k=1$ in (2.9), the following inequalities hold for exponentially $(\theta,h-m)$-convex functions via Caputo fractional derivatives:*

$$\begin{aligned}&\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}((^C D^{\alpha}_{a^+}f)(b)+(-1)^n(^C D^{\alpha}_{b^-}f)(a))\leq\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)\\&\times\int_0^1 t^{n-\alpha-1}h(t^\theta)dt+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\int_0^1 t^{n-\alpha-1}h(1-t^\theta)dt\\&\leq\frac{1}{(np-\alpha p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t^\theta))^q dt\right)^{\frac{1}{q}}\\&\times\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right).\end{aligned}$$

Corollary 2.19. *Taking $\eta = 0$ in (2.9), the following inequalities hold for $(\theta, h-m)$-convex functions via Caputo k-fractional derivatives:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}(({}^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b^-}f)(a)) \leq \left(f^{(n)}(a)+f^{(n)}(b)\right)$$
$$\times \int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt + m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt$$
$$\leq \frac{1}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t^\theta))^q dt\right)^{\frac{1}{q}}$$
$$\times\left(f^{(n)}(a)+f^{(n)}(b)+m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\right).$$

Corollary 2.20. *Choosing $\eta = 0$ and $\theta = 1$ in (2.9), the following inequalities hold for $(h-m)$-convex functions via Caputo k-fractional derivatives defined in [[24], Theorem 2.7]:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}(({}^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b^-}f)(a)) \leq \left(f^{(n)}(a)+f^{(n)}(b)\right)$$
$$\times \int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt + m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t)dt$$
$$\leq \frac{1}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}}$$
$$\times\left(f^{(n)}(a)+f^{(n)}(b)+m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\right).$$

Corollary 2.21. *By setting $\eta = 0$, $\theta = 1$ and $k = 1$ in (2.9), the following inequalities hold for $(h-m)$-convex functions via Caputo fractional derivatives defined in [[24], Corollary 2.8]:*

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}(({}^C D^{\alpha}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha}_{b^-}f)(a)) \leq \left(f^{(n)}(a)+f^{(n)}(b)\right)$$
$$\times \int_0^1 t^{n-\alpha-1}h(t)dt + m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\int_0^1 t^{n-\alpha-1}h(1-t)dt$$
$$\leq \frac{1}{(np-\alpha p-p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}}$$
$$\times\left(f^{(n)}(a)+f^{(n)}(b)+m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\right).$$

Corollary 2.22. *Taking $\theta = 1$ and $h(t) = t^s$ in (2.9), the following inequalities hold for exponentially (s, m)-convex functions via Caputo k-fractional derivatives:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}(({}^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b^-}f)(a))$$
$$\leq \frac{k\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)}{kn-\alpha+ks}+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\beta\left(n-\frac{\alpha}{k},s+1\right)$$
$$\leq \frac{k\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)}{kn-\alpha+ks}+\frac{m\left(\frac{f^{(n)}(\frac{b}{m})}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}(\frac{a}{m})}{e^{\frac{\eta a}{m}}}\right)}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}.$$

Corollary 2.23. *Choosing $\eta=0,\theta=1,m=1$ and $h(t)=t$ in (2.9), the following inequalities hold for convex functions via Caputo k-fractional derivatives:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}(({}^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b^-}f)(a))$$
$$\leq \frac{k\left(f^{(n)}(a)+f^{(n)}(b)\right)}{kn-\alpha}\leq \frac{2(f^{(n)}(a)+f^{(n)}(b))}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}.$$

Theorem 2.24. *Let $\alpha>0$, $k\geq 1$ and $\alpha\notin\{1,2,3,\ldots\}$, $n=[\alpha]+1$ and $[a,b]\subset[0,+\infty)$, $f:[0,+\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,mb]$, where $a<mb$ and h be a superadditive function. Also, assume that $f^{(n)}$ be an exponentially $(\theta,h-m)$-convex function with $(\theta,m)\in(0,1]^2$ and $\eta\in\mathbb{R}$. Then the following inequality for Caputo k−fractional derivatives holds:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}(({}^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b^-}f)(a))$$
$$\leq \frac{h(1)}{2}\left(\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right). \tag{2.11}$$

