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Research article

Exceptional set in Waring–Goldbach problem for sums of one square and five cubes

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Abstract: Let N be a sufficiently large integer. In this paper, it is proved that, with at most $O(N^{4/9+\varepsilon})$ exceptions, all even positive integers up to N can be represented in the form $p_1^2+p_2^3+p_3^3+p_4^3+p_5^3+p_6^3$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers.

Keywords: Waring–Goldbach problem; circle method; exceptional set
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1. Introduction and main result

Waring's problem of mixed powers concerns the representation of sufficiently large integer n in the form

$$n = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}.$$

Among the most interesting cases of mixed powers is that of establishing the representations of sufficiently large integer as the sum of one square and s positive cubes for each $s > 1$, i.e.,

$$n = x^2 + y_1^3 + y_2^3 \cdots + y_s^3. \tag{1.1}$$

In 1930, Stanley [7] showed that (1.1) is solvable for $s > 6$. Afterwards, Stanley [8] and Watson [12] solved the cases $s = 6$ and $s = 5$, respectively. It should be emphasized that Stanley [7] obtained the asymptotic formula for $s > 6$, while Sinnadurai [6] obtained the asymptotic formula for $s = 6$. But in [12], Watson only proved a quite weak lower bound for the number of representation (1.1) with $s = 5$. In 1986, Vaughan [9] enhanced Watson's result and derived a lower bound with the expected order of magnitude. In 2002, Wooley [13] illustrated that, although the excepted asymptotic formula of (1.1) with $s = 5$ can not be established by the technique currently available, the exceptional set is extremely sparse. To be specific, let $E_1(N)$ denote the number of integer $n \leqslant N$ which can not be represented as one square and five positive cubes with expected asymptotic formula, then Wooley [13] showed that $E_1(N) \ll N^{\varepsilon}$.

In view of the results of Vaughan [9] and Wooley [13], it is reasonable to conjecture that, for every sufficiently large even integer N, the following equation

$$N = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 \tag{1.2}$$

is solvable. Here and below, the letter p, with or without subscript, denotes a prime number. But this conjecture is perhaps out of reach at present. However, it is possible to replace a variable by an almost–prime. In 2014, Cai [1] proved that, for every sufficiently large even integer N, the following equation

$$N = x^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 \tag{1.3}$$

is solvable with x being an almost–prime $\mathcal{P}_{36}$ and the $p_j\,(j = 1, 2, 3, 4, 5)$ primes. Later, in 2018, Li and Zhang [3] enhanced the result of Cai [1] and showed that (1.3) is solvable with x being an almost–prime $\mathcal{P}_6$ and the $p_j\,(j = 1, 2, 3, 4, 5)$ primes. Recently, in 2021, Xue, Zhang and Li [15] consider the problem (1.2) with almost equal prime variables, i.e.,

$$\begin{cases} n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3, \quad n \in [N - 6U, N + 6U], \\ \left|p_1^2 - \dfrac{N}{6}\right| \leqslant U, \quad \left|p_i^3 - \dfrac{N}{6}\right| \leqslant U, \ \ i = 2, 3, 4, 5, 6, \end{cases} \tag{1.4}$$

where $U = N^{1-\delta+\varepsilon}$ with $\delta > 0$ hoped to be as large as possible. Let $E(N, U)$ denote the number of all positive even integers n satisfying

$$N - 6U \leqslant n \leqslant N + 6U,$$

which can not be represented as (1.4). One wants to show that there exists $\delta \in (0, 1)$ such that

$$E(N, U) \ll U^{1-\varepsilon}, \quad U = N^{1-\delta+\varepsilon}. \tag{1.5}$$

In [15], they proved that $\delta \leqslant 8/225$.

In this paper, we shall investigate the exceptional set of the problem (1.2) and establish the following result.

Theorem 1.1. *Let $E(N)$ denote the number of positive even integers n up to N, which can not be represented as*

$$n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3. \tag{1.6}$$

Then, for any $\varepsilon > 0$, we have

$$E(N) \ll N^{\frac{4}{9}+\varepsilon}.$$

We will establish Theorem 1.1 by using a pruning process into the Hardy–Littlewood circle method. In the treatment of the integrals over minor arcs, we will employ the methods, which is developed by Wooley in [14], combining with the new estimates for exponential sum over primes developed by Zhao [16]. For the treatment of the integrals on the major arcs, we shall prune the major arcs further and deal with them respectively. The full details will be explained in the following revelent sections.

Notation. Throughout this paper, let p, with or without subscripts, always denote a prime number; ε always denotes a sufficiently small positive constant, which may not be the same at different occurrences. As usual, we use $\varphi(n), d(n)$ and $\omega(n)$ to denote the Euler's function, Dirichlet's divisor function and the number of distinct prime factors of n, respectively. Also, we use χ mod q to denote a Dirichlet character modulo q, and χ^0 mod q the principal character. $e(x) = e^{2\pi ix}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. N is a sufficiently large integer and $n \in [N/2, N]$, and thus $\log N \asymp \log n$. The letter c, with or without subscripts or superscripts, always denote a positive constant, which may not be the same at different occurrences.

2. Outline of the proof of Theorem 1.1

Let N be a sufficiently large positive integer. By a splitting argument, it is sufficient to consider the even integers $n \in (N/2, N]$. For the application of the Hardy–Littlewood method, we need to define the Farey dissection. Let $A > 0$ be a sufficiently large fixed number, which will be determined at the end of the proof. We set

$$Q_0 = \log^A N, \quad Q_1 = N^{\frac{1}{6}}, \quad Q_2 = N^{\frac{5}{6}}, \quad \mathfrak{I}_0 = \left[-\frac{1}{Q_2}, 1 - \frac{1}{Q_2} \right].$$

