AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2680–2694.
DOI: 10.3934/math.2022151
Received: 14 August 2021
Accepted: 20 October 2021
Published: 18 November 2021

Research article

Positive solutions of infinite coupled system of fractional differential equations in the sequence space of weighted means

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Abstract: We first discuss the existence of solutions of the infinite system of $(n-1,n)$-type semipositone boundary value problems (BVPs) of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}_{0_+}u_i(\rho)+\eta f_i(\rho,v(\rho))=0, & \rho\in(0,1),\\ D^{\alpha}_{0_+}v_i(\rho)+\eta g_i(\rho,u(\rho))=0, & \rho\in(0,1),\\ u_i^{(j)}(0)=v_i^{(j)}(0)=0, & 0\le j\le n-2,\\ u_i(1)=\zeta\int_0^1 u_i(\vartheta)d\vartheta,\ v_i(1)=\zeta\int_0^1 v_i(\vartheta)d\vartheta, & i\in\mathbb{N}, \end{cases}$$

in the sequence space of weighted means $c_0(W_1,W_2,\Delta)$, where $n\ge 3$, $\alpha\in(n-1,n]$, η,ζ are real numbers, $0<\eta<\alpha$, $D^{\alpha}_{0_+}$ is the Riemann-Liouville's fractional derivative, and f_i,g_i, $i=1,2,\ldots$, are semipositone and continuous. Our approach to the study of solvability is to use the technique of measure of noncompactness. Then, we find an interval of η such that for each η lying in this interval, the system of $(n-1,n)$-type semipositone BVPs has a positive solution. Eventually, we demonstrate an example to show the effectiveness and usefulness of the obtained result.

Keywords: Darbo's theorem; difference sequence space of weighted means; infinite system of boundary value problems; measure of noncompactness
Mathematics Subject Classification: 47H09, 47H10, 34A12

1. Introduction and preliminaries

We are interested to discuss about the existence of positive solutions of the following infinite coupled system of $(n-1,n)$-type semipositone boundary value problems (BVPs) of nonlinear fractional differential equations (IBVP for short) in the sequence space of weighted means $c_0(W_1, W_2, \Delta)$

$$\begin{cases} D^{\alpha}_{0_+} u_i(\rho) + \eta f_i(\rho, v(\rho)) = 0, & \rho \in (0,1), \\ D^{\alpha}_{0_+} v_i(\rho) + \eta g_i(\rho, u(\rho)) = 0, & \rho \in (0,1), \\ u_i^{(j)}(0) = v_i^{(j)}(0) = 0, & 0 \le j \le n-2, \\ u_i(1) = \zeta \int_0^1 u_i(\vartheta) d\vartheta, \ v_i(1) = \zeta \int_0^1 v_i(\vartheta) d\vartheta, & i \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $n \ge 3$, $\alpha \in (n-1, n]$, η, ζ are real numbers, $0 < \eta < \alpha$, $D^{\alpha}_{0_+}$ is the Riemann-Liouville's (R-L's) fractional derivative, and f_i, g_i, $i = 1, 2, \ldots$, are continuous and sign-changing. This kind of problems that the nonlinearity in (1.1) may change signs is mentioned as semipositone problems in the literature.

Fractional differential equations (FDEs) occur in the various fields of biology [16], economy [20, 38], engineering [24,32], physical phenomena [5,7,8,16,25], applied science, and many other fields [3, 9, 14, 21]. Hristova and Tersian [18] solved an FDE with a different strategy, and Harjani, López, and Sadarangani [17] solved an FDE using a fixed point approach. Now, we intend to solve an FDE by using the technique of measure of noncompactness. On the other hand, we encounter many real world problems, which can be modeled and described using infinite systems of FDEs (see [4, 27, 34, 36, 37]). In the theory of infinite system of FDEs, the measure of noncompactness (MNC) plays a significant role, which was introduced by Kuratowski [23] (see recent works [27, 35, 36]). The MNC has been utilized in sequence spaces for various classes of differential equations, see [2, 6, 11–13, 26, 29, 30, 35, 36].

The difference sequence spaces of weighted means $\lambda(u, v, \Delta)$ ($\lambda = c_0, c$, and l_∞) first have been introduced in [33]. Thereafter, Mursaleen et al. [28] constructed some estimations for the Hausdorff MNC of some matrix operators on these spaces. They also determined several classes of compact operators in such spaces. Motivated by the mentioned papers, in this work, we first discuss the existence of solutions of IBVP (1.1) in the difference sequence space of weighted means $c_0(W_1, W_2, \Delta)$. Then, we find an interval of η such that for any η belongs to this interval, IBVP (1.1) has a positive solution. Eventually, we demonstrate an example illustrating the obtained results. Here, we preliminarily collect some definitions and auxiliary facts applied throughout this paper.

Suppose that $(\Lambda, \|\cdot\|)$ is a real Banach space containing zero element. We mean by $D(z, r)$ the closed ball centered at z with radius r. For a nonempty subset $\mathcal{U}$ of Λ, the symbol $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ and the symbol $\mathrm{Conv}\mathcal{U}$ denotes the closed convex hull of $\mathcal{U}$. We denote by $\mathfrak{M}_\Lambda$ the family of all nonempty, bounded subsets of Λ and by $\mathfrak{N}_\Lambda$ the family consisting of nonempty relatively compact subsets of Λ.

Definition 1.1. [1] The function $\tilde{\mu} : \mathfrak{M}_\Lambda \to [0, +\infty)$ is called an MNC in Λ if for any $\mathcal{U}, \mathcal{V}_1, \mathcal{V}_2 \in \mathfrak{M}_\Lambda$, the properties (i)–(v) hold:

(i) $\ker \tilde{\mu} = \{\mathcal{U} \in \mathfrak{M}_\Lambda : \tilde{\mu}(\mathcal{U}) = 0\} \neq \emptyset$ and $\ker \tilde{\mu} \subseteq \mathfrak{N}_\Lambda$.
(ii) If $\mathcal{V}_1 \subset \mathcal{V}_2$, then $\tilde{\mu}(\mathcal{V}_1) \le \tilde{\mu}(\mathcal{V}_2)$.
(iii) $\tilde{\mu}(\overline{\mathcal{U}}) = \tilde{\mu}(\mathrm{Conv}\mathcal{U}) = \tilde{\mu}(\mathcal{U})$.
(iv) For each $\rho \in [0,1]$, $\tilde{\mu}(\rho\mathcal{U} + (1-\rho)\mathcal{V}) \le \rho\tilde{\mu}(\mathcal{U}) + (1-\rho)\tilde{\mu}(\mathcal{V})$.
(v) If for each natural number n, $\mathcal{U}_n$ is a closed set in $\mathfrak{M}_\Lambda$, $\mathcal{U}_{n+1} \subset \mathcal{U}_n$, and $\lim_{n\to\infty}\tilde{\mu}(\mathcal{U}_n) = 0$, then $\mathcal{U}_\infty = \bigcap_{n=1}^{\infty} \mathcal{U}_n$ is nonempty.

