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Research article

Numerical simulation of time partial fractional diffusion model by Laplace transform

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Abstract: In the present work, the authors developed the scheme for time Fractional Partial Diffusion Differential Equation (FPDDE). The considered class of FPDDE describes the flow of fluid from the higher density region to the region of lower density, macroscopically it is associated with the gradient of concentration. FPDDE is used in different branches of science for the modeling and better description of those processes that involve flow of substances. The authors introduced the novel concept of fractional derivatives in term of both time and space independent variables in the proposed FPDDE. We provided the approximate solution for the underlying generalized non-linear time PFDDE in the sense of Caputo differential operator via Laplace transform combined with Adomian decomposition method known as Laplace Adomian Decomposition Method (LADM). Furthermore, we established the general scheme for the considered model in the form of infinite series by aforementioned techniques. The consequent results obtained by the proposed technique ensure that LADM is an effective and accurate technique to handle nonlinear partial differential equations as compared to the other available numerical techniques. At the end of this paper, the obtained numerical solution is visualized graphically by Matlab to describe the dynamics of desired solution.

Keywords: partial fraction diffusion equations; Laplace transform; numerical approximation; Matlab
Mathematics Subject Classification: Primary: 37A25; Secondary: 34D20, 37M01

1. Introduction

Those dynamical and biological phenomena that involve the rate of change are usually modeled by Ordinary Differential Equations (ODEs) or Partial Differential Equations (PDEs). Differential Equations (DEs) have the ability to predict all types of dynamic phenomena and are used to describe the exponential growth and decay over time. DEs have a wide range of applications in various fields, such as physics, engineering, biology, etc. Furthermore, some useful applications of DEs to model the engineering and physical phenomena can be found in some recent articles, see [1–5]. ODEs often model one dimensional dynamical system, such as moment, flow of electricity, motion of an object and motion of pendulum to explain thermodynamics concepts, for detail, see [6–8]. DEs are also used in the formulation of biological fields, such as to check growth and decay of diseases. On the other hand PDEs are also used to model multidimensional system. One of the important and significant application of PDEs is that it can be used to formulate natural phenomena, such as sound waves, heat transform, electrostatics, electrodynamics, quantum mechanics and flow of fluid, see [9–14].

An essential aspect of the proposed field is to investigate Fractional Partial Differential Equations (FPDEs). The researchers have made considerable attention to investigate the concerned class of DEs from both theoretical and application point of view. Fractional operators have some advantages over conventional derivatives, such as hereditary properties, memory effects, globle in nature and great degree of freedom. Keeping this importance in view, researchers paid more attention to investigate the considered class of PFDEs. The mathematical models involving fractional-order derivatives are more reliable and accurate as compared to traditional derivatives. In some situations, a mathematical model involving integer-order derivative does not describe the real situation. In this connection, fractional-order derivatives are more efficient to describe such a real-world problems, see [15–21].

Diffusion equation is one of the well known PDEs, that is related to the flow of fluid from the higher density region to the region of lower density; macroscopically it is associated with the gradient of concentration. Fractional Partial Diffusion Models (FPDMs) are used in different branches of science for the modeling and better description of those processes that involve flow of substances. For example, the time-fractional diffusion model describes the diffusion of porous media in physics, relaxation phenomena, and model anomalous diffusion. Apart from this, FPDMs are also used in the modeling of a variety of biological processes, like the transport of fusion plasma, see [22].

FPDDEs are the generalization of diffusion models of integer order derivatives. S. Kumar et al. [23, 24] studied the time FPDDEs under external force, that is given by

$$\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}} = K\frac{\partial^{2}\psi(x,t)}{\partial x^{2}} - \frac{\partial}{\partial x}\Big(\psi(x,t)F(x)\Big), \qquad 0 < \alpha \leq 1, \quad K > 0. \tag{1.1}$$

where the probability density function is represented by $\psi(x,t)$ for the particle at a point 'x' and time 't'. The constant $K > 0$ depends upon the universal gas constant, which is a coefficient of fraction, temperature, external force $F(x)$ and Avogadro number.

