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Research article

Note on r-central Lah numbers and r-central Lah-Bell numbers

Hye Kyung Kim*

Department of Mathematics Education, Daegu Catholic University, Gyeongsan 38430, Republic of Korea

* **Correspondence:** Email: hkkim@cu.ac.kr.

Abstract: The r-Lah numbers generalize the Lah numbers to the r-Stirling numbers in the same sense. The Stirling numbers and the central factorial numbers are one of the important tools in enumerative combinatorics. The r-Lah number counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks. In this paper, the r-central Lah numbers and the r-central Lah-Bell numbers ($r \in \mathbb{N}$) are introduced parallel to the r-extended central factorial numbers of the second kind and r-extended central Bell polynomials. In addition, some identities related to these numbers including the generating functions, explicit formulas, binomial convolutions are derived. Moreover, the r-central Lah numbers and the r-central Lah-Bell numbers are shown to be represented by Riemann integral, respectively.

Keywords: Lah numbers; Lah-Bell numbers; r-Lah numbers; r-Lah-Bell polynomials; central factorial numbers of the second kind
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1. Introduction

For $n \geq k \geq 0$, the Stirling numbers of the second kind $S_2(n,k)$ can be interpreted as the number of ways to partition a set with n elements into k non-empty subsets [7, 19] and the n-th Bell number B_n $(n \geq 0)$ as the number of ways to partition a set with n elements into non-empty subsets.

As is known, the n-th Bell polynomial possess the following explicit formula and generating function

$$B_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k, \quad \text{and} \quad e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \ (n \geq 0) \quad \text{(see [7, 22, 26])}, \tag{1.1}$$

respectively.

It is well-known that

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \quad \text{(see [7])}, \tag{1.2}$$

where $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$, $(n \geq 1)$ and $< x >_0 = 1$.

The central factorial numbers of the second kind $T(n,k)$ are the coefficients in the expansion of x^n in terms of central factorials given by

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}, \quad (n \geq k \geq 0), \quad \text{(see [26])}, \tag{1.3}$$

where $x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\cdots\left(x - \frac{n}{2} + 1\right)$, $(n \geq 1)$ and $x^{[0]} = 1$. From (1.3), it is easy to see that the generating function of $T(n,k)$ is

$$\frac{1}{k!}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n,k) \frac{t^n}{n!}, \quad \text{(see [15, 17, 26])}, \tag{1.4}$$

where $k \geq 0$.

Throughout this section, let $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k \geq 0$ and $r \in \mathbb{N}$.

The r-Stirling numbers $S_2^{(r)}(n,k)$ of the second kind are defined by the generating function

$$\frac{1}{k!}(e^t - 1)^k e^{rt} = \sum_{n=k}^{\infty} S_2^{(r)}(n+r, k+r) \frac{t^n}{n!}, \quad \text{(see [3])}.$$

and the r-Stirling numbers enumerate the numbers of partitions of the set $\{1, 2, \cdots, n\}$ into k nonempty disjoint subsets in such a way that $1, 2, \cdots, r$ are in distinct subsets. The generating function of r-Bell polynomials $B_n^{(r)}(x) = \sum_{k=1}^{n} S_2^{(r)}(n,k)$ is given by

$$e^{rt} e^{x(e^t - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [3])}.$$

In particular, when $x = 1$, $B_n^{(c,r)} = B_n^{(c,r)}(1)$, which are called the r-extended Bell numbers.

Kim et al. [10] introduced the r-extended central factorial numbers of the second kind given by

$$\frac{1}{k!} e^{rt} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T_r(n+r, k+r) \frac{t^n}{n!}, \quad (k \geq 0) \tag{1.5}$$

and the r-extended central Bell polynomials $B_n^{(c,r)}(x)$ given by

$$e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)} = \sum_{n=0}^{\infty} B_n^{(c,r)}(x) \frac{t^n}{n!} \quad (r \in \mathbb{N} \cup \nvdash).$$

The unsigned Lah-number $L(n,k)$ counts the number of partitions of a set with n elements into k ordered blocks with no box left empty. Lah-numbers are rarely called Stirling numbers of the third kind. It is well known that the Lah number $L(n,k)$ respectively possess the following explicit formula and generating function

$$L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1}, \quad \text{(see [11, 14, 18])}, \tag{1.6}$$

and

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!}, \quad \text{(see [11, 14, 18])}. \tag{1.7}$$