Proof. Since $f^{(n)}$ is exponentially $(\theta,h-m)$−convex on $[a,b]$, then for $t\in[0,1]$, we get

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)$$
$$\leq \frac{(h(t^\theta)+h(1-t^\theta))}{2}\left(\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right).$$

Since h is superadditive function, then

$$h(t^\theta)+h(1-t^\theta)\leq h(1),\quad \text{for all}\quad \theta\in(0,1]\quad\text{and}\quad t\in[0,1].$$

Therefore

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)$$
$$\leq \frac{h(1)}{2}\left(\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right).$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating with respect to t over $[0,1]$, yield the following

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}(f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb))dt$$
$$\leq \frac{h(1)}{2}\left(\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt.$$

By change of variable, we get the required result. □

Corollary 2.25. *By setting $k=1$ in (2.11), the following inequality holds for exponentially $(\theta, h-m)$-convex functions via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}(({}^C D^{\alpha}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha}_{b^-}f)(a))$$
$$\leq \frac{h(1)}{2}\left(\left(\frac{f^{(n)}(a)}{e^{\eta a}}+\frac{f^{(n)}(b)}{e^{\eta b}}\right)+m\left(\frac{f^{(n)}\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f^{(n)}\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right)\right).$$

Corollary 2.26. *Taking $\eta=0$ and $\theta=1$ in (2.11), the following inequality holds for $(h-m)$-convex functions via Caputo k-fractional derivatives defined in [[24], Theorem 2.9]:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}(({}^C D^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b^-}f)(a))$$
$$\leq \frac{h(1)}{2}\left(\left(f^{(n)}(a)+f^{(n)}(b)\right)+m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\right).$$

Corollary 2.27. *Choosing $\eta=0$, $\theta=1$ and $k=1$ in (2.11), the following inequality holds for $(h-m)$-convex functions via Caputo k-fractional derivatives defined in [[24], Corollary 2.10]:*

$$\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}(({}^C D^{\alpha}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha}_{b^-}f)(a))$$
$$\leq \frac{h(1)}{2}\left(\left(f^{(n)}(a)+f^{(n)}(b)\right)+m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\right).$$

Corollary 2.28. *By setting $\eta=0$, $\theta=1$, $m=1$, $h(t)=t$ and $k=1$ in (2.11), the following inequality holds for convex functions via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}(({}^C D^{\alpha}_{a^+}f)(b)+(-1)^n({}^C D^{\alpha}_{b^-}f)(a))\leq f^{(n)}(a)+f^{(n)}(b).$$

3. Other results

We need the following known lemma to prove our next results.

Lemma 3.1. *[24] Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$ and $f : [a, mb] \to \mathbb{R}$, where $a, b \in [0, +\infty)$ be a differentiable mapping on interval (a, mb), with $a < mb$ and $m \in (0, 1]$. If $f \in AC^{n+1}[a, mb]$, then the following equality for Caputo $k-$fractional derivatives holds:*

$$\frac{f^{(n)}(mb) + f^{(n)}(b)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n - \frac{\alpha}{k}}} \left(({}^C D^{\alpha,k}_{a^+} f)(mb) + ({}^C D^{\alpha,k}_{mb^-} f)(a)\right)$$
$$= \frac{mb - a}{2} \int_0^1 ((1 - t)^{n - \frac{\alpha}{k}} - t^{n - \frac{\alpha}{k}}) f^{(n+1)}(m(1 - t)b + ta)dt.$$

Caputo $k-$fractional derivative inequalities of Hadamard type for exponentially $(\theta, h - m)-$convex function in terms of the $(n+1)$-th derivatives in absolute, is obtained in the following theorem by using above lemma.