The researchers take a keen interest in analysis of FPDDEs, due to numerous applications in physical science, biology and medicine. Moreover, fractional diffusion equations are used to model turbulent flow, groundwater contaminant transport, and the chaotic dynamics of the classical conservative system. Therefore, different analytic and numerical techniques are used for the investigation of such a models [25]. The researchers studied different features of FPDEs by various numerical techniques, such as the New Homotopy Perturbation Method (NHPM) [26], Finite Difference Method (FDM) [27], New Iterative Method (NIM) [28] and many more [29, 30]. Recently, S. Kumar et al., studied the time diffusion equation via Homotopy Perturbation Transform Method (HPTM) [23], with fractional derivative term of independent variable 't' only.

In this paper, the authors have generalized the idea used in [22], and introduced the novel concept of fractional-order derivatives in the sense of both independent variables 't' and 'x'. The newly constructed FPDDE is given by

$$\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}} = K\frac{\partial^{\beta}\psi(x,t)}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(\psi(x,t)F(x)\Big), \qquad 0 < \alpha, \gamma \le 1, \quad 1 < \beta \le 2, \quad K > 0. \tag{1.2}$$

The authors used the tool of LADM, to investigate the considered model. Laplace Adomian decomposition is the best tool to obtain the approximate solution of nonlinear PFDEs. LADM is the combination of two powerful techniques, the Adomian decomposition method and Laplace transform. LT inversion is a well-known ill-posed problem. Here we assume to compute an exact inverse from LT tables. When a closed form for the solution is unknown, LT numerical inversion is needed. It requires attention and different methods, algorithms, and software elements are available in the literature, see [31, 32]. Using the proposed techniques, we can obtain both analytical, if exists and approximate solutions for nonlinear DEs. The considered technique is more powerful as compared to the other available techniques, because it gives us particular solutions without finding general solution for DEs. Furthermore, it does not require predefined size declaration like Runge-Kutta method, possesses less parameters, no need of discretization and linearization. As compared to other analytical techniques, the proposed technique is efficient and simple to investigate numerical solution of nonlinear fractional partial differential equations. The results obtained by this method ensure the capability and reliability of the proposed method for nonlinear fractional partial differential equations, see [21, 33, 34].

2. Preliminaries

The concerned section, is devoted to the well-known definitions related to fractional calculus that will be useful in further correspondence in this work.

Definition 2.1. *The LT of a function $g(x,t)$ defined $\forall\ t \ge 0$, is denoted by $G(x,s) = \mathscr{L}\{g(x,t)\}$ and is given as*

$$G(x,s) = \mathscr{L}\{g(x,t)\} = \int_0^{\infty} e^{-st} g(x,t)dt,$$

where '$\mathscr{L}$' is called LT operator or Laplace transformation and '$\mathbf{s}$' is the transformed variable.

Definition 2.2. *In Caputo sense, the fractional order derivative for the function ψ on the interval $(0,\infty)\times(0,\infty)$ is defined such as*

$${}^{c}\mathcal{D}^{\alpha}\psi(x,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\psi^{m}(x,s)ds, \qquad \alpha \in (m-1,m)\ ,\ m \in \mathbb{N}.$$

where $m = [\alpha]+1$, $[\alpha]$ is the integral part of α. As $\alpha \longrightarrow m$ the Caputo fractional derivative becomes conventional nth order derivative of the function.

Particularly for $\alpha \in (0,1)$,

$${}^{c}\mathcal{D}^{\alpha}\psi(x,t) = \frac{1}{\Gamma(m-1)}\int_0^t \frac{1}{(t-s)^{\alpha}}\frac{\partial}{\partial s}\psi(x,s)ds.$$

Definition 2.3. *The LT of Caputo derivatives is given by*

$$\mathscr{L}\{{}^{c}\mathcal{D}^{\alpha}\psi(x,t)\} = s^{\alpha}\psi(x,s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}\psi^{k}(x,0), \qquad \alpha \in (m-1,m), \qquad m \in N.$$

where $m = [\alpha] + 1$ *and* $[\alpha]$ *denote the integral part of* α.

