Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2973–2988.
DOI: 10.3934/math.2022164
Received: 17 September 2021
Accepted: 15 November 2021
Published: 23 November 2021

Research article

Integral presentations of the solution of a boundary value problem for impulsive fractional integro-differential equations with Riemann-Liouville derivatives

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Abstract: Riemann-Liouville fractional differential equations with impulses are useful in modeling the dynamics of many real world problems. It is very important that there are good and consistent theoretical proofs and meaningful results for appropriate problems. In this paper we consider a boundary value problem for integro-differential equations with Riemann-Liouville fractional derivative of orders from $(1, 2)$. We consider both interpretations in the literature on the presence of impulses in fractional differential equations: With fixed lower limit of the fractional derivative at the initial time point and with lower limits changeable at each impulsive time point. In both cases we set up in an appropriate way impulsive conditions which are dependent on the Riemann-Liouville fractional derivative. We establish integral presentations of the solutions in both cases and we note that these presentations are useful for furure studies of existence, stability and other qualitative properties of the solutions.

Keywords: Riemann-Liouville fractional derivative; impulses; Riemann-Liouville integral; boundary value problem
Mathematics Subject Classification: 34A08, 34A12, 34A38

1. Introduction

Fractional differential equations with various types of fractional derivatives arise in modeling some dynamical processes (see, for example, [15] for the globally projective synchronization of Caputo fractional-order quaternion-valued neural networks with discrete and distributed delays, [18] for the quasi-uniform synchronization issue for fractional-order neural networks with leakage and discrete delays and [11] for Mittag-Leffler stability and adaptive impulsive synchronization of fractional order neural networks in quaternion field). In contrast to the classical derivative the fractional derivative is nonlocal and it depends significantly on its lower limit. As it is mentioned in [13], this leads to some obstacles for studying impulsive fractional differential equations.

Since many phenomena are characterized by abrupt changes at certain moments it is important to consider differential equations with impulses. In the literature there are two main approaches used to introduce impulses to fractional equations:

(i) With a fixed lower limit of the fractional derivative at the initial time- the fractional derivative of the unknown function has a lower limit equal to the initial time point over the whole interval of study;
(ii) With a changeable lower limit of the fractional derivative at each time of impulse- the fractional derivative on each interval between two consecutive impulses is changed because the lower limit of the fractional derivative is equal to the time of impulse.

Both interpretations of impulses are based on corresponding interpretations of impulses in ordinary differential equation, which coincide in the case of integer derivatives. However this is not the case for fractional derivatives. In the literature both types of interpretations are discussed and studied for Caputo fractional differential equations of order $\alpha \in (0, 1)$. We refer the reader to the papers [6, 7, 12, 13, 16] as well as the monograph [3].

We note in the case of the Caputo fractional derivative there is a similarity between both the initial conditions and the impulsive condition between fractional equations and ordinary equations (see, for example, [10] concerning the impulsive control law for the Caputo delay fractional-order neural network model). However for Riemann-Liouville fractional differential equations both the initial condition and impulsive conditions have to be appropriately given (which is different in the case of ordinary derivatives as well as the case of Caputo fractional derivatives). Riemann-Liouville fractional differential equations are considered, for example, in [1, 2] for integral presentation of solutions in the case of the fractional order $\alpha \in (0, 1)$, [5, 8, 17] for the case of the fractional order $\alpha \in (1, 2)$.

In [14] the authors studied the following coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives of the form

$$\begin{cases} \begin{cases} \mathcal{D}^\alpha u(t) - \phi_1(t, \mathcal{I}^\alpha u(t), \mathcal{I}^\beta v(t)) = 0 \quad \text{for } t \in I,\ t \neq t_i,\ \ i = 1, 2, \dots, p, \\ \Delta u(t_j) - \mathcal{E}_j(u(t_j)) = 0, \qquad \Delta u'(t_j) - \mathcal{E}^*_j(u(t_j)) = 0,\ \ j = 1, 2, \dots, p, \\ \nu_1 \mathcal{D}^{\alpha-2} u(t)|_{t=0} = u_1, \qquad \mu_1 u(t)|_{t=T} + \nu_2 \mathcal{I}^{\alpha-1} u(t)|_{t=T} = u_2, \end{cases} \\ \begin{cases} \mathcal{D}^\beta v(t) - \phi_2(t, \mathcal{I}^\alpha u(t), \mathcal{I}^\beta v(t)) = 0 \quad \text{for } t \in I,\ t \neq t_k,\ \ k = 1, 2, \dots, q, \\ \Delta v(t_k) - \mathcal{E}_k(v(t_k)) = 0, \qquad \Delta v'(t_k) - \mathcal{E}^*_j(v(t_k)) = 0,\ \ k = 1, 2, \dots, q, \\ \nu_3 \mathcal{D}^{\beta-2} v(t)|_{t=0} = v_1, \qquad \mu_2 v(t)|_{t=T} + \nu_4 \mathcal{I}^{\beta-1} v(t)|_{t=T} = v_2, \end{cases} \end{cases} \tag{1.1}$$

where $\alpha, \beta \in (1, 2]$, $I = [0, T]$, $\phi_1, \phi_2 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\Delta u(t_j) = u(t_j^+) - u(t_j^-)$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, $\Delta v(t_k) = v(t_k^+) - v(t_k^-)$, $\Delta v'(t_j) = v'(t_k^+) - v'(t_k^-)$, where $u(t_j^+), v(t_k^+)$ and $v(t_j^-), v(t_k^-)$ are the right limits and left limits respectively, $\mathcal{E}_j, \mathcal{E}^*_j, \mathcal{E}_k, \mathcal{E}^*_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\mathcal{D}^\alpha, \mathcal{I}^\alpha$ are the α-order Riemann-Liouville fractional derivative and integral operators respectively and $\mathcal{D}^{\beta-2} = \mathcal{I}^{2-\beta}$.

Since fractional integrals and derivatives have memories, and their lower limits are very important we will use the notations ${}^{RL}D^\alpha_{a,t}$ and $I^\beta_{a,t}$, respectively, instead of $\mathcal{D}^\alpha$ and $\mathcal{I}^\beta$, i.e. the Riemann-Liouville fractional derivative is defined by (see, for example [9])

$$ {}^{RL}D^{\alpha}_{a,t}u(t) = \frac{1}{\Gamma(2-\alpha)}\Big(\frac{d}{dt}\Big)^2 \int_a^t \frac{u(s)}{(t-s)^{\alpha-1}}ds, \quad t > a, \ \ \alpha \in (1,2), \tag{1.2} $$

and the Riemann-Liouville fractional integral $I^{\beta}_{a,t}$ of order $\alpha > 0$ is defined by (see, for example, [9])

$$ I^{\beta}_{a,t}u(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1}u(s)ds, t > a, \tag{1.3} $$

where $a \geq 0, \beta > 0$ are given numbers.