The r-Lah number $L_r(n,k)$ counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks and $L_r(n,k)$ respectively possess the following explicit formula and generating function

$$L_r(n,k)=\binom{n+2r-1}{k+2r-1}\frac{n!}{k!}\ (k\geq 0), \quad \text{(see [14, 15, 23, 24])}, \tag{1.8}$$

and

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k\left(\frac{1}{1-t}\right)^{2r}=\sum_{n=k}^{\infty}L_r(n,k)\frac{t^n}{n!},(k\geq 0), \quad \text{(see [14, 15, 23, 24])}. \tag{1.9}$$

Kim-Kim introduced the r-extended Lah-Bell polynomials given by the generating function

$$\left(\frac{1}{1-t}\right)^{2r}e^{x\left(\frac{1}{1-t}-1\right)}=\sum_{n=k}^{\infty}\mathbf{B}^L_{r,n}(x)\frac{t^n}{n!},\ (k\geq 0), \quad \text{(see [16])}. \tag{1.10}$$

When $x=1$, $\mathbf{B}^L_n=\mathbf{B}^L_n(1)$ and $\mathbf{B}^L_{r,n}=\mathbf{B}^L_{r,n}(1)$, which are called the Lah-Bell numbers and r-extended Lah-Bell numbers, respectively.

Recently, H. K. Kim introduced the central Lah numbers $L^{(C)}(n,k)$ and the central Lah-Bell numbers $LB^{(C)}_n$ by means of the following generating functions

$$\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k=\sum_{n=k}^{\infty}L^{(C)}(n,k)\frac{t^n}{n!}, \quad \text{(see [13])}, \tag{1.11}$$

and

$$exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)=\sum_{n=0}^{\infty}LB^{(C)}_n\frac{t^n}{n!}, \quad \text{(see [13])}, \tag{1.12}$$

respectively. Where $LB^{(c)}_n=\sum_{k=0}^{n}L^{(C)}(n,k)$.

Let f be a non-negative real-valued function on interval $[a, b]$, and let $R = \{(x, y)|a \leq x \leq b,\ \ 0 < y < f(x)\}$ be the region of the plane under $f(x)$ and above interval $[a, b]$. The Riemann integral

$$\int_a^b f(x)\,dx = \lim_{max\triangle x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\triangle x_k$$

is approximations for the area of R, where $\triangle x_k = \frac{(b-a)k}{n}$ and $\triangle x_k^* = \frac{(b-a)k}{2n}$.

The central factorials have been introduced and studied by Steffensen [27]. The central factorial numbers are at least as important as Stirling numbers, and often appear in their properties and applications to difference calculus, spline theory, and to approximation theory, etc [1, 2, 4–6, 9, 10, 12, 13, 17, 16]. Moreover, the Lah numbers appear in various fields of mathematics such as non-crossing partitions, Dyck paths, q-analogues as well as falling and rising factorials [7, 8, 10, 13, 14, 16, 18, 20, 21, 23, 25]. Recently, H.K. Kim introduced and studied the central Lah numbers and the central Lah-Bell numbers. In this paper, we consider the r-central Lah numbers and the r-central Lah-Bell numbers ($r \in \mathbb{N}$), which generalize the central Lah numbers and the central Lah-Bell numbers. We derive the generating function, explicit formulas, binomial convolutions, and Riemann integral representations of the r-central Lah numbers and the r-central Lah-Bell numbers, respectively.

2. r-central Lah numbers and r-central Lah-Bell numbers

In this section, we introduce the r-central Lah numbers and the r-central Lah-Bell numbers, respectively, which are "central" analogues for r-extended central factorial numbers of the second kind and r-extended central Bell polynomials. We investigate some properties of these numbers.