In what follows, we mean by $\mathfrak{M}_Y$, the family of bounded subsets of the metric space (Y, d).

Definition 1.2. [10] Suppose that (Y, d) is a metric space. Also, suppose that $\mathcal{P} \in \mathfrak{M}_Y$. The Kuratowski MNC of $\mathcal{P}$, which is denoted by $\alpha(\mathcal{P})$, is the infimum of the set of positive real numbers ε such that $\mathcal{P}$ can be covered by a finite number of sets of diameter less than to ε. Indeed,

$$\alpha(\mathcal{P}) = \inf\Big\{\varepsilon > 0 : \mathcal{P} \subset \bigcup_{j=1}^{n} K_j, K_j \subset Y, \mathrm{diam}(K_j) < \varepsilon\ (j = 1, \ldots, n);\ n \in \mathbb{N}\Big\},$$

when $\mathrm{diam}(K_j) = \sup\{d(\varsigma, \nu) : \varsigma, \nu \in K_j\}$.

The Hausdorff MNC (ball MNC) of the bounded set $\mathcal{P}$, which is denoted by $\chi(\mathcal{P})$, is defined by

$$\chi(\mathcal{P}) = \inf\Big\{\varepsilon > 0 : \mathcal{P} \subset \bigcup_{j=1}^{n} D(y_j, r_j), y_j \in Y, r_j < \varepsilon\ (j = 1, \ldots, n);\ n \in \mathbb{N}\Big\}.$$

Here, we quote a result contained in [10].

Lemma 1.3. *Let (Y, d) be a metric space and let $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{M}_Y$. Then*

(i) $\beta(\mathcal{P}) = 0$ *if and only if $\mathcal{P}$ is totally bounded,*
(ii) $\mathcal{P}_1 \subset \mathcal{P}_2 \Rightarrow \beta(\mathcal{P}_1) \leq \beta(\mathcal{P}_2)$,
(iii) $\beta(\overline{\mathcal{P}}) = \beta(\mathcal{P})$,
(iv) $\beta(\mathcal{P}_1 \cup \mathcal{P}_2) = \max\{\beta(\mathcal{P}_1), \beta(\mathcal{P}_2)\}$.

Besides, if Y is a normed space, then

(v) $\beta(\mathcal{P}_1 + \mathcal{P}_2) \leq \beta(\mathcal{P}_1) + \beta(\mathcal{P}_2)$,
(vi) for each complex number ρ, $\beta(\rho\mathcal{P}) = |\rho|\beta(\mathcal{P})$.

Now, we state a version of Darbo's theorem [10], which is fundamental in our work.

Theorem 1.4. *[10] Suppose that $\widetilde{\mu}$ is an MNC in a Banach space Λ. Also, suppose that $\emptyset \neq \mathfrak{D} \subseteq \Lambda$ is a bounded, closed, and convex set and that $S : \mathfrak{D} \to \mathfrak{D}$ is a continuous mapping. If a constant $\kappa \in [0, 1)$ exists such that*

$$\widetilde{\mu}(S(\mathcal{X})) \leq \kappa\widetilde{\mu}(\mathcal{X})$$

for any nonempty subset $\mathcal{X}$ of $\mathfrak{D}$, then S has a fixed point in the set $\mathfrak{D}$.

Suppose that $J = [0, s]$ and that Λ is a Banach space. Consider the Banach space $C(J, \Lambda)$ with the norm

$$\|z\|_{C(J,\Lambda)} := \sup\{\|z(\rho)\| : \rho \in J\}, \qquad z \in C(J, \Lambda).$$

Proposition 1.5. *[10] Suppose that $\Omega \subseteq C(J, \Lambda)$ is equicontinuous and bounded. Then $\tilde{\mu}(\Omega(\cdot))$ is continuous on J and*

$$\tilde{\mu}(\Omega) = \sup_{\rho\in J} \tilde{\mu}(\Omega(\rho)), \qquad \tilde{\mu}\Big(\int_0^{\rho} \Omega(\varrho)d\varrho\Big) \leq \int_0^{\rho} \tilde{\mu}(\Omega(\varrho))d\varrho.$$

Definition 1.6. (see [22, 31]) Suppose that $f : (0, \infty) \to \mathbb{R}$ is a continuous function. The R-L's fractional derivative of order ℓ ($\ell > 0$) is defined as

$$D^{\ell}_{0_+} f(\jmath) = \frac{1}{\Gamma(n-\ell)} \Big(\frac{d}{d\jmath}\Big)^n \int_0^{\jmath} \frac{f(\varsigma)}{(\jmath - \varsigma)^{1-n+\ell}} d\varsigma,$$

when $n = [\ell] + 1$ and the right-hand side is pointwise defined on $(0, +\infty)$.

We terminate this section by describing the unique solution of a nonlinear FDE, which will be needed later.