By Dirichlet's lemma on rational approximation (for instance, see Lemma 12 on page 104 of Pan and Pan [4]), each $\alpha \in [-1/Q_2, 1 - 1/Q_2]$ can be written as the form

$$\alpha = \frac{a}{q} + \lambda, \qquad |\lambda| \leqslant \frac{1}{qQ_2}, \tag{2.1}$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q_2$ and $(a, q) = 1$. Define

$$\mathfrak{M}(q,a) = \left[\frac{a}{q} - \frac{Q_1}{qN}, \frac{a}{q} + \frac{Q_1}{qN} \right], \qquad \mathfrak{M} = \bigcup_{1\leqslant q\leqslant Q_1} \bigcup_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} \mathfrak{M}(q,a),$$

$$\mathfrak{M}_0(q,a) = \left[\frac{a}{q} - \frac{Q_0^{200}}{qN}, \frac{a}{q} + \frac{Q_0^{200}}{qN} \right], \qquad \mathfrak{M}_0 = \bigcup_{1\leqslant q\leqslant Q_0^{100}} \bigcup_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} \mathfrak{M}_0(q,a),$$

$$\mathfrak{m}_1 = \mathfrak{I}_0 \setminus \mathfrak{M}, \qquad \mathfrak{m}_2 = \mathfrak{M} \setminus \mathfrak{M}_0.$$

Then we obtain the Farey dissection

$$\mathfrak{I}_0 = \mathfrak{M}_0 \cup \mathfrak{m}_1 \cup \mathfrak{m}_2. \tag{2.2}$$

For $k = 2, 3$, we define

$$f_k(\alpha) = \sum_{X_k<p\leqslant 2X_k} e(p^k\alpha),$$

where $X_k = (N/16)^{\frac{1}{k}}$. Let

$$\mathscr{R}(n) = \sum_{\substack{n=p_1^2+p_2^3+p_3^3+p_4^3+p_5^3+p_6^3\\ X_3<p_2,\ldots,p_6\leqslant 2X_3\\ X_2<p_1\leqslant 2X_2}} 1.$$

From (2.2), one has

$$\begin{aligned}\mathscr{R}(n) &= \int_0^1 f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha = \int_{-\frac{1}{Q_2}}^{1-\frac{1}{Q_2}} f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha \\ &= \left\{\int_{\mathfrak{M}_0} + \int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2}\right\} f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha.\end{aligned}$$

In order to prove Theroem 1.1, we need the two following propositions:

Proposition 2.1. *For $n \in [N/2, N]$, there holds*

$$\int_{\mathfrak{M}_0} f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O\left(\frac{n^{7/6}}{\log^7 n}\right), \tag{2.3}$$

where $\mathfrak{S}(n)$ is the singular series defined in (4.1), which is absolutely convergent and satisfies

$$0 < c^* \leqslant \mathfrak{S}(n) \ll 1 \tag{2.4}$$

for any integer n satisfying $n \equiv 0 \pmod 2$ and some fixed constant $c^* > 0$; while $\mathfrak{J}(n)$ is defined by (4.5) and satisfies

$$\mathfrak{J}(n) \asymp \frac{n^{7/6}}{\log^6 n}.$$

The proof of (2.3) in Proposition 2.1 will be demonstrated in Section 4. For the property (2.4) of singular series, we shall give the proof in Section 5.

Proposition 2.2. *Let $\mathcal{Z}(N)$ denote the number of integers $n \in [N/2, N]$ satisfying $n \equiv 0 \pmod 2$ such that*

$$\sum_{j=1}^{2}\left|\int_{\mathfrak{m}_j} f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha\right| \gg \frac{n^{7/6}}{\log^7 n}.$$

Then we have

$$\mathcal{Z}(N) \ll N^{\frac{4}{9}+\varepsilon}.$$

The proof of Proposition 2.2 will be given in section 6. The remaining part of this section is devoted to establishing Theorem 1.1 by using Proposition 2.1 and Proposition 2.2.

Proof of Theorem 1.1. From Proposition 2.2, we deduce that, with at most $O(N^{\frac{4}{9}+\varepsilon})$ exceptions, all even integers $n \in [N/2, N]$ satisfy

$$\sum_{j=1}^{2}\left|\int_{\mathfrak{m}_j} f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha\right| \ll \frac{n^{7/6}}{\log^7 n},$$

from which and Proposition 2.1, we conclude that, with at most $O(N^{\frac{4}{9}+\varepsilon})$ exceptions, all even integers $n \in [N/2, N]$ can be represented in the form $p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers. By a splitting argument, we get

$$E(N) \ll \sum_{0\leqslant \ell \ll \log N} \mathcal{Z}\left(\frac{N}{2^\ell}\right) \ll \sum_{0\leqslant \ell \ll \log N} \left(\frac{N}{2^\ell}\right)^{\frac{4}{9}+\varepsilon} \ll N^{\frac{4}{9}+\varepsilon}.$$

This completes the proof of Theorem 1.1.

3. Some auxiliary lemmas

Lemma 3.1. *Suppose that α is a real number, and that $|\alpha - a/q| \leqslant q^{-2}$ with $(a, q) = 1$. Let $\beta = \alpha - a/q$. Then we have*

$$f_k(\alpha) \ll d^{\delta_k}(q)(\log x)^c \left(X_k^{1/2} \sqrt{q(1 + N|\beta|)} + X_k^{4/5} + \frac{X_k}{\sqrt{q(1 + N|\beta|)}} \right),$$

where $\delta_k = \frac{1}{2} + \frac{\log k}{\log 2}$ and c is a constant.

Proof. See Theorem 1.1 of Ren [5]. □

Lemma 3.2. *Suppose that α is a real number, and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a, q) = 1, \qquad 1 \leqslant q \leqslant Q \qquad \text{and} \qquad |q\alpha - a| \leqslant Q^{-1}.$$

If $P^{\frac{1}{2}} \leqslant Q \leqslant P^{\frac{5}{2}}$, then one has

$$\sum_{P<p\leqslant 2P} e(p^3\alpha) \ll P^{1-\frac{1}{12}+\varepsilon} + \frac{q^{-\frac{1}{6}} P^{1+\varepsilon}}{(1 + P^3|\alpha - a/q|)^{1/2}}.$$

Proof. See Lemma 8.5 of Zhao [16]. □

Lemma 3.3. *For $\alpha \in \mathfrak{m}_1$, we have*

$$f_3(\alpha) \ll N^{\frac{11}{36}+\varepsilon}.$$

Proof. By Dirichlet's rational approximation (2.1), for $\alpha \in \mathfrak{m}_1$, one has $Q_1 \leqslant q \leqslant Q_2$. From Lemma 3.2, we get

$$f_3(\alpha) \ll X_3^{\frac{11}{12}+\varepsilon} + X_3^{1+\varepsilon} Q_1^{-\frac{1}{6}} \ll N^{\frac{11}{36}+\varepsilon}.$$