Now, in view of (1.5) and (1.11), we introduce the following generating function that defines the r-central Lah numbers $L_r^{(C)}(n, k)$

$$\frac{1}{k!}\Big(2\Big(\frac{1}{2-t} - \frac{1}{2+t}\Big)\Big)^k \Big(\frac{1}{1-t}\Big)^{2r} = \sum_{n=k}^{\infty} L_r^{(C)}(n, k)\frac{t^n}{n!},\ \ (r \in \mathbb{N}). \tag{2.1}$$

Theorem 1. *For $n \geq k \geq 0$, an explicit formula of the r-central Lah-numbers $L_r^{(C)}(n, k)$ is*

$$L_r^{(C)}(n, k) = \sum_{m=k}^{n} \sum_{q=i}^{m} \sum_{i=0}^{k} \binom{n}{m} (-1)^{k-i+m-q} 2^{-m} < 2r >_{n-m} L(q, i) L(m - q, k - i),$$

where $L(0, 0) = 1,\ \ L(n, 0) = 0$, and $L(n, k) = 0$ for all $k > n$.

Proof. We observe that from (1.2), (1.11) and (2.1),

$$\begin{aligned}
\sum_{n=k}^{\infty} L_r^{(C)}(n,k)\frac{t^n}{n!} &= \sum_{n=k}^{\infty} L^{(C)}(n,k)\frac{t^n}{n!}\Big(\frac{1}{t-1}\Big)^{2r} \\
&= \frac{1}{k!}\Big(\frac{\frac{1}{2}t}{1-\frac{1}{2}t} - \frac{-\frac{1}{2}t}{1-(-\frac{1}{2}t)}\Big)^k\Big(\frac{1}{t-1}\Big)^{2r} \\
&= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\Big(\frac{\frac{1}{2}t}{1-\frac{1}{2}t}\Big)^i(-1)^{k-i}\Big(\frac{-\frac{1}{2}t}{1-(-\frac{1}{2}t)}\Big)^{k-i}\Big(\frac{1}{t-1}\Big)^{2r} \\
&= \sum_{i=0}^{k}(-1)^{k-i}\frac{1}{i!}\Big(\frac{\frac{1}{2}t}{1-\frac{1}{2}t}\Big)^i\frac{1}{(k-i)!}\Big(\frac{-\frac{1}{2}t}{1-(-\frac{1}{2}t)}\Big)^{k-i}\Big(\frac{1}{t-1}\Big)^{2r} \qquad (2.2) \\
&= \sum_{i=0}^{k}(-1)^{k-i}\sum_{q=i}^{\infty}L(q,i)\frac{(\frac{1}{2}t)^q}{q!}\sum_{l=k-i}^{\infty}L(l,k-i)\frac{(-\frac{1}{2}t)^l}{l!}\sum_{j=0}^{\infty} < 2r >_j \frac{t^j}{j!} \\
&= \sum_{i=0}^{k}\Big(\sum_{m=k}^{\infty}\sum_{q=i}^{m}(-1)^{k-i+m-q}2^{-m}L(q,i)L(m-q,k-i)\Big)\frac{t^m}{m!}\sum_{j=0}^{\infty} < 2r >_j \frac{t^j}{j!} \\
&= \sum_{n=k}^{\infty}\Big(\sum_{m=k}^{n}\sum_{q=i}^{m}\sum_{i=0}^{k}\binom{n}{m}(-1)^{k-i+m-q}2^{-m} < 2r >_{n-m} L(q,i)L(m-q,k-i)\Big)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of both side of (2.2), we obtain the desired result.

□

Theorem 2. *For $n \geq k \geq 0$, we have*

$$L_r^{(C)}(n,k) = \sum_{l=k}^{n}\binom{n}{l} < 2r >_{n-l} L^{(C)}(l,k),$$

where $L^{(C)}(n,k)$ are the central Lah-numbers.

Proof. From (1.2), (1.11) and (2.1), we observe that

$$\begin{aligned}
\sum_{n=k}^{\infty} L_r^{(C)}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\Big(2\Big(\frac{1}{2-t} - \frac{1}{2+t}\Big)\Big)^k\Big(\frac{1}{1-t}\Big)^{2r} \\
&= \sum_{l=k}^{\infty} L^{(C)}(l,k)\frac{t^l}{l!}\sum_{m=0}^{\infty} < 2r >_m \frac{t^m}{m!} \qquad (2.3) \\
&= \sum_{n=k}^{\infty}\Big(\sum_{l=k}^{n}\binom{n}{l}L^{(C)}(l,k) < 2r >_{n-l}\Big)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of both sides of (2.3), we get the desired result. □

In view of (1.1), we define the r-central Lah-Bell polynomials $LB_{n,r}^{(C)}(x)$ by

$$LB_{n,r}^{(C)}(x) = \sum_{k=0}^{n} L_r^{(C)}(n,k)x^k, \quad (n \geq 0), \qquad (2.4)$$

when $x = 1$, $LB^{(C)}_{n,r}(1) := LB^{(C)}_{n,r}$ are the central r-Lah-Bell numbers.