Note there are some unclear parts in the statement of coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives (1.1), such as:

a). The presence of two different integers p and q in (1.1) leads to different domains of both the unknown functions u and v. For example, in Corollary 1 [14] the solutions $u(t)$ and $v(t)$ are defined on $[0, t_{p+1}]$ and $[0, t_{q+1}]$ respectively, which causes some problems in the definitions of formulas (3.7) or (3.8) ([14]);

b). The impulsive functions $\mathcal{E}_j, \mathcal{E}_k, \ j = 1, 2, \dots, p, \ k = 1, 2, \dots, q$ are assumed different but they are not (it is clear for example, for $j = k = 1$). The same is about the functions $\mathcal{E}^*_j, \mathcal{E}^*_k, \ j = 1, 2, \dots, p, \ k = 1, 2, \dots, q$.

In this paper we sort out the above mentioned points by setting up the cleared statement of the boundary value problem with the Riemann-Liouville (RL) fractional integral for the impulsive Riemann-Liouville fractional differential equation studied in [14], and we prove a new the integral presentations of the solutions. To be more general, we study two different interpretations for the presence of impulses in fractional differential equations. The first one is the case of the fixed lower limit of the RL fractional derivative at the initial time 0 and the second one is the case of the changed lower limit of the fractional RL derivative at any point of impulse. In both cases the integral presentation of the solution is provided.

2. RL-fractional derivatives with fixed lower limit

2.1. Statement of the problem

Define two different sequences of points of impulses

$$ 0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T \ \text{ and } \ 0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau_{q+1} = T, $$

where p, q are given natural numbers.

We will consider the following nonlinear boundary value problem for the coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives with a lower limit at 0

$$ \begin{cases} \begin{cases} {}^{RL}D^{\alpha}_{0,t}u(t) - \phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)) = 0 \quad \text{for } t \in I, \ t \neq t_i, \ \ i = 1, 2, \dots, p, \\ \Delta u(t_j) - \mathcal{E}_j(u(t_j)) = 0, \qquad \Delta u'(t_j) - \mathcal{E}^*_j(u(t_j)) = 0, \ \ j = 1, 2, \dots, p, \\ -\mathcal{I}^{2-\alpha}_{0,t}u(t)|_{t=0} = u_1, \quad \mu_1 u(t)|_{t=T} + \nu_1 \mathcal{I}^{\alpha-1}_{0,t}u(t)|_{t=T} = u_2, \end{cases} \\ \begin{cases} {}^{RL}D^{\beta}_{0,t}v(t) - \phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)) = 0 \quad \text{for } t \in I, \ t \neq \tau_k, \ \ k = 1, 2, \dots, q, \\ \Delta v(\tau_k) - \mathcal{S}_k(v(\tau_k)) = 0, \qquad \Delta v'(\tau_k) - \mathcal{S}^*_k(v(\tau_k)) = 0, \ \ k = 1, 2, \dots, q, \\ \mathcal{I}^{2-\beta}_{0,t}v(t)|_{t=0} = v_1, \quad \mu_2 v(t)|_{t=T} + \nu_2 \mathcal{I}^{\beta-1}_{0,t}v(t)|_{t=T} = v_2, \end{cases} \end{cases} \tag{2.1} $$

where $\alpha,\beta \in (1,2]$, $I = [0,T]$, $\phi_1,\phi_2 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\Delta u(t_j) = u(t_j^+) - u(t_j^-)$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, $\Delta v(\tau_k) = v(\tau_k^+) - v(\tau_k^-)$, $\Delta v'(\tau_k) = v'(\tau_k^+) - v'(\tau_k^-)$, where $u(t_j^+), v(\tau_k^+), u'(t_j^+), v'(\tau_k^+)$ and $u(t_j^-)$, $v(t_k^-)$, $u'(t_j^-)$, $v'(t_k^-)$ are the right limits and left limits respectively, $\mathcal{E}_j, \mathcal{E}_j^*, \mathcal{S}_k, \mathcal{S}_k^* : \mathbb{R} \to \mathbb{R}$ are continuous functions, and ${}^{RL}D^{\beta}_{0,t}, \mathcal{I}^{\alpha}_{0,t}$ are the α-order Riemann-Liouville fractional derivative and integral operators, respectively, $\mu_i, \nu_i, u_k, v_k,\ i = 1,2$, are given constants.

In the statement of the problem (2.1) some parts of (1.1) are cleared: there are two different points of impulses; the lower limits of the fractional integrals and fractional integrals are written; different functions at different points of impulses are used.

In our proofs we will use the following well known properties for fractional integrals (see, for example [9]).

$$
\begin{aligned}
&I^{\alpha}_{a,t} I^{\beta}_{a,t} u(t) = I^{\alpha+\beta}_{a,t} u(t), \quad \alpha,\beta > 0,\\
&I^{\alpha}_{a,t}(t-a)^q = \frac{\Gamma(q+1)}{\Gamma(q+\alpha+1)}(t-a)^{q+\alpha}, \ \ \alpha > 0,\ q > -1.\ t > a.
\end{aligned}
\tag{2.2}
$$

We will apply the following auxiliary result which is a generalization of the result in [4] for an arbitrary lower limit of the fractional derivative:

Lemma 1. *([4]). The general solution of the Riemann-Liouville fractional differential equation*

$$
{}^{RL}D^{\alpha}_{a,t} w(t) = g(t), \quad t \in (a,T], \quad \alpha \in (1,2) \tag{2.3}
$$

is given by

$$
\begin{aligned}
w(t) &= c_1(t-a)^{\alpha-1} + c_0(t-a)^{\alpha-2} + I^{\alpha}_{a,t}\, g(t)\\
&= c_1(t-a)^{\alpha-1} + c_0(t-a)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} g(s)ds,\ t \in (a,T],
\end{aligned}
\tag{2.4}
$$

where $c_0, c_1, a \geq 0$ are arbitrary real constants.

2.2. Integral presentation of the linear problem

We will consider an appropriate boundary value problem for a scalar impulsive linear equation, we will prove a formula for its solution and later we will apply it to obatin the main result.

Consider the following boundary value problem for the linear impulsive fractional differential equation with Riemann-Liouville derivatives of the form

$$
\begin{aligned}
&{}^{RL}D^{\alpha}_{0,t} u(t) = f(t), \quad t \in (0,T],\ \ t \neq t_j,\ j = 1,2,\ldots p,\ \ \alpha \in (1,2),\\
&\Delta u(t_j) = \mathcal{E}_j(u(t_j)), \qquad \Delta u'(t_j) = \mathcal{E}_j^*(u(t_j))\ \ j = 1,2,\ldots,p,\\
&I^{2-\alpha}_{0,t} u(t)|_{t=0} = u_1, \quad \mu_1 u(t)|_{t=T} + \nu_1 I^{\alpha-1}_{0,t} u(t)|_{t=T} = u_2,
\end{aligned}
\tag{2.5}
$$

where $f:\ [0,T] \to \mathbb{R}$ is a continuous function, $\mathcal{E}_j, \mathcal{E}_j^* : \mathbb{R} \to \mathbb{R}$ are continuous functions, $u_1, u_2 \in \mathbb{R}$.