Lemma 1.7. *[39] Let $h \in C[0,1]$. Then the BVP*

$$\begin{cases} D^{\ell}_{0_+}u(\rho) + h(\rho) = 0, & \rho \in (0,1),\ 2 \le n-1 < \ell \le n, \\ u^{(j)}(0) = 0, & j \in [0, n-2], \\ u(1) = \zeta \int_0^1 u(\varrho)d\varrho, \end{cases} \tag{1.2}$$

has a unique solution

$$u(\rho) = \int_0^1 H(\rho,\varrho)h(\varrho)d\varrho,$$

when $H(\rho,\varrho)$ is the Green's function of BVP (1.2) *defined as*

$$H(\rho,\varrho) = \begin{cases} \frac{\rho^{\ell-1}(1-\varrho)^{\ell-1}(\ell-\zeta+\zeta\varrho)-(\ell-\zeta)(\rho-\varrho)^{\ell-1}}{(\ell-\zeta)\Gamma(\ell)}, & 0 \le \varrho \le \rho \le 1, \\ \frac{\rho^{\ell-1}(1-\varrho)^{\ell-1}(\ell-\zeta+\zeta\varrho)}{(\ell-\zeta)\Gamma(\ell)}, & 0 \le \rho \le \varrho \le 1. \end{cases}$$

The function $H(\rho,\varrho)$ has the following properties:

$$\zeta\rho^{\ell-1}q(\varrho) \le H(\rho,\varrho) \le \frac{M_0\rho^{\ell-1}}{(\ell-\zeta)\Gamma(\ell)}, \quad H(\rho,\varrho) \le M_0 q(\varrho), \quad \textit{for}\ \rho,\varrho \in [0,1],$$

where $M_0 = (\ell-\zeta)(\ell-1) + \ell + \zeta$ and $q(\varrho) = \frac{\varrho(1-\varrho)^{\ell-1}}{(\ell-\zeta)\Gamma(\ell)}$.

2. Difference sequence space of weighted means $c_0(W_1, W_2, \Delta)$

Suppose that S is the space of complex or real sequences. Any vector subspace of S is said to be a sequence space. We denote by c the space of convergent sequences and by c_0 the space of null sequences.

A complete linear metric sequence space is called an FK space if it has the property that convergence implies coordinatewise convergence. Moreover, a normed FK space is called a BK space. It is known the spaces c_0 and c are BK spaces with the norm $\|z\|_\infty = \sup_{k\in\mathbb{N}}|z_k|$ (see [12]).

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are sequence spaces. We denote by $(\mathcal{X},\mathcal{Y})$ the class of infinite matrices $\mathcal{B}$ that map $\mathcal{X}$ into $\mathcal{Y}$. We denote by $\mathcal{B} = (b_{mk})_{m,k=0}^{\infty}$ an infinite complex matrix and by $\mathcal{B}_m$ its mth row. Then we can write

$$\mathcal{B}_m(x) = \sum_{k=0}^{\infty} b_{mk}x_k \text{ and } \mathcal{B}(x) = (\mathcal{B}_m(x))_{m=0}^{\infty}.$$

Thus $\mathcal{B} \in (\mathcal{X},\mathcal{Y})$ if and only if $\mathcal{B}_m(x)$ converges for all m and all $x \in \mathcal{X}$ and $\mathcal{B}(x) \in \mathcal{Y}$.

The set

$$\mathcal{X}_{\mathcal{B}} = \{x \in S : \mathcal{B}(x) \in \mathcal{X}\} \tag{2.1}$$

is called the matrix domain of $\mathcal{B}$ in $\mathcal{X}$; see [19]. An infinite matrix $Y = (y_{nl})$ is said to be a triangle if $y_{nn} \neq 0$ and $y_{nl} = 0$ for each $l > n$. The matrix domain of a triangle Y, $\mathcal{X}_Y$, shares many properties with the sequence space $\mathcal{X}$. For instance, if $\mathcal{X}$ is a BK space, then $\mathcal{X}_Y$ is a BK space with the norm $\|Z\|_{\mathcal{X}_Y} = \|YZ\|_{\mathcal{X}}$ for each $Z \in \mathcal{X}_Y$; see [15].

Now, let $W = (w_k)$ be a sequence. The difference sequence of W is denoted by $\Delta W = (w_k - w_{k-1})$. Suppose that $W_1 = (w_k^1)$ and $W_2 = (w_k^2)$ are the sequences of real numbers such that $w_k^1 \neq 0$ and $w_k^2 \neq 0$ for all k. Also, consider the triangle $Y = (y_{nl})$ defined by

$$(y_{nl}) = \begin{cases} w_n^1(w_l^2 - w_{l+1}^2), & l \leq n, \\ w_n^1 w_n^2, & l = n, \\ 0, & l > n. \end{cases}$$

The difference sequence space of weighted means $c_0(W_1, W_2, \Delta)$ is defined as the matrix domain of the triangle Y in the space c_0. Evidently, $c_0(W_1, W_2, \Delta)$ is a BK space with the norm defined by

$$\|x\| = \|Y(x)\|_\infty = \sup_m |Y_m(x)|, \qquad x \in c_0(W_1, W_2, \Delta).$$

Now, we describe the Hausdorff MNC χ in the space $c_0(W_1, W_2, \Delta)$. For this purpose, we quote the following two theorems.

Theorem 2.1. *[26] Suppose that $\mathcal{P} \in \mathfrak{M}_{c_0}$. Also, suppose that $P_m : c_0 \to c_0$ is the operator defined by $P_m(z) = (z_0, z_1, \ldots, z_m, 0, 0, \ldots)$. Then*

$$\chi(\mathcal{P}) = \lim_{m\to\infty} \sup_{z\in\mathcal{P}} \|(\mathcal{I} - P_m)(z)\|_\infty,$$

when $\mathcal{I}$ is the identity operator.

Theorem 2.2. *[19] Let X be a normed sequence space. Also, let χ_Y and χ denote the Hausdorff MNC on $\mathfrak{M}_{X_Y}$ and $\mathfrak{M}_X$, the family of bounded sets in X_Y and X, respectively. Then*

$$\chi_Y(\mathcal{P}) = \chi(Y(\mathcal{P})),$$

where $\mathcal{P} \in \mathfrak{M}_{X_Y}$.

Combining these two facts gives us the following theorem.

Theorem 2.3. *Let $\mathcal{P} \in \mathfrak{M}_{c_0(W_1,W_2,\Delta)}$. Then the Hausdorff MNC χ on the space $c_0(W_1, W_2, \Delta)$ can be defined as the following form:*

$$\chi_Y(\mathcal{P}) = \chi(Y(\mathcal{P})) = \lim_{m\to\infty} \sup_{x\in\mathcal{P}} \|(\mathcal{I} - P_m)(Y(x))\|_\infty.$$

3. Main results

In this section, we first make some sufficient conditions to discuss the existence of solutions of IBVP (1.1) in the space $c_0(W_1, W_2, \Delta)$. Then, we give an interval of η such that any η belongs to this interval and the infinite system (1.1) has a positive solution. Eventually, we demonstrate an example to present the effectiveness of the obtained result.

Here, we consider some assumptions.