Theorem 3. *For $n \geq k \geq 0$, the generating function of the r-central Lah-Bell polynomials is*

$$\exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty} LB^{(C)}_{n,r}(x)\frac{t^n}{n!}.$$

Proof. From (2.1) and (2.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} LB^{(C)}_{n,r}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} L^{(C)}_r(n,k)x^k\right)\frac{t^n}{n!} = \sum_{k=0}^{\infty} x^k\left(\sum_{n=k}^{\infty} L^{(C)}_r(n,k)\right)\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k x^k\left(\frac{1}{1-t}\right)^{2r} = \exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\left(\frac{1}{1-t}\right)^{2r}.\end{aligned} \tag{2.5}$$

By (2.5), we get the desired result. □

Corollary 4. *The generating function of the r-central Lah-Bell numbers $LB^{(C)}_{n,r}$ is*

$$\exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty} LB^{(C)}_{n,r}\frac{t^n}{n!}.$$

Lemma 5. *For $n \geq k \geq 0$, we have*

$$L^{(C)}_r(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{d=0}^{n-l}\binom{k}{j}\binom{n}{l}\binom{n-l}{d}(-1)^{j+n-l-d}2^{l-n} < j >_{n-l-d} < k-j >_d < 2r >_l .$$

Proof. From (1.2) and (2.1), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} L^{(C)}_r(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k\left(\frac{1}{1-t}\right)^{2r}\\ &= \frac{1}{k!}2^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{1}{2+t}\right)^j\left(\frac{1}{2-t}\right)^{k-j}\sum_{l=0}^{\infty} < 2r >_l \frac{t^l}{l!}\\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{c=0}^{\infty} < j >_c \frac{(-\frac{t}{2})^c}{c!}\sum_{d=0}^{\infty} < k-j >_d \frac{(\frac{t}{2})^d}{d!}\sum_{l=0}^{\infty} < 2r >_l \frac{t^l}{l!}\\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{h=0}^{\infty}\sum_{d=0}^{h}\binom{h}{d}2^{-h}(-1)^{h-d} < j >_{h-d} < k-j >_d \frac{t^h}{h!}\sum_{l=0}^{\infty} < 2r >_l \frac{t^l}{l!}\\ &= \frac{1}{k!}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{d=0}^{n-l}\binom{k}{j}\binom{n}{l}\binom{n-l}{d}(-1)^{j+n-l-d}2^{l-n} < j >_{n-l-d} < k-j >_d < 2r >_l\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

By comparing the coefficients of both side of (2.6), we have desired result. □

Theorem 6. *For $n \geq k \geq 0$, we have*

$$L_r^{(C)}(n,k) = \frac{2(n!)}{k!\pi} Im \int_0^{\pi} 2^k \Big(\frac{1}{2-e^{i\theta}} - \frac{1}{2+e^{i\theta}}\Big)^k \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta.$$