Theorem 3.2. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$ and $[a, b] \subset [0, +\infty)$, $f : [0, +\infty) \to \mathbb{R}$ be a function such that $f \in AC^{n+1}[a, mb]$, where $a < mb$. If $\left|f^{(n+1)}\right|$ is an exponentially $(\theta, h - m)-$convex with $(\theta, m) \in (0, 1]^2$ and $\eta \in \mathbb{R}$, then the following inequality for Caputo $k-$fractional derivatives holds:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n - \frac{\alpha}{k}}} ({}^C D^{\alpha,k}_{b^+} f)(mb) + ({}^C D^{\alpha,k}_{mb^-} f)(a)\right| \\
&\leq \frac{mb - a}{2} \left(\frac{(2^{np - \frac{\alpha}{k}p + 1} - 1)^{\frac{1}{p}}}{(2^{np - \frac{\alpha}{k}p + 1}(np - \frac{\alpha}{k}p + 1))^{\frac{1}{p}} - 1}\right) \\
&\times \left(\frac{|f^{(n+1)}(a)|}{e^{\eta a}} \left(\left(\int_0^{\frac{1}{2}} (h(t^\theta))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t^\theta))^q dt\right)^{\frac{1}{q}}\right)\right. \\
&\left. + m\frac{|f^{(n+1)}(b)|}{e^{\eta b}} \left(\left(\int_0^{\frac{1}{2}} (h(1 - t^\theta))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(1 - t^\theta))^q dt\right)^{\frac{1}{q}}\right)\right),
\end{aligned} \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Proof. From Lemma 3.1 and by using the properties of modulus, we get

$$\left|\frac{f^{(n)}(mb) + f^{(n)}(b)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n - \frac{\alpha}{k}}} \left(({}^C D^{\alpha,k}_{a^+} f)(mb) + ({}^C D^{\alpha,k}_{mb^-} f)(a)\right)\right|$$
$$\leq \frac{mb - a}{2} \int_0^1 \left|(1 - t)^{n - \frac{\alpha}{k}} - t^{n - \frac{\alpha}{k}}\right| \left|f^{(n+1)}(m(1 - t)b + ta)\right| dt.$$

By exponentially $(\theta, h - m)-$convexity of $\left|f^{(n+1)}\right|$, we have

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n - \frac{\alpha}{k}}} ({}^C D^{\alpha,k}_{a^+} f)(mb) + ({}^C D^{\alpha,k}_{mb^-} f)(a)\right| \\
&\leq \frac{mb - a}{2} \int_0^{\frac{1}{2}} \left((1 - t)^{n - \frac{\alpha}{k}} - t^{n - \frac{\alpha}{k}}\right) \left(mh(1 - t^\theta) \left|\frac{f^{(n+1)}(b)}{e^{\eta b}}\right| + h(t^\theta) \left|\frac{f^{(n+1)}(a)}{e^{\eta a}}\right|\right) dt
\end{aligned} \tag{3.2}$$

$$+\int_{\frac{1}{2}}^{1}\left(t^{n-\frac{\alpha}{k}}-(1-t)^{n-\frac{\alpha}{k}}\right)\left(mh(1-t^{\theta})\left|\frac{f^{(n+1)}(b)}{e^{\eta b}}\right|+h(t^{\theta})\left|\frac{f^{(n+1)}(a)}{e^{\eta a}}\right|\right)dt$$

$$=\frac{mb-a}{2}\left\{\left|\frac{f^{(n+1)}(a)}{e^{\eta a}}\right|\left(\int_{0}^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(t^{\theta})dt-\int_{0}^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(t^{\theta})dt\right)\right.$$

$$+m\left|\frac{f^{(n+1)}(b)}{e^{\eta b}}\right|\left(\int_{0}^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(1-t^{\theta})dt-\int_{0}^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(1-t^{\theta})dt\right)$$

$$+\left|\frac{f^{(n+1)}(a)}{e^{\eta a}}\right|\left(\int_{\frac{1}{2}}^{1}t^{n-\frac{\alpha}{k}}h(t^{\theta})dt-\int_{\frac{1}{2}}^{1}(1-t)^{n-\frac{\alpha}{k}}h(t^{\theta})dt\right)$$

$$\left.+m\left|\frac{f^{(n+1)}(b)}{e^{\eta b}}\right|\left(\int_{\frac{1}{2}}^{1}t^{n-\frac{\alpha}{k}}h(1-t^{\theta})dt-\int_{\frac{1}{2}}^{1}(1-t)^{n-\frac{\alpha}{k}}h(1-t^{\theta})dt\right)\right\}.$$