$(A1)$ Let $J_1 = [0, 1]$, let $f_i, g_i \in C(J_1 \times \mathbb{R}^\infty, \mathbb{R})$, and let the function $K : J_1 \times C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta)) \to c_0(W_1, W_2, \Delta) \times c_0(W_1, W_2, \Delta)$ be defined by

$$(\varrho, U, V) \to K(U, V)(\varrho) = ((f_i(\varrho, V(\varrho)))_{i=1}^\infty, (g_i(\varrho, U(\varrho)))_{i=1}^\infty),$$

such that the family of functions $(K(U,V)(\varrho))$ is equicontinuous at each point of the space $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$.

$(A2)$ For each $k \in \mathbb{N}$ and $U = (u_i) \in C(J_1, c_0(W_1, W_2, \Delta))$, the following inequalities hold:

$$f_k(\varrho, U(\varrho)) \leq p_k(\varrho)u_k(\varrho),$$

$$g_k(\varrho, U(\varrho)) \leq q_k(\varrho)u_k(\varrho),$$

where $p_k, q_k : J_1 \to \mathbb{R}_+ = [0, +\infty)$ are mappings and the families $\{p_k\}$ and $\{q_k\}$ are equibounded.

$(A3)$ Let f_i, $g_i \in C(J_1 \times \mathbb{R}_+^\infty, \mathbb{R})$ and let a function $\theta \in L^1(J_1, (0, +\infty))$ exist such that $f_i(\rho, \mathfrak{Z}(\rho)) \geq -\theta(\rho)$ and $g_i(\rho, \mathfrak{Z}(\rho)) \geq -\theta(\rho)$, for each $i \in \mathbb{N}, \rho \in J_1$, and nonnegative sequence $(\mathfrak{Z}(\rho))$ in $c_0(W_1, W_2, \Delta)$.

$(A4)$ For any $i \in \mathbb{N}$ and $\rho \in J_1$, let $f_i(\rho, U^0(\rho)) > 0$, where $U^0(\rho) = (u_i^0(\rho))$ and $u_i^0(\rho) = 0$ for all i and all ρ. Also, the sequence $(f_i(\rho, U^0(\rho)))$ is equibounded.

$(A5)$ There exists $\sigma > 0$ such that $g_i(\rho, \mathfrak{Z}(\rho)) > 0$, where $i \in \mathbb{N}$ and $(\rho, \mathfrak{Z}(\rho)) \in J_1 \times ([0, \sigma])^\infty$.

Put

$$P := \sup_{k \in \mathbb{N}} \sup_{\varrho \in J_1} |p_k(\varrho)|,$$

and

$$Q := \sup_{k \in \mathbb{N}} \sup_{\varrho \in J_1} |q_k(\varrho)|.$$

Theorem 3.1. *Assume that IBVP* (1.1) *fulfills the hypotheses* $(A1)$, $(A2)$ *and* $\frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}(Q + P) < 1$*, then it has at least one solution.*

Proof. Let $(U, V) = ((u_i), (v_i))$ be in $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$ and satisfy the initial conditions of IBVP (1.1) and let each u_i and v_i be continuous on J_1. We define the operator T : $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta)) \to C(I, c_0(W_1, W_2, \Delta) \times c_0(W_1, W_2, \Delta))$ by

$$T(U, V)(\rho) = \Big((\eta \int_0^1 H(\rho, \varrho) f_i(\varrho, V(\varrho)) d\varrho)_{i=1}^\infty, (\eta \int_0^1 H(\rho, \varrho) g_i(\varrho, U(\varrho)) d\varrho)_{i=1}^\infty \Big).$$

Note that the product space $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$ is equipped with the norm

$$\|(U, V)\|_{C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))} = \|U\|_{C(J_1, c_0(W_1, W_2, \Delta))} + \|V\|_{C(J_1, c_0(W_1, W_2, \Delta))}$$

for each $(U, V) \in C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$. Then, using our assumptions for any $\rho \in J_1$, we can write

$$\begin{aligned}
&\|T(U, V)(\rho)\|_{c_0(W_1, W_2, \Delta) \times c_0(W_1, W_2, \Delta)} \\
&\quad = \|(\eta \int_0^1 H(\rho, \varrho) f_i(\varrho, V(\varrho)) d\varrho)\|_{c_0(W_1, W_2, \Delta)} \\
&\qquad + \|(\eta \int_0^1 H(\rho, \varrho) g_i(\varrho, U(\varrho)) d\varrho)\|_{c_0(W_1, W_2, \Delta)} \\
&\quad = |\eta| \sup_n |\sum_{k=1}^\infty Y_{nk} \int_0^1 H(\rho, \varrho) f_k(\varrho, V(\varrho)) d\varrho| \\
&\qquad + |\eta| \sup_n |\sum_{k=1}^\infty Y_{nk} \int_0^1 H(\rho, \varrho) g_k(\varrho, U(\varrho)) d\varrho|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}\Big(\sup_n|\sum_{k=1}^{\infty}Y_{nk}\int_0^1 p_k(\varrho)v_k(\varrho)d\varrho| + \sup_n|\sum_{k=1}^{\infty}Y_{nk}\int_0^1 q_k(\varrho)u_k(\varrho)d\varrho|\Big)\\
&\leq \frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}(P+Q)\big(\|U\|_{C(J_1,c_0(W_1,W_2,\Delta))} + \|V\|_{C(J_1,c_0(W_1,W_2,\Delta))}\big)\\
&= \frac{M_0|\eta|(P+Q)}{(\alpha-\zeta)\Gamma(\alpha)}\|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}.
\end{aligned}
$$

Accordingly, we obtain

$$
\begin{aligned}
&\|T(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta))}\\
&\qquad\leq \frac{M_0|\eta|(P+Q)}{(\alpha-\zeta)\Gamma(\alpha)}\|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}.
\end{aligned}
$$

It implies that

$$
r \leq \Big(\frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}(P+Q)r. \tag{3.1}
$$

Let r_0 denote the optimal solution of inequality (3.1). Take

$$
\begin{aligned}
D &= D((U^0,U^0),r_0)\\
&= \Big\{(U,V)\in C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta)) :\\
&\quad \|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))} \leq r_0,\ u_i^{(j)}(0) = v_i^{(j)}(0) = 0,\\
&\quad j\in[0,n-2],\ u_i(1) = \zeta\int_0^1 u_i(\varrho)d\varrho,\quad v_i(1) = \zeta\int_0^1 v_i(\varrho)d\varrho\Big\}.
\end{aligned}
$$