Lemma 2. *The solution of (2.5) satisfies the integral equation*

$$u(t)=\begin{cases} c_0t^{\alpha-1}+\frac{u_1}{\Gamma(\alpha-1)}t^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds, & t\in(0,t_1],\\ c_0t^{\alpha-1}+\Big(\sum_{k=1}^{j}[t_k^{2-\alpha}\mathcal{E}_k^*(u(t_k))]-(\alpha-2)\sum_{k=1}^{j}[t_k^{1-\alpha}\mathcal{E}_k(u(t_k))]\Big)t^{\alpha-1} & \\ \quad+\Big(\frac{u_1}{\Gamma(\alpha-1)}+(\alpha-1)\sum_{k=1}^{j}t_k^{2-\alpha}\mathcal{E}_k(u(t_k))-\sum_{k=1}^{j}t_k^{3-\alpha}\mathcal{E}_k^*(u(t_k))\Big)t^{\alpha-2} & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds, & t\in(t_j,t_{j+1}],\end{cases}$$

where

$$\begin{aligned} c_0=&-\Big(\sum_{k=1}^{p+1}[t_k^{2-\alpha}\mathcal{E}_k^*(u(t_k))]-(\alpha-2)\sum_{k=1}^{p+1}[t_k^{1-\alpha}\mathcal{E}_k(u(t_k))]\Big)\\ &-\Big(\frac{u_1}{\Gamma(\alpha-1)}+(\alpha-1)\sum_{k=1}^{p+1}t_k^{2-\alpha}\mathcal{E}_k(u(t_k))-\sum_{k=1}^{p+1}t_k^{3-\alpha}\mathcal{E}_k^*(u(t_k))\Big)T^{-1}\\ &-\frac{1}{T^{\alpha-1}\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s)ds\\ &-\frac{\nu_1}{\mu_1T^{\alpha-1}\Gamma(\alpha-1)}\int_0^T(T-s)^{\alpha-2}u(s)ds+\frac{u_2}{\mu_1}. \end{aligned}\tag{2.6}$$

Proof. We will use induction.

For $t\in(0,t_1]$ we apply Lemma 1 with $a=0$ and we get

$$u(t)=c_0t^{\alpha-1}+c_1t^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds\tag{2.7}$$

and

$$u'(t)=c_0(\alpha-1)t^{\alpha-2}+c_1(\alpha-2)t^{\alpha-3}+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-2}f(s)ds.\tag{2.8}$$

From the initial condition $I_{0,t}^{2-\alpha}u(t)|_{t=0}=u_1$ and equalities (2.2), (2.7) we get

$$I_{0,t}^{2-\alpha}u(t)|_{t=0}=c_0\frac{\Gamma(\alpha)}{\Gamma(1)}t|_{t=0}+c_1\frac{\Gamma(\alpha-1)}{\Gamma(0)}|_{t=0}+I_{0,t}^{2}f(t)|_{t=0}=c_1\Gamma(\alpha-1),$$

i.e. $c_1=\frac{u_1}{\Gamma(\alpha-1)}$.

For $t\in(t_1,t_2]$ by Lemma 1 with $a=0$ we get

$$u(t)=b_0t^{\alpha-1}+b_1t^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds\tag{2.9}$$

and

$$u'(t)=b_0(\alpha-1)t^{\alpha-2}+b_1(\alpha-2)t^{\alpha-3}+\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s)ds.\tag{2.10}$$

From the impulsive condition $u(t_1+0)-u(t_1-0)=\mathcal{E}_1(u(t_1))$ we obtain

$$\begin{aligned} u(t_1^+)=&b_0t_1^{\alpha-1}+b_1t_1^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds\\ &-c_0t_1^{\alpha-1}-\frac{u_1}{\Gamma(\alpha-1)}t_1^{\alpha-2}-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds\\ =&(b_0-c_0)t_1^{\alpha-1}+(b_1-\frac{u_1}{\Gamma(\alpha-1)})t_1^{\alpha-2}=\mathcal{E}_1(u(t_1)). \end{aligned}\tag{2.11}$$

and from the impulsive condition $u'(t_1+0)-u'(t_1)=\mathcal{E}_1^*(u(t_1))$ we get

$$\begin{aligned}u'(t_1^+)=&b_0(\alpha-1)t_1^{\alpha-2}+b_1(\alpha-2)t_1^{\alpha-3}+\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}f(s)ds\\&-c_0(\alpha-1)t_1^{\alpha-2}-\frac{u_1}{\Gamma(\alpha-1)}(\alpha-2)t_1^{\alpha-3}-\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}f(s)ds\\=&(b_0-c_0)(\alpha-1)t_1^{\alpha-2}+(b_1-\frac{u_1}{\Gamma(\alpha-1)})(\alpha-2)t_1^{\alpha-3}=\mathcal{E}_1^*(u(t_1)).\end{aligned}\tag{2.12}$$

Thus we get the linear system w.r.t. b_0 and b_1

$$\begin{aligned}&(b_0-c_0)(\alpha-1)t_1^{\alpha-2}+(b_1-\frac{u_1}{\Gamma(\alpha-1)})(\alpha-2)t_1^{\alpha-3}=\mathcal{E}_1^*(u(t_1))\\&(b_0-c_0)t_1^{\alpha-1}+(b_1-\frac{u_1}{\Gamma(\alpha-1)})t_1^{\alpha-2}=\mathcal{E}_1(u(t_1))\end{aligned}$$

or

$$\begin{aligned}b_0&=c_0+t_1^{2-\alpha}\mathcal{E}_1^*(u(t_1))-(\alpha-2)t_1^{1-\alpha}\mathcal{E}_1(u(t_1))\\b_1&=(\alpha-1)t_1^{2-\alpha}\mathcal{E}_1(u(t_1))-t_1^{3-\alpha}\mathcal{E}_1^*(u(t_1))+\frac{u_1}{\Gamma(\alpha-1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}u(t)=&\,c_0t^{\alpha-1}+\Big(t_1^{2-\alpha}\mathcal{E}_1^*(u(t_1))-(\alpha-2)t_1^{1-\alpha}\mathcal{E}_1(u(t_1))\Big)t^{\alpha-1}\\&+\Big((\alpha-1)t_1^{2-\alpha}\mathcal{E}_1(u(t_1))-t_1^{3-\alpha}\mathcal{E}_1^*(u(t_1))+\frac{u_1}{\Gamma(\alpha-1)}\Big)t^{\alpha-2}\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad t\in(t_1,t_2].\end{aligned}\tag{2.13}$$