This completes the proof of Lemma 3.3. □

For $1 \leqslant a \leqslant q$ with $(a, q) = 1$, set

$$\mathcal{I}(q, a) = \left[\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0} \right], \qquad \mathcal{I} = \bigcup_{1\leqslant q\leqslant Q_0} \bigcup_{\substack{a=-q \\ (a,q)=1}}^{2q} \mathcal{I}(q, a). \tag{3.1}$$

For $\alpha \in \mathfrak{m}_2$, by Lemma 3.1, we have

$$f_3(\alpha) \ll \frac{N^{\frac{1}{3}} \log^c N}{q^{\frac{1}{2}-\varepsilon}(1 + N|\lambda|)^{1/2}} + N^{\frac{4}{15}+\varepsilon} = V_3(\alpha) + N^{\frac{4}{15}+\varepsilon}, \tag{3.2}$$

say. Then we obtain the following lemma.

Lemma 3.4. *We have*

$$\int_{\mathcal{I}} |V_3(\alpha)|^2 \mathrm{d}\alpha = \sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathcal{I}(q,a)} |V_3(\alpha)|^2 \mathrm{d}\alpha \ll N^{-\frac{1}{3}} \log^{2A} N.$$

Proof. We have

$$\begin{aligned}
&\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|V_3(\alpha)|^2\mathrm{d}\alpha\\
&\ll \sum_{1\leqslant q\leqslant Q_0} q^{-1+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{|\lambda|\leqslant\frac{1}{Q_0}}\frac{N^{\frac{2}{3}}\log^c N}{1+N|\lambda|}\mathrm{d}\lambda\\
&\ll \sum_{1\leqslant q\leqslant Q_0} q^{-1+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\left(\int_{|\lambda|\leqslant\frac{1}{N}}N^{\frac{2}{3}}\log^c N\mathrm{d}\lambda+\int_{\frac{1}{N}\leqslant|\lambda|\leqslant\frac{1}{Q_0}}\frac{N^{\frac{2}{3}}\log^c N}{N|\lambda|}\mathrm{d}\lambda\right)\\
&\ll N^{-\frac{1}{3}}\log^c N\cdot\sum_{1\leqslant q\leqslant Q_0}q^{-1+\varepsilon}\varphi(q)\ll N^{-\frac{1}{3}}Q_0^{1+\varepsilon}\log^c N\ll N^{-\frac{1}{3}}\log^{2A}N.
\end{aligned}$$

This completes the proof of Lemma 3.4. □

4. Proof of Proposition 2.1

In this section, we shall concentrate on establishing Proposition 2.1. We first introduce some notations. For a Dirichlet character χ mod q and $k=2,3$, we define

$$C_k(\chi,a)=\sum_{h=1}^{q}\overline{\chi(h)}e\left(\frac{ah^k}{q}\right),\qquad C_k(q,a)=C_k(\chi^0,a),$$

where χ^0 is the principal character modulo q. Let $\chi_2,\chi_3^{(1)},\chi_3^{(2)},\chi_3^{(3)},\chi_3^{(4)},\chi_3^{(5)}$ be Dirichlet characters modulo q. Define

$$B\left(n,q,\chi_2,\chi_3^{(1)},\chi_3^{(2)},\chi_3^{(3)},\chi_3^{(4)},\chi_3^{(5)}\right)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(C_2(\chi_2,a)\prod_{i=1}^{5}C_3\left(\chi_3^{(i)},a\right)\right)e\left(-\frac{an}{q}\right),$$

$$B(n,q)=B\left(n,q,\chi^0,\chi^0,\chi^0,\chi^0,\chi^0,\chi^0\right),$$

and write

$$A(n,q)=\frac{B(n,q)}{\varphi^6(q)},\qquad \mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q). \tag{4.1}$$

Lemma 4.1. *For $(a,q)=1$ and any Dirichlet character χ mod q, there holds*

$$\left|C_k(\chi,a)\right|\leqslant 2q^{1/2}d^{\beta_k}(q)$$

with $\beta_k=(\log k)/\log 2$.

Proof. See the Problem 14 of Chapter VI of Vinogradov [11]. □

Lemma 4.2. *Let $C_k(q,a)$ be defined as above. Then there holds*

$$\sum_{q\leqslant x}\frac{\left|B(n,q)\right|}{\varphi^6(q)}\ll\log x. \tag{4.2}$$

Proof. By Lemma 4.1, we have

$$B(n,q)\ll\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|C_2(q,a)C_3^5(q,a)\right|\ll q^3\varphi(q)d^9(q).$$

Therefore, the left–hand side of (4.2) is

$$\ll\sum_{q\leqslant x}\frac{q^3\varphi(q)d^9(q)}{\varphi^6(q)}\ll\sum_{q\leqslant x}\frac{(\log\log q)^5d^9(q)}{q^2}\ll(\log\log x)^5\sum_{q\leqslant x}\frac{d^9(q)}{q^2}\ll\log x.$$

This completes the proof of Lemma 4.2. □

In order to treat the integral on the major arcs, we write $f_k(\alpha)$ as follows:

$$f_k(\alpha)=\sum_{\substack{X_k<p\leqslant 2X_k\\(p,q)=1}}e\left(p^k\left(\frac{a}{q}+\lambda\right)\right)+O(\log q)=\sum_{\substack{\ell=1\\(\ell,q)=1}}^{q}e\left(\frac{a\ell^k}{q}\right)\sum_{\substack{X_k<p\leqslant 2X_k\\p\equiv\ell\,(\mathrm{mod}\ q)}}e(p^k\lambda)+O(\log N).$$