Clearly, D is bounded, closed, and convex and T is bounded on D. Now, we prove that T is continuous. Let (U_1,V_1) be a point in D and let ε be an arbitrary positive number. Employing assumption $(A1)$, there exists $\delta > 0$ such that if $(U_2,V_2) \in D$ and $\|(U_1,V_1) - (U_2,V_2)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))} \leq \delta$, then

$$
\|K((U_1,V_1)) - K((U_2,V_2))\|_{C(J_1,c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta))} \leq \frac{(\alpha-\zeta)\Gamma(\alpha)\varepsilon}{M_0|\eta|}.
$$

Therefore, for any ρ in I, we get

$$
\begin{aligned}
&\|T(U_1,V_1)(\rho) - T(U_2,V_2)(\rho)\|_{c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta)}\\
&=\Big\|\Big((\eta\int_0^1 H(\rho,\varrho)f_i(\varrho,V_1(\varrho))d\varrho),(\eta\int_0^1 H(\rho,\varrho)g_i(\varrho,U_1(\varrho))d\varrho)\Big)\\
&\quad-\Big((\eta\int_0^1 H(\rho,\varrho)f_i(\varrho,V_2(\varrho))d\varrho),(\eta\int_0^1 H(\rho,\varrho)g_i(\varrho,U_2(\varrho))d\varrho)\Big)\Big\|_{c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta)}\\
&=\|(\eta\int_0^1 H(\rho,\varrho)(f_i(\varrho,V_1(\varrho)) - f_i(\varrho,V_2(\varrho))d\varrho)\|_{c_0(W_1,W_2,\Delta)}\\
&\quad+\|(\eta\int_0^1 H(\rho,\varrho)(g_i(\varrho,U_1(\varrho)) - g_i(\varrho,U_2(\varrho))d\varrho)\|_{c_0(W_1,W_2,\Delta)}\\
&=|\eta|\sup_n|\sum_{k=1}^{\infty}Y_{nk}\int_0^1 H(\rho,\varrho)(f_k(\varrho,V_1(\varrho)) - f_k(\varrho,V_2(\varrho)))d\varrho|
\end{aligned}
$$

$$
\begin{aligned}
&+|\eta|\sup_n|\sum_{k=1}^{\infty}Y_{nk}\int_0^1 H(\rho,\varrho)(g_k(\varrho,U_1(\varrho))-g_k(\varrho,U_2(\varrho)))d\varrho|\\
\leq&\frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}\Big(\sup_n|\sum_{k=1}^{\infty}Y_{nk}\sup_{\rho\in[0,1]}(f_k(\varrho,V_1(\varrho))-f_k(\varrho,V_2(\varrho)))|\\
&+\sup_n|\sum_{k=1}^{\infty}Y_{nk}\sup_{\rho\in[0,1]}(g_k(\varrho,U_1(\varrho))-g_k(\varrho,U_2(\varrho)))|\Big)\\
=&\frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}\|K(U_1,V_1)-K(U_2,V_2)\|_{C(I,c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta))}\\
\leq&\varepsilon.
\end{aligned}
$$

Accordingly, we get

$$
\|T(U_1,V_1)-T(U_2,V_2)\|_{C(J_1,c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta))}\leq\varepsilon.
$$

Thus, F is continuous.

Next, we show that $T(U,V)$ is continuous on the open interval $(0,1)$. To this aim, let $\rho_1\in(0,1)$ and $\varepsilon>0$ be arbitrary. By applying the continuity of $H(\rho,\varrho)$ with respect to ρ, we are able to find $\delta=\delta(\rho_1,\varepsilon)>0$ such that if $|\rho-\rho_1|<\delta$, then

$$
|H(\rho,\varrho)-H(\rho_1,\varrho)|<\frac{\varepsilon}{|\eta|(P+Q)\|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}}.
$$

We can write

$$
\begin{aligned}
&\|T(U,V)(\rho)-T(U,V)(\rho_1)\|_{c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta)}\\
=&\|(\eta\int_0^1(H(\rho,\varrho)-H(\rho_1,\varrho))f_i(\varrho,V(\varrho))d\varrho)\|_{c_0(W_1,W_2,\Delta)}\\
&+\|(\eta\int_0^1(H(\rho,\varrho)-H(\rho_1,\varrho))g_i(\varrho,U(\varrho))d\varrho)\|_{c_0(W_1,W_2,\Delta)}\\
=&|\eta|\sup_n|\sum_{k=1}^{\infty}Y_{nk}\int_0^1(H(\rho,\varrho)-H(\rho_1,\varrho))f_k(\varrho,V(\varrho))d\varrho|\\
&+|\eta|\sup_n|\sum_{k=1}^{\infty}Y_{nk}\int_0^1(H(\rho,\varrho)-H(\rho_1,\varrho))g_k(\varrho,U(\varrho))d\varrho|\\
\leq&\frac{|\eta|P\varepsilon}{(P+Q)|\eta|\|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}}(\sup_n|\sum_{k=1}^{\infty}Y_{nk}\sup_{\rho\in[0,1]}V_k(\rho)|)\\
&+\frac{|\eta|Q\varepsilon}{(P+Q)|\eta|\|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}}(\sup_n|\sum_{k=1}^{\infty}Y_{nk}\sup_{\rho\in[0,1]}U_k(\rho)|)\\
\leq&\frac{(P+Q)\varepsilon}{(P+Q)\|(U,V)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}}\\
&\times(\|U\|_{C(J_1,c_0(W_1,W_2,\Delta))}+\|V\|_{C(J_1,c_0(W_1,W_2,\Delta))})\\
=&\varepsilon.
\end{aligned}
$$