3. Analysis of time partial fractional diffusion equation

This section of the work, is devoted to the general scheme for solution of time FPDDE (1.2), via LT. The proposed time FPDDE under external force is given by

$$\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}} = K\frac{\partial^{\beta}\psi(x,t)}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(\psi(x,t)F(x)\Big), \qquad 0 < \alpha,\gamma \leq 1, \quad 1 < \beta \leq 2, \quad K > 0 \tag{3.1}$$

subjected to the initial condition(IC)

$$\psi(x,0) = h(x).$$

Where the probability density function is represented by $\psi(x,t)$ for the particle at a point 'x', time 't' and the constant $K > 0$ depends upon universal gas constant, coefficient of fraction, temperature, external force $F(x)$ and Avogadro number.

Applying LT to (3.1), we get

$$\mathscr{L}\Big\{\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}}\Big\} = \mathscr{L}\Big\{K\frac{\partial^{\beta}\psi(x,t)}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(\psi(x,t)F(x)\Big)\Big\}.$$

By using the properties of LT, the above relation becomes

$$\psi(x,s) - \frac{1}{s}\psi(x,0) = \frac{1}{s^{\alpha}}\mathscr{L}\Big\{K\frac{\partial^{\beta}\psi(x,t)}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(\psi(x,t)F(x)\Big)\Big\}.$$

Applying inverse LT and using IC

$$\psi(x,t) = h(x) + \mathscr{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathscr{L}\Big\{K\frac{\partial^{\beta}\psi(x,t)}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(\psi(x,t)F(x)\Big)\Big\}\Big]. \tag{3.2}$$

Assuming the solution $\psi(x,t)$ in the form of infinite series, we have

$$\psi(x,t) = \sum_{i=0}^{\infty}\psi_i(x,t).$$

Thus, Eq (3.2) becomes

$$\sum_{i=0}^{\infty}\psi_i(x,t) = h(x) + \mathscr{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathscr{L}\Big\{K\frac{\partial^{\beta}}{\partial x^{\beta}}\sum_{i=0}^{\infty}\psi_i(x,t) - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(F(x)\sum_{i=0}^{\infty}\psi_i(x,t)\Big)\Big\}\Big]. \tag{3.3}$$

Comparing the i^{th} terms of Eq (3.3), we have

$$\begin{aligned}
\psi_0(x,t) &= h(x),\\
\psi_1(x,t) &= \mathscr{L}^{-1}\Big[\frac{1}{s^\alpha}\mathscr{L}\Big\{K\frac{\partial^\beta\psi_0(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_0(x,t)\Big)\Big\}\Big],\\
\psi_2(x,t) &= \mathscr{L}^{-1}\Big[\frac{1}{s^\alpha}\mathscr{L}\Big\{K\frac{\partial^\beta\psi_1(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_1(x,t)\Big)\Big\}\Big],\\
\psi_3(x,t) &= \mathscr{L}^{-1}\Big[\frac{1}{s^\alpha}\mathscr{L}\Big\{K\frac{\partial^\beta\psi_2(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_2(x,t)\Big)\Big\}\Big],\\
&\vdots
\end{aligned}$$

$$\psi_n(x,t) = \mathscr{L}^{-1}\Big[\frac{1}{s^\alpha}\mathscr{L}\Big\{K\frac{\partial^\beta\psi_{n-1}(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_{n-1}(x,t)\Big)\Big\}\Big]. \qquad n \geq 1 \tag{3.4}$$

By simple computation, we get

$$\begin{aligned}
\psi_0(x,t) &= h(x),\\
\psi_1(x,t) &= \Big\{K\frac{\partial^\beta\psi_0(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_0(x,t)\Big)\Big\}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
\psi_2(x,t) &= \Big\{K\frac{\partial^\beta\xi}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\xi\Big)\Big\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
\psi_3(x,t) &= \Big\{K\frac{\partial^\beta\zeta}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\zeta\Big)\Big\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},
\end{aligned}$$

where

$$\xi = K\frac{\partial^\beta\psi_0(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_0(x,t)\Big)$$

and

$$\zeta = K\frac{\partial^\beta\xi}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\xi\Big).$$