Assume the integral presentation of $u(t)$ is correct on $(t_{j-1},t_j]$, i.e

$$\begin{aligned}u(t)=&\,c_0t^{\alpha-1}+\Big(\sum_{k=1}^{j-1}t_k^{2-\alpha}\mathcal{E}_k^*(u(t_k))-(\alpha-2)\sum_{k=1}^{j-1}t_k^{1-\alpha}\mathcal{E}_k(u(t_k))\Big)t^{\alpha-1}\\&+\Big((\alpha-1)\sum_{k=1}^{j-1}t_k^{2-\alpha}\mathcal{E}_k(u(t_k))-\sum_{k=1}^{j-1}t_k^{3-\alpha}\mathcal{E}_k^*(u(t_k))+\frac{u_1}{\Gamma(\alpha-1)}\Big)t^{\alpha-2}\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad t\in(t_{j-1},t_j].\end{aligned}\tag{2.14}$$

Denote

$$m_0=c_0+\sum_{k=1}^{j-1}t_k^{2-\alpha}\mathcal{E}_k^*(u(t_k))-(\alpha-2)\sum_{k=1}^{j-1}t_k^{1-\alpha}\mathcal{E}_k(u(t_k))$$

and

$$m_1=(\alpha-1)\sum_{k=1}^{j-1}t_k^{2-\alpha}\mathcal{E}_k(u(t_k))-\sum_{k=1}^{j-1}t_k^{3-\alpha}\mathcal{E}_k^*(u(t_k))+\frac{u_1}{\Gamma(\alpha-1)}.$$

Let $t \in (t_j, t_{j+1}]$, $j = 2, \ldots, p$,. By Lemma 1 with $a = 0$ we get

$$u(t) = k_0 t^{\alpha-1} + k_1 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.15}$$

and

$$u'(t) = k_0(\alpha-1) t^{\alpha-2} + k_1(\alpha-2) t^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s) ds. \tag{2.16}$$

From the impulsive conditions and the equality (2.14) we obtain the linear system w.r.t. k_0 and k_1

$$\begin{aligned} &(k_0 - m_0)(\alpha-1) t_j^{\alpha-2} + (k_1 - m_1)(\alpha-2) t_j^{\alpha-3} = \mathcal{E}_2^*(u(t_2)) \\ &(k_0 - m_0) t_j^{\alpha-1} + (k_1 - m_1) t_j^{\alpha-2} = \mathcal{E}_j(u(t_j)) \end{aligned}$$

or

$$k_0 = c_0 + \sum_{k=1}^{j} [t_k^{2-\alpha} \mathcal{E}_k^*(u(t_k))] - (\alpha-2) \sum_{k=1}^{j} [t_k^{1-\alpha} \mathcal{E}_k(u(t_k))]$$

$$k_1 = \frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{j} t_k^{2-\alpha} \mathcal{E}_k(u(t_k)) - \sum_{k=1}^{j} t_k^{3-\alpha} \mathcal{E}_k^*(u(t_k)).$$

Therefore,

$$\begin{aligned} u(t) = {} & c_0 t^{\alpha-1} + \Big(\sum_{k=1}^{j} [t_k^{2-\alpha} \mathcal{E}_k^*(u(t_k))] - (\alpha-2) \sum_{k=1}^{j} [t_k^{1-\alpha} \mathcal{E}_k(u(t_k))] \Big) t^{\alpha-1} \\ & + \Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{j} t_k^{2-\alpha} \mathcal{E}_k(u(t_k)) - \sum_{k=1}^{j} t_k^{3-\alpha} \mathcal{E}_k^*(u(t_k)) \Big) t^{\alpha-2} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t \in (t_j, t_{j+1}],\ j = 1, 2, \ldots, p. \end{aligned} \tag{2.17}$$

From the boundary condition $\mu_1 u(t)|_{t=T} + \nu_1 \mathcal{I}^{\alpha-1} u(t)|_{t=T} = u_2$ we get

$$\mathcal{I}^{\alpha-1} u(t)|_{t=T} = \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} u(s) ds$$

and

$$\begin{aligned} & \mu_1 u(t)|_{t=T} + \nu_1 \mathcal{I}^{\alpha-1} u(t)|_{t=T} \\ & = \mu_1 c_0 T^{\alpha-1} + \mu_1 \Big(\sum_{k=1}^{p+1} [t_k^{2-\alpha} \mathcal{E}_k^*(u(t_k))] - (\alpha-2) \sum_{k=1}^{p+1} [t_k^{1-\alpha} \mathcal{E}_k(u(t_k))] \Big) T^{\alpha-1} \\ & \quad + \mu_1 \Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{p+1} t_k^{2-\alpha} \mathcal{E}_k(u(t_k)) - \sum_{k=1}^{p+1} t_k^{3-\alpha} \mathcal{E}_k^*(u(t_k)) \Big) T^{\alpha-2} \\ & \quad + \mu_1 \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_1(s, I_{0,s}^{\alpha} u(s), I_{0,s}^{\beta} v(s)) ds \\ & \quad + \nu_1 \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} u(s) ds \\ & = u_2. \end{aligned} \tag{2.18}$$

From (2.18) we have (2.6).

□

We will give an example to illustrate the claim of Lemma 2.

Example 1. *Consider the following boundary value problem for the scalar RL fractional differential equation with an impulse at* $t = 1$

$$\begin{aligned} &{}^{RL}D^{1.5}_{0,t}u(t) = t, \quad t \in (0,1] \cup (1,2], \\ &\Delta u(1) = 1, \qquad \Delta u'(1) = 0, \\ &I^{0.5}_{0,t}u(t)|_{t=0} = 0, \qquad u(t)|_{t=2} + I^{0.5}_{0,t}u(t)|_{t=2} = 1. \end{aligned} \tag{2.19}$$

The solution of (2.19) satisfies the integral equation

$$u(t) = \begin{cases} c_0 t^{0.5} + \frac{t^{-0.5}}{\Gamma(0.5)} + \frac{0.266667t^{2.5}}{\Gamma(1.5)}, & t \in (0,1] \\ c_0 t^{0.5} + 0.5t^{0.5} + \Big(\frac{1}{\Gamma(0.5)} + 0.5\Big)t^{-0.5} + \frac{0.266667t^{2.5}}{\Gamma(1.5)}, & t \in (1,2], \end{cases}$$

where

$$c_0 = 0.25 - \frac{1.50849}{2^{0.5}\Gamma(1.5)} - \frac{1}{2^{0.5}\Gamma(0.5)} \int_0^2 (2-s)^{-0.5}u(s)ds. \tag{2.20}$$

Consider the boundary value problem for the nonlinear impulsive fractional integro-differential equations with Riemann-Liouville derivatives of the form

$$\begin{aligned} &{}^{RL}D^{\alpha}_{0,t}u(t) = \phi_1(t, I^{\alpha}_{0,t}u(t)), \quad t \in (0,T], \ t \neq t_j, \ j = 1,2,\dots p, \ \alpha \in (1,2) \\ &\Delta u(t_j) = \mathcal{E}_j(u(t_j)), \qquad \Delta u'(t_j) = \mathcal{E}^*_j(u(t_j)) \ \ j = 1,2,\dots,p, \\ &I^{2-\alpha}_{0,t}u(t)|_{t=0} = u_1, \qquad \mu_1 u(t)|_{t=T} + \nu_1 I^{\alpha-1}_{0,t}u(t)|_{t=T} = u_2, \end{aligned} \tag{2.21}$$

where $\phi_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{E}_j, \mathcal{E}^*_j : \mathbb{R} \to \mathbb{R}$ are continuous functions, $u_1, u_2 \in \mathbb{R}$.