Eventually, we are going to show that $T : D \to D$ fulfills the conditions of Theorem 1.4. Due to Proposition 1.5 and Theorem 2.3, for any nonempty subset $\mathcal{X}_1 \times \mathcal{X}_2 \subset D$, we obtain

$$
\begin{aligned}
&\widetilde{\mu}(T(\mathcal{X}_1 \times \mathcal{X}_2))\\
&\quad = \sup_{\rho\in J_1} \sup_{(U,V)\in\mathcal{X}_1\times\mathcal{X}_2} \widetilde{\mu}(T(U,V)(\rho))\\
&\quad = \sup_{\rho\in[0,1]} \sup_{(U,V)\in\mathcal{X}_1\times\mathcal{X}_2} \widetilde{\mu}((\eta \int_0^1 H(\rho,\varrho) f_i(\varrho, V(\varrho))d\varrho), (\eta \int_0^1 H(\rho,\varrho) g_i(\varrho, U(\varrho))d\varrho))\\
&\quad =|\eta| \sup_{\rho\in[0,1]} \lim_{r\to\infty} \sup_{V\in\mathcal{X}_2} \sup_{n>r} |\sum_{k=1}^{\infty} Y_{nk} \int_0^1 H(\rho,\varrho) f_k(\varrho, V(\varrho))d\varrho|\\
&\qquad + |\eta| \sup_{\rho\in[0,1]} \lim_{r\to\infty} \sup_{U\in\mathcal{X}_1} \sup_{n>r} |\sum_{k=1}^{\infty} Y_{nk} \int_0^1 H(\rho,\varrho) g_k(\varrho, U(\varrho))d\varrho|\\
&\quad \le \frac{M_0|\eta|P}{(\alpha-\zeta)\Gamma(\alpha)} \sup_{\rho\in[0,1]} \lim_{r\to\infty} \sup_{V\in\mathcal{X}_2} \sup_{n>r} |\sum_{k=1}^{\infty} Y_{nk} v_k(\rho)|\\
&\qquad + \frac{M_0|\eta|Q}{(\alpha-\zeta)\Gamma(\alpha)} \sup_{\rho\in[0,1]} \lim_{r\to\infty} \sup_{U\in\mathcal{X}_1} \sup_{n>r} |\sum_{k=1}^{\infty} Y_{nk} u_k(\rho)|\\
&\quad = \frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)} (P+Q)\widetilde{\mu}(\mathcal{X}_1 \times \mathcal{X}_2).
\end{aligned}
$$

Using Theorem 1.4, we conclude that T has a fixed point in D, and hence IBVP (1.1) admits at least one solution in $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$. □

We are now in a position to discuss about the existence of positive solutions of IBVP (1.1) in the space $c_0(W_1, W_2, \Delta)$. To this end, consider the following IBVP

$$
\begin{cases}
-D_{0_+}^{\alpha} x_i(\rho) = \eta(f_i(\rho, (y_i(\rho) - K(\rho))^*) + \theta(\rho)), & \rho \in (0,1),\\
-D_{0_+}^{\alpha} y_i(\rho) = \eta(g_i(\rho, (x_i(\rho) - K(\rho))^*) + \theta(\rho)), & \rho \in (0,1),\\
x_i^{(j)}(0) = y_i^{(j)}(0) = 0, & j \in [0, n-2],\\
x_i(1) = \zeta \int_0^1 x_i(\vartheta)d\vartheta,\ y_i(1) = \zeta \int_0^1 y_i(\vartheta)d\vartheta, & i \in \mathbb{N},
\end{cases}
\tag{3.2}
$$

where

$$
\mathcal{Z}(\rho)^* = \begin{cases} \mathcal{Z}(\rho), & \mathcal{Z}(\rho) \ge 0,\\ 0, & \mathcal{Z}(\rho) < 0, \end{cases}
$$

and $K(\rho) = \eta \int_0^1 H(\rho, \vartheta)\theta(\vartheta)d\vartheta$, which is the solution of the BVP

$$
\begin{cases}
-D_{0_+}^{\alpha} K(\rho) = \eta\theta(\rho), & \rho \in (0,1),\\
K^{(j)}(0) = 0, & j \in [0, n-2],\\
K(1) = \zeta \int_0^1 K(\vartheta)d\vartheta.
\end{cases}
$$

We are going to show that there exists a solution $(x, y) = ((x_i), (y_i))$ for IBVP (1.1) with $x_i(\rho) \ge K(\rho)$ and $y_i(\rho) \ge K(\rho)$ for each $i \in \mathbb{N}$ and for each $\rho \in [0, 1]$.

Accordingly, (U, V) is a nonnegative solution of IBVP (1.1), where $U(\rho) = (x_i(\rho) - K(\rho))$ and $V(\rho) = (y_i(\rho) - K(\rho))$. Indeed, for any $i \in \mathbb{N}$ and each $\rho \in (0, 1)$, we have

$$\begin{cases} -D^{\alpha}_{0_+} x_i(\rho) = -D^{\alpha}_{0_+} u_i(\rho) + (-D^{\alpha}_{0_+} K(\rho)) = \eta(f_i(\rho, v(\rho)) + \theta(\rho)), \\ -D^{\alpha}_{0_+} y_i(\rho) = -D^{\alpha}_{0_+} v_i(\rho) + (-D^{\alpha}_{0_+} K(\rho)) = \eta(g_i(\rho, u(\rho)) + \theta(\rho)). \end{cases}$$

It implies that

$$\begin{cases} -D^{\alpha}_{0_+} u_i(\rho) = \eta(f_i(\rho, v(\rho)), \\ -D^{\alpha}_{0_+} v_i(\rho) = \eta(g_i(\rho, v(\rho)). \end{cases}$$

Therefore, we concentrate our attention to the study of IBVP (3.2). We know that (3.2) is equal to

$$x_i(\rho) = \eta \int_0^1 H(\rho, \vartheta)(f_i(\vartheta, (y_i(\vartheta) - K(\vartheta))^*) + \theta(\vartheta))d\vartheta,$$

$$y_i(\rho) = \eta \int_0^1 H(\rho, \vartheta)(g_i(\vartheta, (x_i(\vartheta) - K(\vartheta))^*) + \theta(\vartheta))d\vartheta. \tag{3.3}$$

In view of (3.3), we get

$$x_i(\rho) = \eta \int_0^1 H(\rho, \vartheta)(f_i(\vartheta, (\eta \int_0^1 H(\vartheta, \varsigma) g_i(\varsigma, (x_i(\varsigma) - K(\varsigma))^*) d\varsigma)^*) + \theta(\vartheta))d\vartheta. \tag{3.4}$$

In what follows, we demonstrate our main result.