Continuing on the same fashion, we obtain the series solution as

$$\begin{aligned}
\psi(x,t) =& h(x) + \Big\{K\frac{\partial^\beta\psi_0(x,t)}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\psi_0(x,t)\Big)\Big\}\frac{t^\alpha}{\Gamma(\alpha+1)} + \Big\{K\frac{\partial^\beta\xi}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\xi\Big)\Big\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&+ \Big\{K\frac{\partial^\beta\zeta}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\zeta\Big)\Big\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots
\end{aligned}$$

The error in the infinite series can be calculated from the n_{th} term of the series truncated by $n-1$ terms. Following truncation error in Taylor series, power series etc we can say that the order of error is $O(z^{n\alpha})$. Consequently, we can say that the error in the series after truncating it to first four terms is given by

$$Error = \Big\{K\frac{\partial^\beta\tau}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\tau\Big)\Big\}\frac{z^{4\alpha}}{\Gamma(4\alpha+1)}, \qquad z \in (t_0, t_f),$$

where

$$\tau = K\frac{\partial^\beta\zeta}{\partial x^\beta} - \frac{\partial^\gamma}{\partial x^\gamma}\Big(F(x)\zeta\Big)$$

Now the truncated approximate solution can be expressed as

$$\psi(x,t) = h(x) + \left\{K\frac{\partial^{\beta}\psi_0(x,t)}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(F(x)\psi_0(x,t)\Big)\right\}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left\{K\frac{\partial^{\beta}\xi}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(F(x)\xi\Big)\right\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ \left\{K\frac{\partial^{\beta}\zeta}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(F(x)\zeta\Big)\right\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + Error,$$

where

$$Error = \left\{K\frac{\partial^{\beta}\tau}{\partial x^{\beta}} - \frac{\partial^{\gamma}}{\partial x^{\gamma}}\Big(F(x)\tau\Big)\right\}\frac{z^{4\alpha}}{\Gamma(4\alpha+1)}, \qquad z \in (t_0, t_f).$$

4. Numerical discussion

This section is comprised of few examples to elaborate the constructed scheme for considered model, we have also presented plots to illustrate the computation for proposed examples.

Example 1. *Consider the time PFDE with $K = 1$, $F(x) = -x$, $\beta = 2$ and $\gamma = 1$, the proposed equation is given as*

$$\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}} = \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi(x,t)\Big), \tag{4.1}$$

subjected to IC:

$$\psi(x,0) = x^2.$$

In applying LT to (4.1), we get

$$\mathscr{L}\left\{\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}}\right\} = \mathscr{L}\left\{\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi(x,t)\Big)\right\}.$$

By using properties of LT, the above relation becomes

$$\psi(x,s) - \frac{1}{s}\psi(x,0) = \frac{1}{s^{\alpha}}\mathscr{L}\left\{\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi(x,t)\Big)\right\}.$$

From the inverse LT and using the IC, we obtain

$$\psi(x,t) = x^2 + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi(x,t)\Big)\right\}\right]. \tag{4.2}$$

Let us assume the solution $\psi(x,t)$ in terms of infinite series as given by

$$\psi(x,t) = \sum_{i=0}^{\infty}\psi_i(x,t).$$

Equation (4.2) becomes

$$\sum_{i=0}^{\infty}\psi_i(x,t) = x^2 + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left\{\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\psi_i(x,t) + \frac{\partial}{\partial x}\Big(x\sum_{i=0}^{\infty}\psi_i(x,t)\Big)\right\}\right]. \tag{4.3}$$

Comparing the terms of Eq (3.3), we have

$$
\begin{aligned}
&\psi_0(x,t) = x^2,\\
&\psi_1(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{\frac{\partial^2\psi_0(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi_0(x,t)\Big)\right\}\right],\\
&\psi_2(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{\frac{\partial^2\psi_1(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi_1(x,t)\Big)\right\}\right],\\
&\psi_3(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{\frac{\partial^2\psi_2(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi_2(x,t)\Big)\right\}\right],\\
&\vdots
\end{aligned}
$$

$$
\psi_n(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{\frac{\partial^2\psi_{n-1}(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(x\psi_{n-1}(x,t)\Big)\right\}\right]. \qquad n \geq 1 \tag{4.4}
$$