Corollary 1. *The solution of (2.21) satisfies the integral equation*

$$u(t) = \begin{cases} c_0 t^{\alpha-1} + \frac{u_1}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_1(s, I^{\alpha}_{0,s}u(s),)ds, \quad t \in (0,t_1] \\ c_0 t^{\alpha-1} + \Big(\sum_{k=1}^{j}[t_k^{2-\alpha}\mathcal{E}^*_k(u(t_k))] - (\alpha-2)\sum_{k=1}^{j}[t_k^{1-\alpha}\mathcal{E}_k(u(t_k))]\Big)t^{\alpha-1} \\ \quad +\Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1)\sum_{k=1}^{j} t_k^{2-\alpha}\mathcal{E}_k(u(t_k)) - \sum_{k=1}^{j} t_k^{3-\alpha}\mathcal{E}^*_k(u(t_k))\Big)t^{\alpha-2} \\ \quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_1(s, I^{\alpha}_{0,s}u(s))ds, \qquad t \in (t_j, t_{j+1}], \end{cases} \tag{2.22}$$

where

$$\begin{aligned} c_0 = &-\Big(\sum_{k=1}^{p+1}[t_k^{2-\alpha}\mathcal{E}^*_k(u(t_k))] - (\alpha-2)\sum_{k=1}^{p+1}[t_k^{1-\alpha}\mathcal{E}_k(u(t_k))]\Big) \\ &-\Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1)\sum_{k=1}^{p+1} t_k^{2-\alpha}\mathcal{E}_k(u(t_k)) - \sum_{k=1}^{p+1} t_k^{3-\alpha}\mathcal{E}^*_k(u(t_k))\Big)T^{-1} \\ &-\frac{1}{T^{\alpha-1}\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\phi_1(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds \\ &-\frac{\nu_1}{\mu_1 T^{\alpha-1}\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}u(s)ds + \frac{u_2}{\mu_1}. \end{aligned} \tag{2.23}$$

The proof of Corollary 1 is similar to the one of Lemma 2 with $f(t) = \phi_1(t, I^{\alpha}_{0,t}u(t))$ and we omit it.

Remark 1. *Corollary 1 and the integral presentation (2.22) correct Theorem 3.1 and the formula (3.2) [14]. The main mistake in the proof of formula (3.2) [14] is the incorrect application of Lemma 1 with* $a = 0$ *on* $(t_1, t_2]$ *and taking the lower limit of the integral in (3.5) [14] incorrectly at* $t_1(\sigma_1)$ *instead of 0. A similar comment applies to all the other intervals* $(t_j, t_{j+1}]$.

2.3. Integral presentation of the problem (2.1)

Following the proof of Lemma 2 and the integral presentation (2.22) of problem (2.21), we have the following result:

Theorem 1. *The solution of (2.1) satisfies the integral equations*

$$u(t) = \begin{cases} c_0 t^{\alpha-1} + \frac{u_1}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_1(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds, \quad t \in (0, t_1], \\ c_0 t^{\alpha-1} + \Big(\sum_{k=1}^{j} [t_k^{2-\alpha} \mathcal{E}_k^*(u(t_k))] - (\alpha-2) \sum_{k=1}^{j} [t_k^{1-\alpha} \mathcal{E}_k(u(t_k))] \Big) t^{\alpha-1} \\ \quad + \Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{j} t_k^{2-\alpha} \mathcal{E}_k(u(t_k)) - \sum_{k=1}^{j} t_k^{3-\alpha} \mathcal{E}_k^*(u(t_k)) \Big) t^{\alpha-2} \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_1(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds, \quad t \in (t_j, t_{j+1}], \end{cases} \tag{2.24}$$

and

$$v(t) = \begin{cases} b_0 t^{\alpha-1} + \frac{v_1}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_2(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds, \quad t \in (0, t_1] \\ b_0 t^{\alpha-1} + \Big(\sum_{k=1}^{j} [t_k^{2-\alpha} \mathcal{S}_k^*(v(t_k))] - (\alpha-2) \sum_{k=1}^{j} [t_k^{1-\alpha} \mathcal{S}_k(v(t_k))] \Big) t^{\alpha-1} \\ \quad + \Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{j} t_k^{2-\alpha} \mathcal{S}_k(v(t_k)) - \sum_{k=1}^{j} t_k^{3-\alpha} \mathcal{S}_k^*(v(t_k)) \Big) t^{\alpha-2} \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_2(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds, \quad t \in (t_j, t_{j+1}], \end{cases} \tag{2.25}$$

where

$$\begin{aligned} c_0 = & -\Big(\sum_{k=1}^{p+1} [t_k^{2-\alpha} \mathcal{E}_k^*(u(t_k))] - (\alpha-2) \sum_{k=1}^{p+1} [t_k^{1-\alpha} \mathcal{E}_k(u(t_k))] \Big) \\ & - \Big(\frac{u_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{p+1} t_k^{2-\alpha} \mathcal{E}_k(u(t_k)) - \sum_{k=1}^{p+1} t_k^{3-\alpha} \mathcal{E}_k^*(u(t_k)) \Big) T^{-1} \\ & - \frac{1}{T^{\alpha-1}\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_1(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds \\ & - \frac{\nu_1}{\mu_1 T^{\alpha-1}\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} u(s)ds + \frac{u_2}{\mu_1} \\ b_0 = & -\Big(\sum_{k=1}^{p+1} [t_k^{2-\alpha} \mathcal{S}_k^*(v(t_k))] - (\alpha-2) \sum_{k=1}^{p+1} [t_k^{1-\alpha} \mathcal{S}_k(v(t_k))] \Big) \\ & - \Big(\frac{v_1}{\Gamma(\alpha-1)} + (\alpha-1) \sum_{k=1}^{p+1} t_k^{2-\alpha} \mathcal{S}_k(v(t_k)) - \sum_{k=1}^{p+1} t_k^{3-\alpha} \mathcal{S}_k^*(v(t_k)) \Big) T^{-1} \\ & - \frac{1}{T^{\alpha-1}\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \phi_2(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds \\ & - \frac{\nu_2}{\mu_2 T^{\alpha-1}\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} v(s)ds + \frac{v_2}{\mu_2}. \end{aligned} \tag{2.26}$$

The proof is similar to the one of Lemma 2 applied twice to each of the both components u and v of the coupled system (2.1) for impulsive points $t_i,\ i = 1, 2, \dots, p$ and $\tau_i,\ i = 1, 2, \dots, p$ and the functions $f(t) = \phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))$ and $f(t) = \phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))$ respectively.