Theorem 3.2. *Assume that IBVP* (1.1) *fulfills the hypotheses* $(A1)$–$(A5)$ *and* $\frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}(Q+P) < 1$. *Then there exists a positive real constant* $\widetilde{\eta}$ *such that for each* $0 < \eta \leq \widetilde{\eta}$, *IBVP* (1.1) *has at least one positive solution.*

Proof. Take any $\delta \in (0, 1)$. Regarding assumptions $(A4)$ and $(A5)$, we are able to find $0 < \varepsilon < \min\{1, \sigma\}$ such that for each $i \in \mathbb{N}$, $\rho \in J_1$ and the nonnegative sequence $\mathfrak{Z}$ in $C(J_1, c_0(W_1, W_2, \Delta))$ with $\|\mathfrak{Z}\|_{C(J_1, c_0(W_1, W_2, \Delta))} < \varepsilon$, we have

$$f_i(\rho, \mathfrak{Z}(\rho)) \geq \delta f_i(\rho, U^0(\rho)), \qquad g_i(\rho, \mathfrak{Z}(\rho)) > 0.$$

Suppose that

$$0 < \eta < \widetilde{\eta} := \min\Big\{\frac{\varepsilon}{2\Upsilon \widetilde{f}(\varepsilon)}, \frac{1}{Q\Upsilon}\Big\},$$

where $\widetilde{f}(\varepsilon) = \max\{f_i(\rho, \mathfrak{Z}(\rho)) + \theta(\rho), \quad i \in \mathbb{N}, \ 0 \leq \rho \leq 1, \ 0 \leq \|\mathfrak{Z}\|_{C(J_1, c_0(W_1, W_2, \Delta))} \leq \varepsilon\}$ and $\Upsilon = \int_0^1 M_0 q(\vartheta) d\vartheta$. Since $\lim_{\varsigma \to 0} \frac{\widetilde{f}(\varsigma)}{\varsigma} = +\infty$ and $\frac{\widetilde{f}(\varepsilon)}{\varepsilon} < \frac{1}{2\Upsilon\eta}$, then there exists $R_0 \in (0, \varepsilon)$ such that $\frac{\widetilde{f}(R_0)}{R_0} = \frac{1}{2\Upsilon\eta}$. Let

$$D_0 = \{x = (x_i) \in C(J_1, c_0(W_1, W_2, \Delta)) : \ \|x - K\|_{C(J_1, c_0(W_1, W_2, \Delta))} < R_0, \ x_i^{(j)}(0) = 0,$$
$$0 \leq j \leq n-2, \ x_i(1) = \zeta \int_0^1 x_i(\vartheta) d\vartheta, \ \text{ for all } i \in \mathbb{N}\}$$

Now, for any $x \in D_0$ and $\rho \in J_1$, we have

$$\|(\eta \int_0^1 H(\rho, \vartheta)(g_i(\rho, (x_i(\rho) - K(\rho))^*)) d\vartheta)\|_{c_0(W_1, W_2, \Delta)}$$

$$
\begin{aligned}
&= \sup_{n\in\mathbb{N}} |\sum_{k=1}^{\infty} Y_{nk}\eta \int_0^1 H(\rho,\vartheta)(g_k(\vartheta,(x_k(\vartheta)-K(\vartheta))^*))d\vartheta| \\
&\le \eta \int_0^1 M_0 q(\vartheta) Q |\sum_{k=1}^{\infty} Y_{nk}(x_k(\vartheta)-K(\vartheta))^* d\vartheta| \\
&= \eta \int_0^1 M_0 q(\vartheta) Q \|x-K\|_{C(J_1,c_0(W_1,W_2,\Delta))} \\
&\le \eta \int_0^1 M_0 q(\vartheta) Q R_0 d\vartheta \\
&< R_0 < \varepsilon.
\end{aligned}
$$

Thus, using (3.4), we deduce that

$$
\begin{aligned}
x_i(\rho) &= \eta \int_0^1 H(\rho,\vartheta)(f_i(\vartheta,(\eta \int_0^1 H(\rho,\varsigma) g_i(\varsigma,(x_i(\varsigma)-K(\varsigma)^*)d\varsigma)^*)+\theta(\vartheta)))d\vartheta \\
&\ge \eta \int_0^1 H(\rho,\vartheta)(\delta f_i(\vartheta,U^0(\vartheta))+\theta(\vartheta))d\vartheta \\
&= \eta(\delta \int_0^1 H(\rho,\vartheta) f_i(\vartheta,U^0(\vartheta))d\vartheta + \int_0^1 H(\rho,\vartheta)\theta(\vartheta)d\vartheta) \\
&> \eta \int_0^1 H(\rho,\vartheta)\theta(\vartheta)d\vartheta = K(\rho),
\end{aligned}
$$

for any $\rho \in J_1$, and any $i \in \mathbb{N}$.

Thanks to relation (3.3), we get

$$
\begin{aligned}
y_i(\rho) &= \eta \int_0^1 H(\rho,\vartheta)(g_i(\vartheta,(x(\vartheta)-K(\vartheta))^*)+\theta(\vartheta))d\vartheta \\
&= \eta \int_0^1 H(\rho,\vartheta)(g_i(\vartheta,x(\vartheta)-K(\vartheta))+\theta(\vartheta))d\vartheta \\
&> \eta \int_0^1 H(\rho,\vartheta)\theta(\vartheta)d\vartheta = K(\rho),
\end{aligned}
$$

for any $\rho \in J_1$.

Thus, if $0 < \eta \le \widetilde{\eta}$, then (x,y) is a positive solution of IBVP (3.2) with $x_i(\rho) \ge K(\rho)$ and $y_i(\rho) \ge K(\rho)$ for each $i \in \mathbb{N}$ and for each $\rho \in J_1$.