Calculating the terms, we have

$$
\begin{aligned}
&\psi_0(x,t) = x^2,\\
&\psi_1(x,t) = \frac{(2+3x^2)t^\alpha}{\Gamma(\alpha+1)},\\
&\psi_2(x,t) = \frac{(8+9x^2)t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&\psi_3(x,t) = \frac{(26+27x^2)t^{3\alpha}}{\Gamma(3\alpha+1)},
\end{aligned}
$$

Continuing this manner, the desired solution of Eq (4.1), is

$$
\psi(x,t) = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(\alpha+1)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(26+27x^2)t^{3\alpha}}{\Gamma(3\alpha+1)} + ...
$$

The truncated solution after four terms is

$$
\psi(x,t) = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(\alpha+1)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(26+27x^2)t^{3\alpha}}{\Gamma(3\alpha+1)} + Error.
$$

For the considered example, Error is

$$
Error = \Big(80x + 81x^2\Big)\frac{z^{4\alpha}}{\Gamma(4\alpha+1)}.
$$

For classical order $\alpha = 1$, the obtained solution is

$$
\psi(x,t) = x^2 + (2+3x^2)t + \frac{(8+9x^2)t^2}{2} + \frac{(26+27x^2)t^3}{6} + \Big(80x + 81x^2\Big)\frac{z^4}{\Gamma(24)}.
$$

Example 2. *Consider the time PFDE with $K = 2$, $F(x) = -e^{-x}$, $\beta = 2$ and $\gamma = 1$, the proposed equation is given as*

$$
\frac{\partial^\alpha\psi(x,t)}{\partial t^\alpha} = 2\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\Big(e^{-x}\psi(x,t)\Big), \tag{4.5}
$$

subjected to IC:

$$\psi(x,0) = e^{-x} + x.$$

Applying LT to (4.5), we get

$$\mathscr{L}\left\{\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha}\right\} = \mathscr{L}\left\{2\frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi(x,t)\right)\right\}.$$

By using properties of LT, the above relation becomes

$$\psi(x,s) - \frac{1}{s}\psi(x,0) = \frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi(x,t)\right)\right\}.$$

Applying now the inverse LT and using IC, we have

$$\psi(x,t) = e^{-x} + x + \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi(x,t)\right)\right\}\right]. \tag{4.6}$$

Assuming the solution $\psi(x,t)$ in terms of infinite series

$$\psi(x,t) = \sum_{i=0}^{\infty} \psi_i(x,t).$$

Thus, the Eq (4.6) becomes

$$\sum_{i=0}^{\infty} \psi_i(x,t) = e^{-x} + x + \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} \psi_i(x,t) + \frac{\partial}{\partial x}\left(e^{-x}\sum_{i=0}^{\infty} \psi_i(x,t)\right)\right\}\right]. \tag{4.7}$$

Comparing both sides of Eq (4.7), we have

$$\begin{aligned}
\psi_0(x,t) &= e^{-x} + x,\\
\psi_1(x,t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2 \psi_0(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi_0(x,t)\right)\right\}\right],\\
\psi_2(x,t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2 \psi_1(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi_1(x,t)\right)\right\}\right],\\
\psi_3(x,t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2 \psi_2(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi_2(x,t)\right)\right\}\right],\\
&\vdots
\end{aligned}$$

$$\psi_n(x,t) = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left\{2\frac{\partial^2 \psi_{n-1}(x,t)}{\partial x^2} + \frac{\partial}{\partial x}\left(e^{-x}\psi_{n-1}(x,t)\right)\right\}\right]. \qquad n \geq 1 \tag{4.8}$$