Proof. For $k \geq j \geq 0$,

By using Lemma 5, we obtain

$$\begin{aligned}
&\frac{1}{k!} Im \int_0^{\pi} 2^k \Big(\frac{1}{2-e^{i\theta}} - \frac{1}{2+e^{i\theta}}\Big)^k \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Im \int_0^{\pi} 2^k \Big(\frac{1}{2+e^{i\theta}}\Big)^j \Big(\frac{1}{2-e^{i\theta}}\Big)^{k-j} \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Im \int_0^{\pi} 2^k \Big(\frac{\frac{1}{2}}{1+\frac{e^{i\theta}}{2}}\Big)^j \Big(\frac{\frac{1}{2}}{1-\frac{e^{i\theta}}{2}}\Big)^{k-j} \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j Im \int_0^{\pi} \sum_{h=0}^{\infty} \sum_{d=0}^{h} \binom{h}{d} (-1)^{h-d} 2^{-h} < j >_{h-d} < k-j >_d \frac{(e^{i\theta})^h}{h!} \\
&\qquad \times \sum_{l=0}^{\infty} < 2r >_l \frac{(e^{i\theta})^l}{l!} \sin n\theta d\theta \qquad (2.7) \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{m=0}^{\infty} \sum_{l=0}^{m} \sum_{d=0}^{m-l} \binom{m}{l} \binom{m-l}{d} 2^{l-m} (-1)^{m-l-d} \\
&\qquad \times < j >_{m-l-d} < k-j >_d < 2r >_l \frac{1}{m!} Im \int_0^{\pi} \sin m\theta \sin n\theta d\theta \\
&= \frac{1}{k!} \sum_{j=0}^{k} \sum_{l=0}^{n} \sum_{d=0}^{n-l} \binom{k}{j} \binom{n}{l} \binom{n-l}{d} (-1)^{j+n-l-d} 2^{l-n} < j >_{n-l-d} < k-j >_d < 2r >_l \frac{1}{n!} \frac{\pi}{2} \\
&= L_r^{(C)}(n,k) \frac{1}{n!} \frac{\pi}{2}.
\end{aligned}$$

From (2.7), we have desired result.

□

Theorem 7. *For $n \geq 1$, we have*

$$LB_{n,r}^{(C)} = \frac{2(n!)}{\pi} Im \int_0^{\pi} \exp\Big(2\Big(\frac{1}{2-e^{i\theta}} - \frac{1}{2+e^{i\theta}}\Big)\Big) \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta.$$

Proof. By (2.4), Theorem 3 and Theorem 6, we observe that

$$\begin{aligned}
&Im \int_0^{\pi} \exp\Big(2\Big(\frac{1}{2-e^{i\theta}} - \frac{1}{2+e^{i\theta}}\Big)\Big)^k \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} Im \int_0^{\pi} 2^k \Big(\frac{1}{2-e^{i\theta}} - \frac{1}{2+e^{i\theta}}\Big) \Big(\frac{1}{1-e^{i\theta}}\Big)^{2r} \sin n\theta d\theta \qquad (2.8) \\
&= \frac{\pi}{n!2} \sum_{k=0}^{\infty} L_r^{(C)}(n,k) = \frac{\pi}{n!2} LB_{n,r}^{(C)}.
\end{aligned}$$

By (2.8), we have desired result. □

Remark.

Corollary 8. *For $n \geq k \geq 0$, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} L_r^{(C)}(n,k)x^{[k]}\frac{t^n}{n!} = \left(\frac{1}{2-t} - \frac{1}{2+t} + \sqrt{\left(\frac{1}{2-t} - \frac{1}{2+t}\right)^2 + 1} \right)^{2x}\left(\frac{1}{1-t}\right)^{2r},$$

and

$$\sum_{k=0}^{n} L_r^{(C)}(n,k)x^{[k]} = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{d=0}^{n}\sum_{l=0}^{d}\binom{n}{d}\binom{2x}{m}\binom{2x-m+2i}{j}\binom{m}{i}\binom{d}{l}(-1)^{j+d-l}$$
$$\times 2^{2x-m+2i-n} < 2x-m+2i-j >_l < j >_{d-l} < 2r >_{n-d} .$$

Proof. It is well known that the generating function of central factorial is given by

$$\left(\frac{t}{2} + \sqrt{\frac{1}{4}t^2 + 1} \right)^{2x} = \sum_{n=0}^{\infty} x^{[n]}\frac{t^n}{n!}, \quad \text{(see [15, 26])}, \tag{2.9}$$

By replacing t with $2\left(\frac{1}{2-t} - \frac{1}{2+t}\right)$ in (2.9), we get

$$\left(\frac{1}{2-t} - \frac{1}{2+t} + \sqrt{\left(\frac{1}{2-t} - \frac{1}{2+t}\right)^2 + 1} \right)^{2x} = \sum_{n=0}^{\infty} x^{[n]}\frac{1}{n!}\left(2\left(\frac{1}{2-t} - \frac{1}{2+t}\right)\right)^n. \tag{2.10}$$