For the innermost sum on the right–hand side of the above equation, by Siegel–Walfisz theorem, we have

$$\begin{aligned}\sum_{\substack{X_k<p\leqslant 2X_k\\p\equiv\ell\,(\mathrm{mod}\ q)}}e(p^k\lambda)&=\int_{X_k}^{2X_k}e(u^k\lambda)\mathrm{d}\,\pi(u,q,\ell)\\&=\int_{X_k}^{2X_k}e(u^k\lambda)\mathrm{d}\left(\frac{1}{\varphi(q)}\int_2^u\frac{\mathrm{d}t}{\log t}+O\left(ue^{-c\sqrt{\log u}}\right)\right)\\&=\frac{1}{\varphi(q)}\int_{X_k}^{2X_k}\frac{e(u^k\lambda)}{\log u}\mathrm{d}u+O\left(X_ke^{-c\sqrt{\log N}}\right)\\&=\frac{v_k(\lambda)}{\varphi(q)}+O\left(X_ke^{-c\sqrt{\log N}}\right),\end{aligned}$$

say. Therefore, we have

$$f_k(\alpha)=\frac{C_k(q,a)}{\varphi(q)}v_k(\lambda)+O\left(X_ke^{-c\sqrt{\log N}}\right),$$

and thus

$$f_2(\alpha)f_3^5(\alpha)=\frac{C_2(q,a)C_3^5(q,a)}{\varphi^6(q)}v_2(\lambda)v_3^5(\lambda)+O\left(N^{\frac{13}{6}}e^{-c\sqrt{\log N}}\right).$$

Then we derive that

$$\int_{\mathfrak{M}_0}f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha$$

$$= \sum_{1\leqslant q\leqslant Q_0^{100}} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right)\int_{-\frac{Q_0^{200}}{qN}}^{\frac{Q_0^{200}}{qN}} \left(\frac{C_2(q,a)C_3^5(q,a)}{\varphi^6(q)} v_2(\lambda)v_3^5(\lambda) + O\left(N^{\frac{13}{6}} e^{-c\sqrt{\log N}}\right)\right)e(-n\lambda)\mathrm{d}\lambda$$

$$= \sum_{1\leqslant q\leqslant Q_0^{100}} \frac{B(n,q)}{\varphi^6(q)} \int_{-\frac{Q_0^{200}}{qN}}^{\frac{Q_0^{200}}{qN}} v_2(\lambda)v_3^5(\lambda)e(-n\lambda)\mathrm{d}\lambda + O\left(N^{\frac{7}{6}} e^{-c\sqrt{\log N}}\right). \tag{4.3}$$

Noting that

$$v_k(\lambda) = \int_{X_k^k}^{(2X_k)^k} \frac{x^{\frac{1}{k}-1}e(\lambda x)}{\log x}\mathrm{d}x,$$

hence the innermost integral in (4.3) can be written as

$$\int_{-\frac{Q_0^{200}}{qN}}^{\frac{Q_0^{200}}{qN}} \left(\int_{X_2^2}^{(2X_2)^2} \frac{x^{-\frac{1}{2}}e(\lambda x)}{\log x}\mathrm{d}x\right)\left(\int_{X_3^3}^{(2X_3)^3} \frac{x^{-\frac{2}{3}}e(\lambda x)}{\log x}\mathrm{d}x\right)^5 e(-n\lambda)\mathrm{d}\lambda. \tag{4.4}$$

By using the elementary estimate

$$v_k(\lambda) = \int_{X_k^k}^{(2X_k)^k} \frac{x^{\frac{1}{k}-1}e(\lambda x)}{\log x}\mathrm{d}x \ll \frac{N^{\frac{1}{k}-1}}{\log N} \min\left(N, \frac{1}{|\lambda|}\right),$$

we know that if we extend the interval of the integral in (4.4) to $[-1/2, 1/2]$, then the resulting error is

$$\ll \int_{\frac{Q_0^{200}}{qN}}^{\frac{1}{2}} \frac{N^{-\frac{23}{6}}}{\log^6 N} \cdot \frac{\mathrm{d}\lambda}{\lambda^6} \ll \frac{N^{-\frac{23}{6}}}{\log^6 N} \cdot \frac{q^5N^5}{Q_0^{1000}} \ll \frac{N^{\frac{7}{6}}}{(\log N)^{500A}} \ll \frac{n^{\frac{7}{6}}}{(\log n)^{500A}}.$$

Hence we obtain

$$\int_{-\frac{Q_0^{200}}{qN}}^{\frac{Q_0^{200}}{qN}} v_2(\lambda)v_3^5(\lambda)e(-n\lambda)\mathrm{d}\lambda = \mathfrak{J}(n) + O\left(\frac{n^{\frac{7}{6}}}{(\log n)^{500A}}\right),$$

where

$$\begin{aligned}
\mathfrak{J}(n) = & \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{X_2^2}^{(2X_2)^2} \frac{x^{-\frac{1}{2}}e(\lambda x)}{\log x}\mathrm{d}x\right)\left(\int_{X_3^3}^{(2X_3)^3} \frac{x^{-\frac{2}{3}}e(\lambda x)}{\log x}\mathrm{d}x\right)^5 e(-n\lambda)\mathrm{d}\lambda \\
= & \int_{X_2^2}^{(2X_2)^2}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{X_3^3}^{(2X_3)^3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x_1^{-\frac{1}{2}}(x_2x_3x_4x_5x_6)^{-\frac{2}{3}}}{(\log x_1)(\log x_2)\cdots(\log x_6)} \\
& \times e((x_1+x_2+x_3+x_4+x_5+x_6-n)\lambda)\mathrm{d}\lambda\mathrm{d}x_1\cdots\mathrm{d}x_6 \\
\asymp & \frac{X_2^{-1}X_3^{-10}}{(\log N)^6}\int_{X_2^2}^{(2X_2)^2}\int_{X_3^3}^{(2X_3)^3}\cdots\int_{X_3^3}^{(2X_3)^3}\int_{-\frac{1}{2}}^{\frac{1}{2}} e((x_1+x_2+\cdots+x_6-n)\lambda)\mathrm{d}\lambda\mathrm{d}x_1\cdots\mathrm{d}x_6 \\
\asymp & \frac{X_2^{-1}X_3^{-10}}{(\log N)^6}N^5 \asymp \frac{N^{\frac{7}{6}}}{(\log N)^6} \asymp \frac{n^{\frac{7}{6}}}{(\log n)^6}.
\end{aligned} \tag{4.5}$$

Therefore, by (5.4), (4.3) becomes

$$\int_{\mathfrak{M}_0} f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O\left(\frac{n^{\frac{7}{6}}}{\log^7 n}\right),$$

which completes the proof of Proposition 2.1.