Remark 2. *Note the integral presentation (2.24), (2.25) of the solutions of the coupled system is the correction of Corollary 1 and integral presentation (3.7), (3.8) in [14].*

3. RL fractional derivatives with changeable lower limits

3.1. Statement of the problem with changeable lower limit of the RL fractional derivatives

Consider the following nonlinear boundary value problem for the coupled impulsive fractional integro-differential equations with Riemann-Liouville derivatives with lower limits at impulsive points $t_i,\ i = 0, 1, 2, \dots, p-1$ and $\tau_k,\ k = 0, 1, 2, \dots, q-1$, respectively,

$$\begin{cases} \begin{cases} {}^{RL}D^{\alpha}_{t_i,t}u(t) - \phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)) = 0 \quad \text{for } t \in (t_i, t_{i+1}] \ \ i = 0, 1, 2, \dots, p, \\ I^{2-\alpha}_{t_j,t}u'(t)|_{t=t_j} = \mathcal{P}_j u(t_j) + \mathcal{Q}_j, \quad j = 1, 2, \dots, p, \\ I^{2-\alpha}_{0,t}u(t)|_{t=0} = u_1, \qquad \mu_1 u(t)|_{t=T} + \nu_1 I^{\alpha-1}_{0,t}u(t)|_{t=T} = u_2, \end{cases} \\ \begin{cases} {}^{RL}D^{\beta}_{\tau_k,t}v(t) - \phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)) = 0 \quad \text{for } t \in (\tau_k, \tau_{k+1}] \ \ k = 0, 1, 2, \dots, q, \\ I^{2-\alpha}_{\tau_k,t}u'(t)|_{t=\tau_k} = \mathcal{P}^*_k v(\tau_k) + \mathcal{Q}_j k^*, \quad k = 1, 2, \dots, q, \\ I^{2-\beta}_{0,t}v(t)|_{t=0} = v_1, \qquad \mu_2 v(t)|_{t=T} + \nu_2 I^{\beta-1}_{0,t}v(t)|_{t=T} = v_2, \end{cases} \end{cases} \tag{3.1}$$

where $\alpha, \beta \in (1, 2]$, $\phi_1, \phi_2 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\mathcal{P}_j, \mathcal{Q}_j,\ j = 1, 2, \dots, p$, and $\mathcal{P}^*_j, \mathcal{Q}^*_j,\ j = 1, 2, \dots, q$, are real numbers, ${}^{RL}D^{\alpha}_{t_i,t}$ and ${}^{RL}D^{\beta}_{\tau_k,t}$ are the α-order Riemann-Liouville fractional derivatives with lower limits at t_i and τ_k, respectively, $\mu_k, \nu_k, u_k, v_k,\ i = 1, 2$, are given constants.

Remark 3. *Note problem (3.1) differs from problem (2.1):*

- *The lower limits of the RL fractional derivatives ${}^{RL}D^{\alpha}_{t_j,t}$ and ${}^{RL}D^{\beta}_{\tau_k,t}$ in (3.1) are changed at any time of impulse t_j and τ_k, respectively.*
- *The impulsive conditions are changed in (3.1). This is because the values of the unknown functions after the impulse, $u(t_j + 0)$ and $v(\tau_k + 0)$, respectively, are considered as initial values at that point. But the RL fractional derivative has a singularity at its lower limit. It requires the chang of the impulsive conditions for the unknown functions.*

3.2. Integral presentation of the linear problem

Consider the following boundary value problem for the scalar linear impulsive fractional equation with Riemann-Liouville derivatives of the form

$$\begin{aligned} &{}^{RL}D^{\alpha}_{t_i,t}u(t) = f(t), \quad \text{for } t \in (t_i, t_{i+1}], \ \ i = 0, 1, 2, \dots, p, \ \ \alpha \in (1, 2) \\ &I^{2-\alpha}_{t_j,t}u'(t)|_{t=t_j} = \mathcal{P}_j u(t_j) + \mathcal{Q}_j, \quad j = 1, 2, \dots, p, \\ &I^{2-\alpha}_{0,t}u(t)|_{t=0} = u_1, \qquad \mu_1 u(t)|_{t=T} + \nu_1 I^{\alpha-1}_{0,t}u(t)|_{t=T} = u_2, \end{aligned} \tag{3.2}$$

where the function $f : [0, T] \to \mathbb{R}$ is a continuous function, $\mathcal{P}_j, \mathcal{Q}_j,\ j = 1, 2, \dots, p$ are real numbers, $u_1, u_2 \in \mathbb{R}$.

Now we will provide an integral presentation of the solution of (3.2).

Lemma 3. *The solution of (3.2) satisfies the integral equation*

$$u(t)=\begin{cases} c_0 t^{\alpha-1}+\frac{u_1}{\Gamma(\alpha-1)}t^{\alpha-2}+I^{\alpha}_{0,t}f(t), \quad t\in(0,t_1] \\ \left(c_0+\frac{u_1}{t_1\Gamma(\alpha-1)}\right)(t-t_m)^{\alpha-1}\prod_{k=1}^{m}\frac{\mathcal{P}_k(t_k-t_{k-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\ \quad +I^{\alpha}_{t_m,t}f(t)+(t-t_m)^{\alpha-1}\sum_{k=1}^{m}\frac{\mathcal{P}_k I^{\alpha}_{t_{k-1},t}f(t)|_{t=t_k}+\mathcal{Q}_k}{(\alpha-1)\Gamma(\alpha-1)}\prod_{j=k+1}^{m}\frac{\mathcal{P}_j(t_j-t_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)}, \\ \quad for \quad t\in(t_m,t_{m+1}], m=1,2,\ldots,p, \end{cases}$$

where

$$\begin{aligned} c_0 &= \frac{u_2}{\mu_1}(T-t_p)^{1-\alpha}\mathcal{M}-\frac{u_1}{t_1\Gamma(\alpha-1)}-(T-t_p)^{1-\alpha}\mathcal{M}I^{\alpha}_{t_p,t}f(t)|_{t=T} \\ &\quad -\mathcal{M}\sum_{k=1}^{p}\frac{\mathcal{P}_k I^{\alpha}_{t_{k-1},t}f(t)|_{t=t_k}+\mathcal{Q}_k}{(\alpha-1)\Gamma(\alpha-1)}\prod_{j=k+1}^{m}\frac{\mathcal{P}_j(t_j-t_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\ &\quad -\frac{\nu_1}{\mu_1}(T-t_p)^{1-\alpha}\mathcal{M}\frac{1}{\Gamma(\alpha-1)}\int_0^T(T-s)^{\alpha-2}u(s)ds, \\ \mathcal{M} &= (\alpha-1)^p\Gamma^p(\alpha-1)\prod_{k=1}^{p}\mathcal{P}_k(t_k-t_{k-1})^{1-\alpha}. \end{aligned} \tag{3.3}$$

Proof. We will use induction.