Now, using the Hölder's inequality in the right hand side of (3.2), we obtain

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^{C}D_{b^+}^{\alpha,k}f)(mb)+({}^{C}D_{mb^-}^{\alpha,k}f)(a)\right|$$

$$\leq\frac{mb-a}{2}\left\{\left|\frac{f^{(n+1)}(a)}{e^{\eta a}}\right|\left(\frac{(2^{np-\frac{\alpha}{k}p+1}-1)^{\frac{1}{p}}-1}{(2^{np-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1))^{\frac{1}{p}}}\left(\int_{0}^{\frac{1}{2}}(h(t^{\theta}))^{q}dt\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\frac{(2^{np-\frac{\alpha}{k}p+1}-1)^{\frac{1}{p}}-1}{(2^{np-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1))^{\frac{1}{p}}}\left(\int_{\frac{1}{2}}^{1}(h(t^{\theta}))^{q}dt\right)^{\frac{1}{q}}\right)$$

$$+m\left|\frac{f^{(n+1)}(b)}{e^{\eta b}}\right|\left(\frac{(2^{np-\frac{\alpha}{k}p+1}-1)^{\frac{1}{p}}-1}{(2^{np-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1))^{\frac{1}{p}}}\left(\int_{0}^{\frac{1}{2}}(h(1-t^{\theta}))^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\frac{(2^{np-\frac{\alpha}{k}p+1}-1)^{\frac{1}{p}}-1}{(2^{np-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1))^{\frac{1}{p}}}\left(\int_{\frac{1}{2}}^{1}(h(1-t^{\theta}))^{q}dt\right)^{\frac{1}{q}}\right)\right\}.$$

After a little computation one can get inequality (3.1). □

Corollary 3.3. *By setting $k=1$ in (3.1), the following inequality holds for exponentially $(\theta,h-m)$-convex functions via Caputo fractional derivatives:*

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}({}^{C}D_{b^+}^{\alpha}f)(mb)+({}^{C}D_{mb^-}^{\alpha}f)(a)\right|$$

$$\leq\frac{mb-a}{2}\left(\frac{(2^{np-\alpha p+1}-1)^{\frac{1}{p}}}{(2^{np-\alpha p+1}(np-\alpha p+1))^{\frac{1}{p}}-1}\right)$$

$$\times\left(\frac{|f^{(n+1)}(a)|}{e^{\eta a}}\left(\left(\int_{0}^{\frac{1}{2}}(h(t^{\theta}))^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(h(t^{\theta}))^{q}dt\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+m\frac{|f^{(n+1)}(b)|}{e^{\eta b}}\left(\left(\int_{0}^{\frac{1}{2}}(h(1-t^{\theta}))^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(h(1-t^{\theta}))^{q}dt\right)^{\frac{1}{q}}\right)\right).$$