Combining with (2.1), (2.4) and (2.10), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n} L_r^{(C)}(n,k)x^{[k]}\frac{t^n}{n!} = \sum_{k=0}^{\infty} x^{[k]}\sum_{n=k}^{\infty} L_r^{(C)}(n,k)\frac{t^n}{n!}\\
&= \sum_{k=0}^{\infty} x^{[k]}\frac{1}{k!}\left(2\left(\frac{1}{2-t} - \frac{1}{2+t}\right)\right)^k\left(\frac{1}{1-t}\right)^{2r}\\
&= \left(\frac{1}{2-t} - \frac{1}{2+t} + \sqrt{\left(\frac{1}{2-t} - \frac{1}{2+t}\right)^2 + 1} \right)^{2x}\left(\frac{1}{1-t}\right)^{2r}\\
&= \sum_{m=0}^{\infty}\binom{2x}{m}\left(\frac{1}{2-t} - \frac{1}{2+t}\right)^{2x-m}\left(\left(\frac{1}{2-t} - \frac{1}{2+t}\right)^2 + 1\right)^{\frac{m}{2}}\left(\frac{1}{1-t}\right)^{2r}\\
&= \sum_{m=0}^{\infty}\binom{2x}{m}\sum_{i=0}^{\infty}\binom{\frac{m}{2}}{i}\left(\frac{1}{2-t} - \frac{1}{2+t}\right)^{2x-m+i}\left(\frac{1}{1-t}\right)^{2r}\\
&= \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\binom{2x}{m}\binom{\frac{m}{2}}{i}\sum_{j=0}^{\infty}\binom{2x-m+i}{j}(-1)^j\left(\frac{2}{1-(-\frac{t}{2})}\right)^j\left(\frac{2}{1-(\frac{t}{2})}\right)^{2x-m-i-j}\left(\frac{1}{1-t}\right)^{2r}
\end{aligned} \tag{2.11}$$

$$= \sum_{n=0}^{\infty}\Bigg(\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{d=0}^{n}\sum_{l=0}^{d}\binom{n}{d}\binom{2x}{m}\binom{2x-m+i}{j}\binom{m}{i}\binom{d}{l}(-1)^{j+d-l}$$
$$\times 2^{2x-m+i-n} < 2x-m+i-j >_l < j >_{d-l} < 2r >_{n-d}\Bigg)\frac{t^n}{n!}.$$

By comparing the coefficients of both sides of (2.11), we attain the desired result.

□

3. Conclusions

In this paper, we introduced the r-central Lah numbers and the r-central Lah numbers ($r \in \mathbb{N}$), respectively. We derived the generating functions and combinatorial identities of the r-central Lah-Bell numbers and polynomials. Furthermore, by showing that each of these two numbers is expressed as Riemann integral in Theorems 6 and 7, respectively, we can infer approximate values for each of them. We also expressed the relation of the r-central Lah-numbers and the central factorials numbers in Corollary 8.

As one of our next projects, we would like to find some interesting applications of the r-central Lah numbers and polynomials, and the r-central Lah-Bell numbers and polynomials introduced in this paper.

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Conflict of interest

The author declares no conflict of interest.

References

1. H. Belbachir, Y. Djemmada, On central Fubini-like numbers and polynomials, *Miskolc Math. Notes*, **22** (2021), 77–90. doi: 10.18514/MMN.2021.2809.

2. H. Belbachir, Y. Djemmada, Generalized geometric polynomials via Steffensen's generalized factorials and Tanny's operators, *Indian J. Pure Appl. Math.*, **51** (2020), 1713–1727. doi: 10.1007/s13226-020-0491-8.

3. A. Z. Broder, The r-Stirling numbers, *Discrete Math.*, **49** (1984), 241–259. doi: 10.1016/0012-365X(84)90161-4.

4. P. L. Butzer, M. Schmidt, E. L. Stark, L. Vogt, Central factorial numbers;their main properties and some applications, *Numer. Funct. Anal. Optim.*, **10** (1989), 419–488. doi: 10.1080/01630568908816313.