5. The singular series

In this section, we shall concentrate on investigating the properties of the singular series which appear in Proposition 2.1.

Lemma 5.1. *Let p be a prime and $p^\alpha \| k$. For $(a,p)=1$, if $\ell \geqslant \gamma(p)$, we have $C_k(p^\ell, a) = 0$, where*

$$\gamma(p) = \begin{cases} \alpha+2, & \text{if } p \neq 2 \text{ or } p=2,\ \alpha=0; \\ \alpha+3, & \text{if } p=2,\ \alpha>0. \end{cases}$$

Proof. See Lemma 8.3 of Hua [2]. □

For $k \geqslant 1$, we define

$$S_k(q,a) = \sum_{m=1}^{q} e\left(\frac{am^k}{q}\right).$$

Lemma 5.2. *Suppose that $(p,a)=1$. Then*

$$S_k(p,a) = \sum_{\chi \in \mathscr{A}_k} \overline{\chi(a)}\tau(\chi),$$

where $\mathscr{A}_k$ denotes the set of non–principal characters χ modulo p for which χ^k is principal, and $\tau(\chi)$ denotes the Gauss sum

$$\sum_{m=1}^{p} \chi(m) e\left(\frac{m}{p}\right).$$

Also, there hold $|\tau(\chi)| = p^{1/2}$ and $|\mathscr{A}_k| = (k, p-1) - 1$.

Proof. See Lemma 4.3 of Vaughan [10]. □

Lemma 5.3. *For $(p,n)=1$, we have*

$$\left|\sum_{a=1}^{p-1} \frac{S_2(p,a) S_3^5(p,a)}{p^6} e\left(-\frac{an}{p}\right)\right| \leqslant 32 p^{-\frac{5}{2}}. \tag{5.1}$$

Proof. We denote by $\mathcal{S}$ the left–hand side of (5.1). By Lemma 5.2, we have

$$\mathcal{S} = \frac{1}{p^6} \sum_{a=1}^{p-1} \left(\sum_{\chi_2 \in \mathscr{A}_2} \overline{\chi_2(a)}\tau(\chi_2)\right)\left(\sum_{\chi_3 \in \mathscr{A}_3} \overline{\chi_3(a)}\tau(\chi_3)\right)^5 e\left(-\frac{an}{p}\right).$$

If $|\mathscr{A}_k| = 0$ for some $k \in \{2,3\}$, then $\mathcal{S}=0$. If this is not the case, then

$$\begin{aligned} \mathcal{S} = \frac{1}{p^6} & \sum_{\chi_2 \in \mathscr{A}_2} \sum_{\chi_3^{(1)} \in \mathscr{A}_3} \sum_{\chi_3^{(2)} \in \mathscr{A}_3} \sum_{\chi_3^{(3)} \in \mathscr{A}_3} \sum_{\chi_3^{(4)} \in \mathscr{A}_3} \sum_{\chi_3^{(5)} \in \mathscr{A}_3} \\ & \times \tau(\chi_2)\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_3^{(3)})\tau(\chi_3^{(4)})\tau(\chi_3^{(5)}) \\ & \times \sum_{a=1}^{p-1} \overline{\chi_2(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_3^{(3)}(a)\chi_3^{(4)}(a)\chi_3^{(5)}(a)} e\left(-\frac{an}{p}\right). \end{aligned}$$

From Lemma 5.2, the sextuple outer sums have not more than $((2, p-1)-1)\times((3, p-1)-1)^5 \leqslant 1\times 2^5 = 32$ terms. In each of these terms, we have

$$\left|\tau(\chi_2)\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_3^{(3)})\tau(\chi_3^{(4)})\tau(\chi_3^{(5)})\right| = p^3.$$

Since in any one of these terms

$$\overline{\chi_2(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_3^{(3)}(a)\chi_3^{(4)}(a)\chi_3^{(5)}(a)}$$

is a Dirichlet character $\chi \pmod p$, the inner sum is

$$\sum_{a=1}^{p-1}\chi(a)e\left(-\frac{an}{p}\right) = \overline{\chi(-n)}\sum_{a=1}^{p-1}\chi(-an)e\left(-\frac{an}{p}\right) = \overline{\chi(-n)}\tau(\chi).$$

From the fact that $\tau(\chi^0) = -1$ for principal character χ^0 mod p, we have

$$\left|\overline{\chi(-n)}\tau(\chi)\right| \leqslant p^{\frac{1}{2}}.$$

By the above arguments, we obtain

$$\left|S\right| \leqslant \frac{1}{p^6}\cdot 32\cdot p^3\cdot p^{\frac{1}{2}} = 32p^{-\frac{5}{2}}.$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let $\mathcal{L}(p, n)$ denote the number of solutions of the congruence*

$$x_1^2 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 \equiv n \pmod p, \qquad 1 \leqslant x_1, x_2, \ldots, x_6 \leqslant p-1.$$

Then, for $n \equiv 0 \pmod 2$, we have $\mathcal{L}(p, n) > 0$.

Proof. We have

$$p\cdot\mathcal{L}(p, n) = \sum_{a=1}^{p} C_2(p, a)C_3^5(p, a)e\left(-\frac{an}{p}\right) = (p-1)^6 + E_p,$$

where

$$E_p = \sum_{a=1}^{p-1} C_2(p, a)C_3^5(p, a)e\left(-\frac{an}{p}\right).$$

By Lemma 5.2, we obtain

$$|E_p| \leqslant (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)^5.$$

It is easy to check that $|E_p| < (p-1)^6$ for $p \geqslant 13$. Therefore, we obtain $\mathcal{L}(p, n) > 0$ for $p \geqslant 13$. For $p = 2, 3, 5, 7, 11$, we can check $\mathcal{L}(p, n) > 0$ directly. □