Corollary 3.4. *Taking $\eta = 0$ in (3.1), the following inequality holds for $(\theta, h - m)$-convex functions via Caputo k-fractional derivatives:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n-\frac{\alpha}{k}}}({}^C D^{\alpha,k}_{b^+} f)(mb) + ({}^C D^{\alpha,k}_{mb^-} f)(a)\right| \\
&\leq \frac{mb - a}{2}\left(\frac{(2^{np-\frac{\alpha}{k}p+1} - 1)^{\frac{1}{p}}}{(2^{np-\frac{\alpha}{k}p+1}(np - \frac{\alpha}{k}p + 1))^{\frac{1}{p}} - 1}\right) \\
&\times\left(|f^{(n+1)}(a)|\left(\left(\int_0^{\frac{1}{2}} (h(t^\theta))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t^\theta))^q dt\right)^{\frac{1}{q}}\right)\right. \\
&\left.+m|f^{(n+1)}(b)|\left(\left(\int_0^{\frac{1}{2}} (h(1 - t^\theta))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(1 - t^\theta))^q dt\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Corollary 3.5. *Choosing $\eta = 0$ and $\theta = 1$ in (3.1), the following inequality holds for $(h - m)$-convex functions via Caputo k-fractional derivatives defined in [[24], Theorem 3.1]:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n-\frac{\alpha}{k}}}({}^C D^{\alpha,k}_{b^+} f)(mb) + ({}^C D^{\alpha,k}_{mb^-} f)(a)\right| \\
&\leq \frac{(mb - a)(|f^{(n+1)}(a)| + m|f^{(n+1)}(b)|)[(2^{np-\frac{\alpha}{k}p+1} - 1)^{\frac{1}{p}}]}{2[(2^{np-\frac{\alpha}{k}p+1}(np - \frac{\alpha}{k}p + 1))^{\frac{1}{p}} - 1]} \\
&\times\left(\left(\int_0^{\frac{1}{2}} (h(t))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 3.6. *By setting $\eta = 0$, $\theta = 1$ and $k = 1$ in (3.1), the following inequality holds for $(h - m)$-convex functions via Caputo fractional derivatives defined in [[24], Corollary 3.2]:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{\Gamma(n - \alpha + 1)}{2(mb - a)^{n-\alpha}}({}^C D^{\alpha}_{b^+} f)(mb) + ({}^C D^{\alpha}_{mb^-} f)(a)\right| \\
&\leq \frac{(mb - a)(|f^{(n+1)}(a)| + m|f^{(n+1)}(b)|)[(2^{np-\alpha p+1} - 1)^{\frac{1}{p}}]}{2[(2^{np-\alpha p+1}(np - \alpha p + 1))^{\frac{1}{p}} - 1]} \\
&\times\left(\left(\int_0^{\frac{1}{2}} (h(t))^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 3.7. *Taking $\theta = 1$ and $h(t) = t^s$ in (3.1), the following inequality holds for exponentially (s, m)-convex functions:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(mb - a)^{n-\frac{\alpha}{k}}}({}^C D^{\alpha,k}_{b^+} f)(mb) + (-1)^n({}^C D^{\alpha,k}_{mb^-} f)(a)\right| \\
&\leq \frac{(mb - a)\left(\left|\frac{f^{(n+1)}(a)}{e^{\eta a}}\right| + m\left|\frac{f^{(n+1)}(b)}{e^{\eta b}}\right|\right)}{2} \\
&\times\left\{\left(\frac{2^{n-\frac{\alpha}{k}+s} - 1}{2^{n-\frac{\alpha}{k}+s}(n - \frac{\alpha}{k} + s + 1)} + \frac{(2^{np-\frac{\alpha}{k}p+1} - 1)^{\frac{1}{p}} - (2^{qs+1} - 1)^{\frac{1}{q}}}{(2^{np-\frac{\alpha}{k}p+1}(np - \frac{\alpha}{k}p + 1))^{\frac{1}{p}}(2^{qs+1}(qs + 1))^{\frac{1}{q}}}\right)\right\}.
\end{aligned}$$

Corollary 3.8. *Choosing $\eta = 0$, $\theta = 1$, $m = 1$, $h(t) = t$ and $k = 1$ in (3.1), the following inequality holds for convex functions via Caputo fractional derivatives:*

$$\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}({}^{C}D^{\alpha}_{b^{+}}f)(b) + ({}^{C}D^{\alpha}_{b^{-}}f)(a)\right|$$
$$\leq \frac{(b-a)(|f^{(n+1)}(a)| + |f^{(n+1)}(b)|)[(2^{np-\alpha p+1}-1)^{\frac{1}{p}}((2^{q+1}-1)^{\frac{1}{q}}+1)]}{2[((2^{np-\alpha p+1}(np-\alpha p+1))^{\frac{1}{p}}-1)(2^{q+1}(q+1))^{\frac{1}{q}}]}.$$

4. Conclusions

In this paper, some inequalities of Hadamard type for exponentially $(\alpha, h-m)$–convex functions via Caputo k–fractional derivatives are obtained. By applied integral identity including the $(n+1)$–order derivative of a given function via Caputo k–fractional derivatives, we given some new of its related integral inequalities results. Some new results are given and know results are recaptured as special cases from our results. Since convexity and (exponentially $(\alpha, h-m)$–convexity) have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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Conflict of interest

The authors declare no conflict of interest.

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