5. C. A. Charalambides, Central factorial numbers and related expansions, *Fibonacci Quart.*, **19** (1981), 451–456.
6. M. W. Coffet, A set of identites for a class of alternating binomial sums arising in computing applications, *arXiv*. Available from: `https://arxiv.org/abs/math-ph/0608049`.
7. L. Comtet, *Advanced combinatorics: The art of finite and infinite expansions*, Dordrecht: D. Reidel Publishing Company, 1974.
8. M. Eastwood, H. Goldschmidt, Zero-energy felds on complex projective space, *J. Differ. Geom.*, **94** (2013), 129–157. doi: 10.4310/jdg/1361889063.
9. D. S. Kim, D. V. Dolgy, D. Kim, T. Kim, Some identities on r-central factorial numbers and r-central Bell polynomials, *Adv. Differ. Equ.*, **2019** (2019), 245. doi: 10.1186/s13662-019-2195-0.
10. D. S. Kim, H. Y. Kim, D. Kim, T. Kim, On r-central incomplete and complete Bell polynomials, *Symmetry*, **11** (2019), 724. doi: 10.3390/sym11050724.
11. D. S. Kim, T. Kim, Lah-Bell numbers and polynomials, *Proc. Jangjeon Math. Soc.*, **23** (2020), 577–586.
12. D. S. Kim, J. Kwon, D. V. Dolgy, T. Kim, On central Fubini polynomials associated with central factorial numbers of the second kind, *Proc. Jangjeon Math. Soc.*, **21** (2018), 589–598.
13. H. K. Kim, Central Lah numbers and central Lah-Bell numbers, 2021. doi: 10.13140/RG.2.2.24556.08321/1.
14. H. K. Kim, D. S. Lee, Note on extended Lah-Bell polynomials and degenerate extended Lah-Bell polynomials. *Adv. Stud. Contemp. Math.*, **30** (2020), 547–558.
15. T. Kim, A note on central factorial numbers, *Proc. Jangjeon. Math. Soc.*, **21** (2018), 575–588.
16. T. Kim, D. S. Kim, r-extended Lah-Bell numbers and polynomials associated with r-Lah numbers, *Proc. Jangjeon Math. Soc.*, **24** (2021), 507–514.
17. T. Kim, D. S. Kim, A note on central Bell numbers and polynomials. *Russ. J. Math. Phys.*, **27** (2020), 76–81. doi: 10.1134/S1061920820010070.
18. I. Lah, A new kind of numbers and its application in the actuarial mathematics, *Bol. Inst. Acyuar. Port.*, **9** (1954), 7–15.
19. A. F. Loureiro, New results on the Bochner condition about classical orthogonal polynomials, *J. Math. Anal. Appl.*, **364** (2010), 307–323. doi: 10.1016/j.jmaa.2009.12.003.
20. Y. Ma, D. S. Kim, T. Kim, H. Kim, H. Lee, Some identities of Lah-Bell polynomials, *Adv. Differ. Equ.*, **2020** (2020), 510. doi: 10.1186/s13662-020-02966-6.
21. I. Martinjak, R. Skrekovski, Lah numbers and Lindstrom's lemma, *C. R. Math.*, **356** (2018), 5–7. doi: 10.1016/j.crma.2017.11.010.
22. M. Mihoubi, Bell polynomials and binomial type sequences, *Discrete Math.*, **308** (2008), 2450–2459. doi: 10.1016/j.disc.2007.05.010.
23. G. Nyul, G. Racz, The r-Lah numbers, *Discrete Math.*, **338** (2015), 1660–1666, doi: 10.1016/j.disc.2014.03.029.
24. G. Nyul, G. Racz, Sums of r-Lah numbers and r-Lah polynomials, *Ars Math. Contemp,* **18** (2020), 211–222. doi: 10.26493/1855-3974.1793.c4d.

25. C. Ramirez, M. Shattuck, A (p, q)-analogue of the r-Whitney-Lah numbers, *J. Integer Seq.*, **19** (2016), 16.5.6.

26. J. Riordan, *Combinatorial identities*, New York: John Wiley and Sons, Inc., 1968.

27. J. F. Steffensen, *Interpolation*, Baltimore 1927.

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