Calculating the terms, we have

$$\begin{aligned}
\psi_0(x,t) &= e^{-x} + x,\\
\psi_1(x,t) &= \frac{(3e^{-x} - 2e^{-2x} - xe^{-x})t^\alpha}{\Gamma(\alpha+1)},\\
\psi_2(x,t) &= \frac{(10e^{-x} - 23e^{-2x} + 6e^{-3x} - 2xe^{-x} + 2xe^{-2x})t^{2\alpha}}{\Gamma(2\alpha+1)},
\end{aligned}$$

Continuing the same fashion, the solution of Eq (4.5), is obtained as

$$\psi(x,t) = e^{-x} + x + \frac{(3e^{-x} - 2e^{-2x} - xe^{-x})t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(10e^{-x} - 23e^{-2x} + 6e^{-3x} - 2xe^{-x} + 2xe^{-2x})t^{2\alpha}}{\Gamma(2\alpha+1)} + ...$$

The solution truncated after three terms is

$$\psi(x,t) = e^{-x} + x + \frac{(3e^{-x} - 2e^{-2x} - xe^{-x})t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(10e^{-x} - 23e^{-2x} + 6e^{-3x} - 2xe^{-x} + 2xe^{-2x})t^{2\alpha}}{\Gamma(2\alpha+1)} + Error,$$

where

$$Error = \frac{(6e^{-x} - 122e^{-2x} + 126e^{-3x} - 24e^{-4x} + 2xe^{-x} + 8xe^{-2x} - 6xe^{-3x})z^{3\alpha}}{\Gamma(3\alpha+1)}, \qquad z \in (t_0, t_f).$$

For classical order $\alpha = 1$, the obtained solution is

$$\begin{aligned}\psi(x,t) =& e^{-x} + x + (3e^{-x} - 2e^{-2x} - xe^{-x})t + (10e^{-x} - 23e^{-2x} + 6e^{-3x} - 2xe^{-x} + 2xe^{-2x})\frac{t^2}{2} + \\ & (6e^{-x} - 122e^{-2x} + 126e^{-3x} - 24e^{-4x} + 2xe^{-x} + 8xe^{-2x} - 6xe^{-3x})\frac{z^3}{6}, \qquad z \in (t_0, t_f).\end{aligned}$$

5. Discussion

The obtained results reveal the complete agreement with the results obtained by Das [24] and S. Kumar, et al. [23] by VIM and LHPM respectively. The plots obtained for the LADM solution and VIM solution also show complete similarity, which justifies the claim of complete agreement. LADM solution is almost valid for a wide range of nonlinear DEs. The consequences obtained by the proposed technique ensure that the scheme is very effective and easy to implement in considering a class of nonlinear PFDDE models. In comparison with other analytical techniques, the proposed technique is an efficient and simple tool to investigate numerical solution of nonlinear fractional partial differential equations with a high degree of accuracy.

The plots illustrate the possible concentration of finding the flowing particles of a fluid at any point and time, based on the behavior of the approximated solution. Figures 1 and 2 show the behavior of the solution of example 1, in which the dependent variable has direct linkage with space and time variables. Where as Figures 3 and 4 show the behavior of solution of example 2, in which the dependent variable has direct linkage with time and inverse with space variables. For different values of α, the two dimensional behavior of the corresponding approximated solutions is also displayed in the given plots.

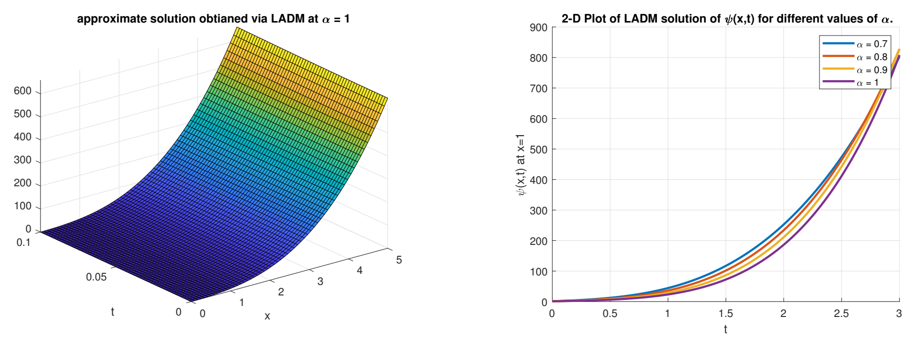

Figure 1. 3-D and 2-D plots of approximate solution of Example 1, obtained via LADM.