For $t\in(0,t_1]$ similar to Lemma 2 we get

$$u(t)=c_0t^{\alpha-1}+c_1t^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds \tag{3.4}$$

where $c_1=\frac{u_1}{\Gamma(\alpha-1)}$.

For $t\in(t_1,t_2]$ by Lemma 1 with $a=t_1$ we get

$$u(t)=b_0(t-t_1)^{\alpha-1}+b_1(t-t_1)^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}f(s)ds \tag{3.5}$$

and

$$u'(t)=b_0(\alpha-1)(t-t_1)^{\alpha-2}+b_1(\alpha-2)(t-t_1)^{\alpha-3}+\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t(t-s)^{\alpha-2}f(s)ds. \tag{3.6}$$

From the impulsive condition $I^{2-\alpha}_{t_1,t}u'(t)|_{t=t_1}=\mathcal{P}_1u(t_1)+\mathcal{Q}_1$, equalities (2.2) and $I^{\alpha}_{t_1,t}f(t))|_{t=t_1}=0$ we obtain

$$\begin{aligned} I^{2-\alpha}_{t_1,t}u'(t) &= b_0(\alpha-1)I^{2-\alpha}_{t_1,t}(t-t_1)^{\alpha-2}+b_1(\alpha-2)I^{2-\alpha}_{t_1,t}(t-t_1)^{\alpha-3}+I^{2-\alpha}_{t_1,t}I^{\alpha-1}_{t_1,t}f(t) \\ &= b_0(\alpha-1)\Gamma(\alpha-1)+b_1(\alpha-2)I^{2-\alpha}_{t_1,t}(t-t_1)^{\alpha-3}+I^{1}_{t_1,t}f(t) \end{aligned} \tag{3.7}$$

and

$$I^{2-\alpha}_{t_1,t}u'(t)|_{t=t_1}=b_0(\alpha-1)\Gamma(\alpha-1)+b_1(\alpha-2)I^{2-\alpha}_{t_1,t}(t-t_1)^{\alpha-3}|_{t=t_1}=\mathcal{P}_1u(t_1)+\mathcal{Q}_1<\infty. \tag{3.8}$$

Since the integral $I^{2-\alpha}_{t_1,t}(t-t_1)^{\alpha-3}$ does not converge, it follows that $b_1=0$ and

$$b_0=\frac{\mathcal{P}_1}{(\alpha-1)\Gamma(\alpha-1)}(c_0(t_1-t_0)^{\alpha-1}+\frac{u_1}{t_1\Gamma(\alpha-1)}(t_1-t_0)^{\alpha-1}+I^{\alpha}_{t_0,t}f(t)|_{t=t_1})+\frac{\mathcal{Q}_1}{(\alpha-1)\Gamma(\alpha-1)}.$$

Thus,

$$
\begin{aligned}
u(t) = &\Big(c_0 + \frac{u_1}{t_1\Gamma(\alpha-1)}\Big)\frac{\mathcal{P}_1(t_1-t_0)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)}(t-t_1)^{\alpha-1} \\
&+ \frac{\mathcal{P}_1}{(\alpha-1)\Gamma(\alpha-1)} I^{\alpha}_{t_0,t} f(t)|_{t=t_1}(t-t_1)^{\alpha-1} \\
&+ \frac{\mathcal{Q}_1}{(\alpha-1)\Gamma(\alpha-1)}(t-t_1)^{\alpha-1} + I^{\alpha}_{t_1,t} f(t).
\end{aligned}
\tag{3.9}
$$

Similarly, for $t \in (t_j, t_{j+1}]$, $j = 1, 2, \ldots, p$, by Lemma 1 with $a = t_j$ we get

$$
u(t) = k_0(t-t_j)^{\alpha-1} + k_1(t-t_j)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\phi_1(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds \tag{3.10}
$$

and

$$
u'(t) = k_0(\alpha-1)(t-t_j)^{\alpha-2} + k_1(\alpha-2)(t-t_j)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\int_{t_j}^{t}(t-s)^{\alpha-2}\phi_1(s, I^{\alpha}_{0,s}u(s), I^{\beta}_{0,s}v(s))ds. \tag{3.11}
$$

From the impulsive conditions we obtain $k_1 = 0$ and

$$
k_0 = \frac{\mathcal{P}_j}{(\alpha-1)\Gamma(\alpha-1)}u(t_j) + \frac{\mathcal{Q}_j}{(\alpha-1)\Gamma(\alpha-1)}.
$$

From the boundary condition $\mu_1 u(t)|_{t=T} + \nu_1 \mathcal{I}^{\alpha-1}u(t)|_{t=T} = u_2$ we get

$$
\mathcal{I}^{\alpha-1}u(t)|_{t=T} = \frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}u(s)ds
$$

and

$$
\begin{aligned}
&\mu_1 u(t)|_{t=T} + \nu_1 \mathcal{I}^{\alpha-1}u(t)|_{t=T} \\
&= \mu_1\Big(c_0 + \frac{u_1}{t_1\Gamma(\alpha-1)}\Big)(T-t_p)^{\alpha-1}\prod_{k=1}^{p}\frac{\mathcal{P}_k(t_k-t_{k-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\
&\quad + \mu_1 I^{\alpha}_{t_p,t}f(t)|_{t=T} + \mu_1(T-t_p)^{\alpha-1}\sum_{k=1}^{p}\frac{\mathcal{P}_k I^{\alpha}_{t_{k-1},t}f(t)|_{t=t_k} + \mathcal{Q}_k}{(\alpha-1)\Gamma(\alpha-1)}\prod_{j=k+1}^{m}\frac{\mathcal{P}_j(t_j-t_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\
&\quad + \nu_1\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}u(s)ds \\
&= u_2
\end{aligned}
\tag{3.12}
$$

and we obtain (3.3).