Let $U(\rho) = (u_i(\rho)) = (x_i(\rho) - K(\rho))$ and let $V(\rho) = (v_i(\rho)) = (y_i(\rho) - K(\rho))$. Then (U,V) is a nonnegative solution of IBVP (1.1). □

Example 3.3. Consider the following IBVP of FDEs

$$
\begin{cases}
D_{0_+}^{\frac{39}{2}} u_i(\rho) + \frac{1}{40}\sum_{j=i}^{+\infty} \frac{e^{-2\rho}\left(\arctan^2(v_j(\rho)+1)+\frac{\pi}{2}\sin^2(v_j(\rho)-1)\right)\cos(\rho)}{j(j+1)(\rho+1)} = 0, & 0<\rho<1, \\
D_{0_+}^{\frac{39}{2}} v_i(\rho) + \frac{1}{40}\sum_{j=i}^{+\infty} \frac{e^{-5\rho}\left(1+u_j(\rho)+\sin^2(u_j(\rho)-1)\right)}{j^2\cosh(\rho)(2\rho+3)} = 0, & 0<\rho<1, \\
u_i^{(j)}(0) = v_i^{(j)}(0) = 0, & 0 \le j \le 18, \\
u_i(1) = 19.4\int_0^1 u_i(\vartheta)d\vartheta,\ v_i(1) = 19.4\int_0^1 v_i(\vartheta)d\vartheta, & i \in \mathbb{N},
\end{cases}
\tag{3.5}
$$

in the space $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$. By taking $\alpha = \frac{39}{2}$, $\eta = \frac{1}{40}$, $\zeta = 19.4$,

$$f_i(\rho, V(\rho)) = \sum_{j=i}^{+\infty} \frac{e^{-2\rho}(\arctan^2(v_j(\rho) + 1) + \frac{\pi}{2}\sin^2(v_j(\rho) - 1))\cos(\rho)}{j(j+1)(\rho+1)},$$

and

$$g_i(\rho, U(\rho)) = \sum_{j=i}^{+\infty} \frac{e^{-5\rho}(1 + u_j(\rho) + \sin^2(u_j(\rho) - 1))}{j^2\cosh(\rho)(2\rho+3)},$$

system (3.5) is a special case of IBVP (1.1). Clearly, $f_i, g_i \in C(J_1 \times \mathbb{R}_+^\infty, \mathbb{R})$ for each $i \in \mathbb{N}$. It can be easily verified that condition $(A1)$ holds. Indeed, suppose that $(U, V), (U^1, V^1) \in C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$ and that $\varepsilon > 0$ is arbitrary. Now if $\|(U, V)-(U^1, V^1)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))} \leq \frac{6\varepsilon}{\pi^2+12\pi}$, then for each $\rho \in [0, 1]$, we obtain

$$\begin{aligned}
&\|K(U, V)(\rho) - K(U^1, V^1)(\rho)\|_{c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta)}\\
=&\|((f_i(\rho, V(\rho)) - f_i(\rho, V^1(\rho))), (g_i(\rho, U(\rho)) - g_i(\rho, U^1(\rho))))\|_{c_0(W_1,W_2,\Delta)\times c_0(W_1,W_2,\Delta)}\\
=&\|(f_i(\rho, V(\rho)) - f_i(\rho, V^1(\rho)))\|_{c_0(W_1,W_2,\Delta)} + \|(g_i(\rho, U(\rho)) - g_i(\rho, U^1(\rho)))\|_{c_0(W_1,W_2,\Delta)}\\
=&\sup_n |\sum_{i=1}^{\infty} Y_{ni} \sum_{j=i}^{+\infty} \frac{e^{-2\rho}\cos(\rho)}{j(j+1)(\rho+1)}((\arctan^2(v_j(\rho) + 1) - \arctan^2(v_j^1(\rho) + 1))\\
&+ \frac{\pi}{2}(\sin^2(v_j(\rho) - 1) - \sin^2(v_j^1(\rho) - 1)))|\\
&+ \sup_n |\sum_{i=1}^{\infty} Y_{ni} \sum_{j=i}^{+\infty} \frac{e^{-5\rho}}{j^2\cosh(\rho)(2\rho+3)}\\
&\qquad((1 + u_j(\rho) - 1 - u_j^1(\rho)) + (\sin^2(u_j(\rho) - 1) - \sin^2(u_j^1(\rho) - 1)))|\\
\leq&\sup_n |\sum_{i=1}^{\infty} 2\pi Y_{ni}(v_j(\rho) - v_j^1(\rho))| + \sup_n |\sum_{i=1}^{\infty} \frac{\pi^2}{6} Y_{ni}(u_j(\rho) - u_j^1(\rho))|\\
\leq&(\frac{\pi^2 + 12\pi}{6})(\|U(\rho) - U^1(\rho)\|_{c_0(W_1,W_2,\Delta)} + \|V(\rho) - V^1(\rho)\|_{c_0(W_1,W_2,\Delta)})\\
=&(\frac{\pi^2 + 12\pi}{6})\|(U, V) - (U^1, V^1)\|_{C(J_1,c_0(W_1,W_2,\Delta))\times C(J_1,c_0(W_1,W_2,\Delta))}\\
\leq&\varepsilon.
\end{aligned}$$

Also, we get

$$f_i(\rho, V(\rho)) \leq \pi v_i(\rho), \qquad g_i(\rho, U(\rho)) \leq \frac{\pi^2}{9} u_i(\rho).$$

For each natural number i and $\rho \in [0, 1]$, we put $p_i(\rho) = \pi$ and $q_i(\rho) = \frac{\pi^2}{9}$. Thus $(p_i(\rho))$ and $(q_i(\rho))$ are equibounded on the interval I. Moreover, $P = \pi$ and $Q = \frac{\pi^2}{9}$. Note that

$$f_i(\rho, V(\rho)) + \theta(\rho) > 0, \text{ and } g_i(\rho, U(\rho)) + \theta(\rho) > 0,$$

where $\theta(\rho) = \tan(\rho)$ for each $\rho \in I$. Evidently, $f_i(\rho, U^0(\rho)) > 0$, the sequence $(f_i(\rho, U^0(\rho)))$ is equibounded, and $g_i(\rho, U(\rho)) > 0$. Moreover, $\frac{M_0|\eta|}{(\alpha-\zeta)\Gamma(\alpha)}(P + Q) = \frac{101.875\times\sqrt{\pi}}{18.5\times17.5\times\cdots\times1.5\times9} < 1$. Therefore, we conclude from Theorem 3.2 that (3.5) has a positive solution (U, V) in the space $C(J_1, c_0(W_1, W_2, \Delta)) \times C(J_1, c_0(W_1, W_2, \Delta))$.

4. Conclusions

Mursaleen et al. [28] constructed a measure of noncompactness in the difference sequence space of weighted means $\lambda(u, v, \Delta)$. Also, a fractional differential equation was studied by Yuan [39]. Now, in this work, we discuss the existence of solutions of the infinite coupled system of $(n-1, n)$-type semipositone boundary value problem of nonlinear fractional differential Eq (1.1) in the difference sequence space of weighted means $c_0(W_1, W_2, \Delta)$.

Acknowledgments

We would like to thank the referees for their useful comments and suggestions which have significantly improved the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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