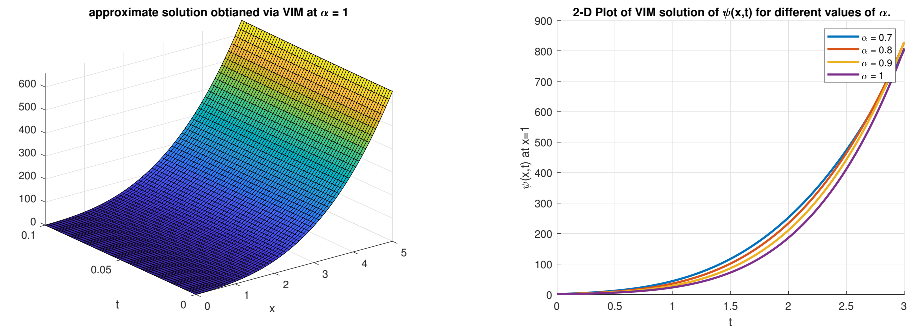

Figure 2. 3-D and 2-D plots of approximate solution of Example 1, obtained via VIM.

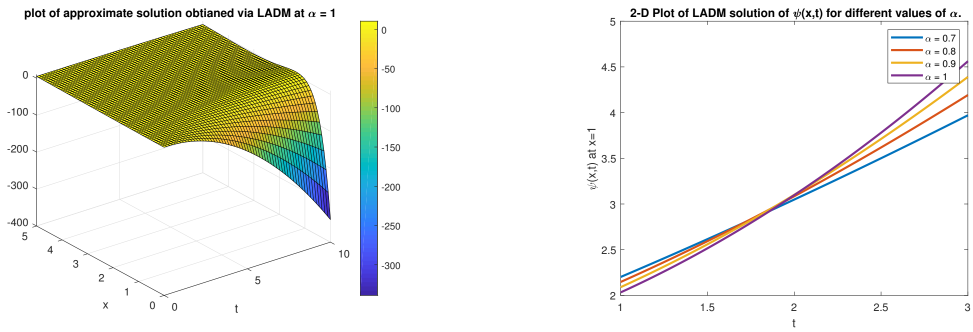

Figure 3. 3-D and 2-D plots of approximate solution of Example 2, obtained via LADM.

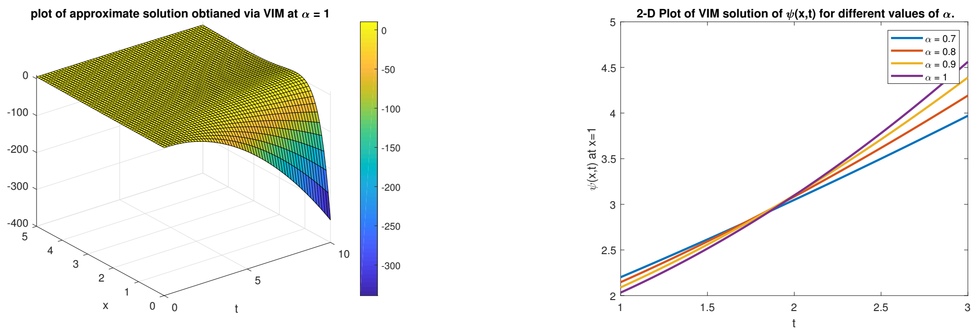

Figure 4. 3-D and 2-D plots of approximate solution of Example 2, obtained via VIM.

6. Conclusions

In this paper, the authors developed the scheme for generalized time FPDDE involving more than one fractional derivative. In order to obtain the desired results, the authors utilized the tools of well-known numerical techniques so-called LADM. To elaborate our main results, we provided some numerical problems for illustrative purposes in few iterations. Which provides the reliability of proposed techniques. To describe the dynamics of the proposed model, we visualized the obtained results graphically via Matlab.

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Conflict of interest

There exist no conflict of interest.

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