Lemma 5.5. *$A(n, q)$ is multiplicative in q.*

Proof. By the definition of $A(n,q)$ in (4.1), we only need to show that $B(n,q)$ is multiplicative in q. Suppose $q=q_1q_2$ with $(q_1,q_2)=1$. Then we have

$$\begin{aligned}B(n,q_1q_2) &= \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2} C_2(q_1q_2,a)C_3^5(q_1q_2,a)e\bigg(-\frac{an}{q_1q_2}\bigg)\\ &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1}\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_1q_2,a_1q_2+a_2q_1)C_3^5(q_1q_2,a_1q_2+a_2q_1)e\bigg(-\frac{a_1n}{q_1}\bigg)e\bigg(-\frac{a_2n}{q_2}\bigg). \end{aligned}\tag{5.2}$$

For $(q_1,q_2)=1$ and $k\in\{2,3\}$, there holds

$$\begin{aligned}C_k(q_1q_2,a_1q_2+a_2q_1) &= \sum_{\substack{m=1\\(m,q_1q_2)=1}}^{q_1q_2} e\bigg(\frac{(a_1q_2+a_2q_1)m^k}{q_1q_2}\bigg)\\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1}\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\bigg(\frac{(a_1q_2+a_2q_1)(m_1q_2+m_2q_1)^k}{q_1q_2}\bigg)\\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1} e\bigg(\frac{a_1(m_1q_2)^k}{q_1}\bigg)\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\bigg(\frac{a_2(m_2q_1)^k}{q_2}\bigg)\\ &= C_k(q_1,a_1)C_k(q_2,a_2). \end{aligned}\tag{5.3}$$

Putting (5.3) into (5.2), we deduce that

$$\begin{aligned}&B(n,q_1q_2)\\ &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} C_2(q_1,a_1)C_3^5(q_1,a_1)e\bigg(-\frac{a_1n}{q_1}\bigg)\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_2,a_2)C_3^5(q_2,a_2)e\bigg(-\frac{a_2n}{q_2}\bigg)\\ &= B(n,q_1)B(n,q_2).\end{aligned}$$

This completes the proof of Lemma 5.5. □

Lemma 5.6. *Let $A(n,q)$ be as defined in (4.1). Then*
(i) *We have*

$$\sum_{q>Z} A(n,q) \ll Z^{-\frac{3}{2}+\varepsilon}d(n). \tag{5.4}$$

(ii) *There exists an absolute positive constant $c^*>0$, such that, for $n\equiv 0 \pmod 2$, there holds*

$$0<c^*\leqslant \mathfrak{S}(n)\ll 1.$$

Proof. From Lemma 5.5, we know that $B(n,q)$ is multiplicative in q. Therefore, there holds

$$B(n,q)=\prod_{p^t\|q} B(n,p^t)=\prod_{p^t\|q}\sum_{\substack{a=1\\(a,p)=1}}^{p^t} C_2(p^t,a)C_3^5(p^t,a)e\bigg(-\frac{an}{p^t}\bigg). \tag{5.5}$$

From (5.5) and Lemma 5.1, we deduce that $B(n,q)=\prod_{p\|q}B(n,p)$ or 0 according to whether q is square–free or not. Thus, one has

$$\sum_{q=1}^{\infty}A(n,q)=\sum_{\substack{q=1\\ q\text{ square–free}}}^{\infty}A(n,q). \tag{5.6}$$

Write

$$\mathcal{R}(p,a):=C_2(p,a)C_3^5(p,a)-S_2(p,a)S_3^5(p,a).$$

Then

$$A(n,p)=\frac{1}{(p-1)^6}\sum_{a=1}^{p-1}S_2(p,a)S_3^5(p,a)e\left(-\frac{an}{p}\right)+\frac{1}{(p-1)^6}\sum_{a=1}^{p-1}\mathcal{R}(p,a)e\left(-\frac{an}{p}\right). \tag{5.7}$$

Applying Lemma 4.1 and noticing that $S_k(p,a)=C_k(p,a)+1$, we get $S_k(p,a)\ll p^{\frac{1}{2}}$, and thus $\mathcal{R}(p,a)\ll p^{\frac{5}{2}}$. Therefore, the second term in (5.7) is $\leqslant c_1p^{-\frac{5}{2}}$. On the other hand, from Lemma 5.3, we can see that the first term in (5.7) is $\leqslant 2^6\cdot 32p^{-\frac{5}{2}}=2048p^{-\frac{5}{2}}$. Let $c_2=c_1+2048$. Then we have proved that, for $p\nmid n$, there holds

$$|A(n,p)|\leqslant c_2p^{-\frac{5}{2}}. \tag{5.8}$$

Moreover, if we use Lemma 4.1 directly, it follows that

$$\begin{aligned}\big|B(n,p)\big|&=\left|\sum_{a=1}^{p-1}C_2(p,a)C_3^5(p,a)e\left(-\frac{an}{p}\right)\right|\leqslant\sum_{a=1}^{p-1}\Big|C_2(p,a)C_3^5(p,a)\Big|\\&\leqslant(p-1)\cdot 2^6\cdot p^3\cdot 486=31104p^3(p-1),\end{aligned}$$

and therefore

$$\big|A(n,p)\big|=\frac{|B(n,p)|}{\varphi^6(p)}\leqslant\frac{31104p^3}{(p-1)^5}\leqslant\frac{2^5\cdot 31104p^3}{p^5}=\frac{995328}{p^2}. \tag{5.9}$$

Let $c_3=\max(c_2,995328)$. Then for square–free q, we have

$$\begin{aligned}\big|A(n,q)\big|&=\left(\prod_{\substack{p|q\\ p\nmid n}}\big|A(n,p)\big|\right)\left(\prod_{\substack{p|q\\ p|n}}\big|A(n,p)\big|\right)\leqslant\left(\prod_{\substack{p|q\\ p\nmid n}}(c_3p^{-\frac{5}{2}})\right)\left(\prod_{\substack{p|q\\ p|n}}(c_3p^{-2})\right)\\&=c_3^{\omega(q)}\left(\prod_{p|q}p^{-\frac{5}{2}}\right)\left(\prod_{p|(n,q)}p^{\frac{1}{2}}\right)\ll q^{-\frac{5}{2}+\varepsilon}(n,q)^{\frac{1}{2}}.\end{aligned}$$