□

Example 2. *Consider the equation*

$$
\begin{aligned}
&{}^{RL}D^{1.5}_{0,t}u(t) = t, \quad t \in (0,1], \qquad {}^{RL}D^{1.5}_{1,t}u(t) = t, \ t \in (1,2], \\
&I^{0.5}_{1,t}u'(t)|_{t=1} = 1, \\
&I^{0.5}_{0,t}u(t)|_{t=0} = 0, \quad u(t)|_{t=2} + I^{0.5}_{0,t}u(t)|_{t=2} = 1.
\end{aligned}
\tag{3.13}
$$

Note the Eq (3.13) is similar to (2.19) but the lower limit of the fractional derivative is changed at the point of the impulse. The solution of (3.13) satisfies the integral equation

$$u(t) = \begin{cases} \frac{0.266667t^{2.5}}{\Gamma(1.5)}, & t \in (0,1], \\ \frac{1}{\Gamma(1.5)}(-0.4(t-1)^{2.5} + \frac{2(t-1)^{1.5}t}{3}) + \frac{(t-1)^{0.5}}{(0.5)\Gamma(0.5)}, & t \in (1,2]. \end{cases}$$

It is clear the change of the lower limits of the fractional derivatives has a huge influence on the solution of the equation.

3.3. Integral presentation of the problem (3.1)

Based on the integral presentation of the linear problem (3.2) and Lemma 3, we obtain the following result:

Theorem 2. *The solution of (3.1) satisfies the integral equations*

$$u(t) = \begin{cases} c_0 t^{\alpha-1} + \frac{u_1}{\Gamma(\alpha-1)} t^{\alpha-2} + I^{\alpha}_{0,t}\phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)), \quad t \in (0, t_1] \\ \Big(c_0 + \frac{u_1}{t_1\Gamma(\alpha-1)}\Big)(t-t_m)^{\alpha-1} \prod_{k=1}^{m} \frac{\mathcal{P}_k(t_k-t_{k-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\ \quad + I^{\alpha}_{t_m,t}\phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)) \\ \quad + (t-t_m)^{\alpha-1} \sum_{k=1}^{m} \frac{\mathcal{P}_k\ I^{\alpha}_{t_{k-1},t}\phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))|_{t=t_k} + \mathcal{Q}_k}{(\alpha-1)\Gamma(\alpha-1)} \prod_{j=k+1}^{m} \frac{\mathcal{P}_j(t_j-t_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)}, \\ \quad \textit{for} \quad t \in (t_m, t_{m+1}], m = 1, 2, \ldots, p, \end{cases}$$

and

$$v(t) = \begin{cases} b_0 t^{\alpha-1} + \frac{v_1}{\Gamma(\alpha-1)} t^{\alpha-2} + I^{\alpha}_{0,t}\phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)), \quad t \in (0, \tau_1] \\ \Big(b_0 + \frac{v_1}{\tau_1\Gamma(\alpha-1)}\Big)(t-\tau_m)^{\alpha-1} \prod_{k=1}^{m} \frac{\mathcal{P}^*_k(\tau_k-\tau_{k-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\ \quad + I^{\alpha}_{\tau_m,t}\phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t)) \\ \quad + (t-\tau_m)^{\alpha-1} \sum_{k=1}^{m} \frac{\mathcal{P}^*_k\ I^{\alpha}_{\tau_{k-1},t}\phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))|_{t=\tau_k} + \mathcal{Q}^*_k}{(\alpha-1)\Gamma(\alpha-1)} \prod_{j=k+1}^{m} \frac{\mathcal{P}^*_j(\tau_j-\tau_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)}, \\ \quad \textit{for} \quad t \in (\tau_m, \tau_{m+1}], m = 1, 2, \ldots, q, \end{cases}$$

where

$$\begin{aligned} c_0 = & \frac{u_2}{\mu_1}(T-t_p)^{1-\alpha}\mathcal{M} - \frac{u_1}{t_1\Gamma(\alpha-1)} \\ & - (T-t_p)^{1-\alpha}\mathcal{M} I^{\alpha}_{t_q,t}\phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))|_{t=T} \\ & - \mathcal{M}\sum_{k=1}^{p} \frac{\mathcal{P}_k I^{\alpha}_{t_{k-1},t}\phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))|_{t=t_k} + \mathcal{Q}_k}{(\alpha-1)\Gamma(\alpha-1)} \prod_{j=k+1}^{p} \frac{\mathcal{P}_j(t_j-t_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\ & - \frac{v_1}{\mu_1}(T-t_p)^{1-\alpha}\mathcal{M}\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}u(s)ds, \\ \mathcal{M} = & (\alpha-1)^p\Gamma^p(\alpha-1)\prod_{k=1}^{p}\mathcal{P}_k(t_k-t_{k-1})^{1-\alpha}, \end{aligned}$$

$$
\begin{aligned}
b_0 = &\frac{\nu_2}{\mu_2}(T-\tau_q)^{1-\alpha}C - \frac{\nu_1}{\tau_1\Gamma(\alpha-1)} \\
&-(T-\tau_q)^{1-\alpha}C\, I^{\alpha}_{\tau_q,t}\phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))_{t=T} \\
&-C\sum_{k=1}^{q}\frac{\mathcal{P}^*_k\, I^{\alpha}_{\tau_{k-1},t}\phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))|_{t=\tau_k}+Q^*_k}{(\alpha-1)\Gamma(\alpha-1)}\prod_{j=k+1}^{q}\frac{\mathcal{P}^*_j(\tau_j-\tau_{j-1})^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} \\
&-\frac{\nu_2}{\mu_2}(T-t_q)^{1-\alpha}C\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}u(s)ds, \\
C = &(\alpha-1)^q\Gamma^q(\alpha-1)\prod_{k=1}^{q}\mathcal{P}^*_k(\tau_k-\tau_{k-1})^{1-\alpha}.
\end{aligned}
$$

The proof is similar to the one of Lemma 3 applied twice to each of the both components u and v of the coupled system (3.1) for impulsive points $t_i,\ i=1,2,\dots,p$ and $\tau_i,\ i=1,2,\dots,p$ and the functions $f(t)=\phi_1(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))$ and $f(t)=\phi_2(t, I^{\alpha}_{0,t}u(t), I^{\beta}_{0,t}v(t))$ respectively.

4. Conclusions

In this paper we set up and study a scalar nonlinear integro-differential equation with Riemann-Liouville fractional derivative and impulses. We consider a boundary value problem for the studied equation with Riemann-Liouville fractional derivative of order in $(1,2)$. Note for Riemann-Liouville fractional differential equations both the initial condition and impulsive conditions have to be appropriately given (which is different in the case of ordinary derivatives as well as the case of Caputo fractional derivatives). We consider both interpretations in the literature on the presence of impulses in fractional differential equations: With fixed lower limit of the fractional derivative at the initial time point and with lower limits changeable at each impulsive time point. In both cases we set up in an appropriate way impulsive conditions which are dependent on the Riemann-Liouville fractional derivative. We obtain integral presentations of the solutions in both cases. These presentations could be successfully used for furure studies of existence, stability and other qualitative properties of the solutions of the integro-differential equations with Riemann-Liouville fractional derivative of order in $(1,2)$ and impulses.

Acknowledgments

S. H. is partially supported by the Bulgarian National Science Fund under Project KP-06-N32/7.

Conflict of interest

The authors declare that they have no competing interests.

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