Hence, by (5.6), we obtain

$$\begin{aligned}\sum_{q>Z}|A(n,q)|&\ll\sum_{q>Z}q^{-\frac{5}{2}+\varepsilon}(n,q)^{\frac{1}{2}}=\sum_{d|n}\sum_{q>\frac{Z}{d}}(dq)^{-\frac{5}{2}+\varepsilon}d^{\frac{1}{2}}=\sum_{d|n}d^{-2+\varepsilon}\sum_{q>\frac{Z}{d}}q^{-\frac{5}{2}+\varepsilon}\\&\ll\sum_{d|n}d^{-2+\varepsilon}\left(\frac{Z}{d}\right)^{-\frac{3}{2}+\varepsilon}=Z^{-\frac{3}{2}+\varepsilon}\sum_{d|n}d^{-\frac{1}{2}}\ll Z^{-\frac{3}{2}+\varepsilon}d(n),\end{aligned}$$

which proves (5.4), and hence gives the absolutely convergence of $\mathfrak{S}(n)$. In order to prove (ii) of Lemma 5.6, by Lemma 5.5, we first note that

$$\begin{aligned}\mathfrak{S}(n) &= \prod_{p}\left(1+\sum_{t=1}^{\infty}A(n,p^t)\right)=\prod_{p}(1+A(n,p))\\ &= \left(\prod_{p\leqslant c_3}(1+A(n,p))\right)\left(\prod_{\substack{p>c_3\\ p\nmid n}}(1+A(n,p))\right)\left(\prod_{\substack{p>c_3\\ p\mid n}}(1+A(n,p))\right).\end{aligned}\tag{5.10}$$

From (5.8), we have

$$\prod_{\substack{p>c_3\\ p\nmid n}}(1+A(n,p))\geqslant\prod_{p>c_3}\left(1-\frac{c_3}{p^{5/2}}\right)\geqslant c_4>0.\tag{5.11}$$

By (5.9), we obtain

$$\prod_{\substack{p>c_3\\ p\mid n}}(1+A(n,p))\geqslant\prod_{p>c_3}\left(1-\frac{c_3}{p^{2}}\right)\geqslant c_5>0.\tag{5.12}$$

On the other hand, it is easy to see that

$$1+A(n,p)=\frac{p\cdot\mathcal{L}(p,n)}{\varphi^6(p)}.$$

By Lemma 5.4, we know that $\mathcal{L}(p,n)>0$ for all p with $n\equiv 0 \pmod 2$, and thus $1+A(n,p)>0$. Therefore, there holds

$$\prod_{p\leqslant c_3}(1+A(n,p))\geqslant c_6>0.\tag{5.13}$$

Combining the estimates (5.10)–(5.13), and taking $c^*=c_4c_5c_6>0$, we derive that

$$\mathfrak{S}(n)\geqslant c^*>0.$$

Moreover, by (5.8) and (5.9), we have

$$\mathfrak{S}(n)\leqslant\prod_{p\nmid n}\left(1+\frac{c_3}{p^{5/2}}\right)\cdot\prod_{p\mid n}\left(1+\frac{c_3}{p^2}\right)\ll 1.$$

This completes the proof Lemma 5.6. □

6. Proof of Proposition 2.2

In this section, we shall give the proof of Proposition 2.2. We denote by $\mathcal{Z}_j(N)$ the set of integers n satisfying $n\in[N/2,N]$ and $n\equiv 0 \pmod 2$ for which the following estimate

$$\left|\int_{\mathfrak{m}_j}f_2(\alpha)f_3^5(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|\gg\frac{n^{\frac{7}{6}}}{\log^7 n}\tag{6.1}$$

holds. For convenience, we use $\mathcal{Z}_j$ to denote the cardinality of $\mathcal{Z}_j(N)$ for abbreviation. Also, we define the complex number $\xi_j(n)$ by taking $\xi_j(n) = 0$ for $n \notin \mathcal{Z}_j(N)$, and when $n \in \mathcal{Z}_j(N)$ by means of the equation

$$\left| \int_{\mathfrak{m}_j} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \mathrm{d}\alpha \right| = \xi_j(n) \int_{\mathfrak{m}_j} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \mathrm{d}\alpha. \tag{6.2}$$

Plainly, one has $|\xi_j(n)| = 1$ whenever $\xi_j(n)$ is nonzero. Therefore, we obtain

$$\sum_{n \in \mathcal{Z}_j(N)} \xi_j(n) \int_{\mathfrak{m}_j} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{m}_j} f_2(\alpha) f_3^5(\alpha) \mathcal{K}_j(\alpha) \mathrm{d}\alpha, \tag{6.3}$$

where the exponential sum $\mathcal{K}_j(\alpha)$ is defined by

$$\mathcal{K}_j(\alpha) = \sum_{n \in \mathcal{Z}_j(N)} \xi_j(n) e(-n\alpha).$$

For $j = 1, 2$, set

$$I_j = \int_{\mathfrak{m}_j} f_2(\alpha) f_3^5(\alpha) \mathcal{K}_j(\alpha) \mathrm{d}\alpha.$$

From (6.1)–(6.3), we derive that

$$I_j \gg \sum_{n \in \mathcal{Z}_j(N)} \frac{n^{\frac{7}{6}}}{\log^7 n} \gg \frac{\mathcal{Z}_j N^{\frac{7}{6}}}{\log^7 N}, \qquad j = 1, 2. \tag{6.4}$$

By Lemma 2.1 of Wooley [14] with $k = 2$, we know that, for $j = 1, 2$, there holds

$$\int_0^1 \left| f_2(\alpha) \mathcal{K}_j(\alpha) \right|^2 \mathrm{d}\alpha \ll N^{\varepsilon} \left(\mathcal{Z}_j N^{\frac{1}{2}} + \mathcal{Z}_j^2 \right). \tag{6.5}$$

It follows from Cauchy's inequality, Lemma 2.5 of Vaughan [10], Lemma 3.3 and (6.5) that

$$\begin{aligned} I_1 &\ll \left(\sup_{\alpha \in \mathfrak{m}_1} \left| f_3(\alpha) \right| \right) \times \left(\int_0^1 \left| f_3(\alpha) \right|^8 \mathrm{d}\alpha \right)^{\frac{1}{2}} \left(\int_0^1 \left| f_2(\alpha) \mathcal{K}_1(\alpha) \right|^2 \mathrm{d}\alpha \right)^{\frac{1}{2}} \\ &\ll N^{\frac{11}{36}+\varepsilon} \cdot \left(N^{\frac{5}{3}+\varepsilon} \right)^{\frac{1}{2}} \cdot \left(N^{\varepsilon} \left(\mathcal{Z}_1 N^{\frac{1}{2}} + \mathcal{Z}_1^2 \right) \right)^{\frac{1}{2}} \\ &\ll N^{\frac{41}{36}+\varepsilon} \left(\mathcal{Z}_1^{\frac{1}{2}} N^{\frac{1}{4}} + \mathcal{Z}_1 \right) \ll \mathcal{Z}_1^{\frac{1}{2}} N^{\frac{25}{18}+\varepsilon} + \mathcal{Z}_1 N^{\frac{41}{36}+\varepsilon}. \end{aligned} \tag{6.6}$$

Combining (6.4) and (6.6), we get

$$\mathcal{Z}_1 N^{\frac{7}{6}} \log^{-7} N \ll I_1 \ll \mathcal{Z}_1^{\frac{1}{2}} N^{\frac{25}{18}+\varepsilon} + \mathcal{Z}_1 N^{\frac{41}{36}+\varepsilon},$$

which implies

$$\mathcal{Z}_1 \ll N^{\frac{4}{9}+\varepsilon}. \tag{6.7}$$

Next, we give the upper bound for $\mathcal{Z}_2$. By (3.2), we obtain

$$I_2 \ll \int_{\mathfrak{m}_2} \left| f_2(\alpha) f_3^4(\alpha) V_3(\alpha) \mathcal{K}_2(\alpha) \right| \mathrm{d}\alpha + N^{\frac{4}{15}+\varepsilon} \times \int_{\mathfrak{m}_2} \left| f_2(\alpha) f_3^4(\alpha) \mathcal{K}_2(\alpha) \right| \mathrm{d}\alpha = I_{21} + I_{22}, \tag{6.8}$$

say. For $\alpha \in \mathfrak{m}_2$, then either one has $Q_0^{100} < q \leqslant Q_1$ or $Q_0^{100} < N|q\alpha - a| \leqslant NQ_2^{-1} = Q_1$. Therefore, by Lemma 3.1, we get

$$\sup_{\alpha\in\mathfrak{m}_2} \left|f_2(\alpha)\right| \ll X_2^{\frac{4}{5}+\varepsilon} + \frac{X_2(\log N)^c}{(q(1+N|\alpha-a/q|))^{\frac{1}{2}-\varepsilon}} \ll \frac{X_2(\log N)^c}{Q_0^{50-\varepsilon}} \ll \frac{N^{\frac{1}{2}}}{\log^{40A} N}. \tag{6.9}$$

In view of the fact that $\mathfrak{m}_2 \subseteq \mathcal{I}$, where $\mathcal{I}$ is defined by (3.1), Cauchy's inequality, the trivial estimate $\mathcal{K}_2(\alpha) \ll \mathcal{Z}_2$, Theorem 4 of Hua (See [2], P 19), Lemma 3.4 and (6.9), we obtain

$$\begin{aligned} I_{21} &\ll \mathcal{Z}_2 \cdot \sup_{\alpha\in\mathfrak{m}_2} \left|f_2(\alpha)\right| \times \left(\int_0^1 \left|f_3(\alpha)\right|^8 \mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_{\mathcal{I}} \left|V_3(\alpha)\right|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\ &\ll \mathcal{Z}_2 \cdot \left(\frac{N^{\frac{1}{2}}}{\log^{40A} N}\right) \cdot (N^{\frac{5}{3}} \log^c N)^{\frac{1}{2}} \cdot (N^{-\frac{1}{3}} \log^{2A} N)^{\frac{1}{2}} \ll \frac{\mathcal{Z}_2 N^{\frac{7}{6}}}{\log^{20A} N}, \end{aligned} \tag{6.10}$$

where the parameter A is chosen sufficiently large for above bound to work. Moreover, it follows from Cauchy's inequality, (6.5) and Theorem 4 of Hua (See [2], P 19) that

$$\begin{aligned} I_{22} &\ll N^{\frac{4}{15}+\varepsilon} \times \left(\int_0^1 \left|f_3(\alpha)\right|^8 \mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_0^1 \left|f_2(\alpha)\mathcal{K}_2(\alpha)\right|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\ &\ll N^{\frac{4}{15}+\varepsilon} \cdot \left(N^{\frac{5}{3}+\varepsilon}\right)^{\frac{1}{2}} \cdot \left(N^{\varepsilon}\left(\mathcal{Z}_2 N^{\frac{1}{2}} + \mathcal{Z}_2^2\right)\right)^{\frac{1}{2}} \\ &\ll N^{\frac{11}{10}+\varepsilon} \cdot \left(\mathcal{Z}_2^{\frac{1}{2}} N^{\frac{1}{4}} + \mathcal{Z}_2\right) \ll \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{27}{20}+\varepsilon} + \mathcal{Z}_2 N^{\frac{11}{10}+\varepsilon}. \end{aligned} \tag{6.11}$$

Combining (6.4), (6.8), (6.10) and (6.11), we deduce that

$$\frac{\mathcal{Z}_2 N^{\frac{7}{6}}}{\log^7 N} \ll I_2 = I_{21} + I_{22} \ll \frac{\mathcal{Z}_2 N^{\frac{7}{6}}}{\log^{20A} N} + \mathcal{Z}_2^{\frac{1}{2}} N^{\frac{27}{20}+\varepsilon} + \mathcal{Z}_2 N^{\frac{11}{10}+\varepsilon},$$

which implies

$$\mathcal{Z}_2 \ll N^{\frac{11}{30}+\varepsilon}. \tag{6.12}$$

From (6.7) and (6.12), we have

$$\mathcal{Z}(N) \ll \mathcal{Z}_1 + \mathcal{Z}_2 \ll N^{\frac{4}{9}+\varepsilon}.$$

This completes the proof of Proposition 2.2.

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Conclict of interest

The authors declare that no conflict of interest exists in this manuscript.

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