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Research article

The category of affine algebraic regular monoids

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Abstract: The main aim of this study is to characterize affine weak k-algebra H whose affine k-variety $S = M_k(H, k)$ admits a regular monoid structure. As preparation, we determine some results of weak Hopf algebras morphisms, and prove that the anti-function from the category C of weak Hopf algebras whose weak antipodes are anti-algebra morphisms is adjoint. Then, we prove the main result of this study: the anti-equivalence between the category of affine algebraic k-regular monoids and the category of finitely generated commutative reduced weak k-Hopf algebras.

Keywords: weak Hopf algebra; affine algebraic regular monoid; affine weak Hopf algebra
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1. Introduction

The definition of a monoidal category first appeared in 1963 in the work of MacLane [8], and later in his classic book [9] (first published in 1971). The notion of a Hopf algebra first appeared in topology (more specifically, in the work [5] of Hopf in 1941, as an algebraic construction capturing the structure of cohomology rings of H-spaces, in particular, Lie groups). Around that time, it was realized that Hopf algebras could be viewed as algebraic structures arising from tensor categories with a fiber functor, i.e., a tensor functor to the category of vector spaces, through the so-called reconstruction theory (which takes its origins in [4, 19]). Since then, the theory of Hopf algebras has gradually became a part of the theory of tensor categories, and the theory of categories has been used in proving some of the more recent results on Hopf algebras (such as the classification of semisimple Hopf algebras of prime power dimension and the classification of triangle Hopf algebras). Linear algebraic groups have been studied by a number of well known mathematicians [2, 6] for details. Very special classes of linear algebraic semigroups were studied in detail in [3, 14, 15]. A general study of linear algebraic semigroups was begun by Mohan. S. Putcha in the papers [16, 17]. As known, the category of affine algebraic groups is anti-equivalent to the category of finitely generated commutative reduced Hopf algebras [1]. The weak Hopf algebra proposed by Fang Li is an extension of Hopf algebras. So, we can define the so-called affine algebraic regular monoid such that the category of affine algebraic regular monoids is anti-equivalent to the category of finitely generated commutative reduced weak Hopf algebras. This is the main work of this paper.

2. Preliminary

Throughout this paper, $\mathbf{R}, \mathbf{Z}^+$ will denote the sets of all reals and positive integers, respectively.

As preparation, we introduce several concepts that related to a semigroup S. For an element $x \in S$, we call x regular if there exists some $x^* \in S$ satisfying

$$xx^*x = x, \quad x^*xx^* = x^*,$$

x^* is called the regular inverse of x. We denote

$$V(x) = \{x^* \in S \mid xx^*x = x,\ x^*xx^* = x^*\},$$

the set of all regular inverse elements of x and

$$E(S) = \{x \in S \mid x^2 = x\},$$

the set of all idempotents of S. If all the elements of S are regular, we call S a regular semigroup. If $V(S)$ has only one element for any $x \in S$, S is called an inverse semigroup; in this instance, we usually denote $x^{-1} = x^*$. A bialgebra H over k is called a *weak Hopf algebra* [10] if there exists $T \in \mathrm{Hom}_k(H, H)$ (the convolution algebra) satisfying

$$id * T * id = id, \quad T * id * T = T,$$

where T is called a *weak antipode* of H.

The same name weak Hopf algebra was also used as another kind of generalization of Hopf algebra in [7, 12, 13] which comultiplication is no longer required to preserve the unit equivalently, the counit is not required to be an algebra homomorphism. We must point out that these two kinds of weak Hopf algebra are completely different generalizations of each other in various directions, since the only common subclass just consists of Hopf algebras see [11]. The initial motivation of the latter was its connection with the theory of algebra extension.

k will denote a fixed algebraically closed field, $k^* = k \setminus \{0\}$. If $n \in \mathbf{Z}^+$, then $k^n = k \times k \times \cdots \times k$ is the affine n-space, $M_n(k)$ the multiplicative monoid of all $n \times n$ matrices over k and $GL(n, k)$ its group units. If $X \subset k^n$, the X is closed if it is the set of zeroes of a system of polynomials on k^n. If $X \subset k^n$, the $\bar{X}$ denotes the closure of X. By an algebraic semigroup we mean a closed subset of k^n along with an associative operation which is also a polynomial map. By a homomorphism between algebraic semigroups, we mean a semigroup homomorphism which is also a morphism of varieties (a polynomial map). By a connected semigroup we mean an algebraic semigroup such that the underlying closed set is irreducible (i.e. is not a union of two proper closed subsets).

3. Affine algebraic semigroup

Firstly, comparing with the concept of affine algebraic group, we will introduce the concept of affine algebraic semigroup.

Definition 2.1. [1] If A is a commutative k-algebra, the set $X_A = M_k(A, k)$ of all k-algebra morphisms from A to k is called an affine k-variety.

In particular, when A is a finitely generated k-algebra, we say that X_A is an affine algebraic k-variety.

Definition 2.2. [1] Let A, B be commutative k-algebras, and let $\mu : A \to B$ be a k-algebra morphism. Then the map

$$^a\mu : X_B \to X_A, \ \ ^a\mu(x) = x \circ \mu, \ \ x \in X_B$$

is called a morphism of affine k-varieties.

Definition 2.3. [1] If A is a commutative k-algebra,

$$\text{nil}A = \{f \in A : \text{there exists a natural number } n \text{ for which } f^n = 0\}.$$

If nil$A = \{0\}$, A is called reduced.

Let A, B be commutative k-algebras, V_k denotes the category of affine k-varieties, and the set of all morphisms from an affine k-variety X_B to X_A is written by $V_k(X_B, X_A)$. So, we can get the following result.

Proposition 2.4. [1] Let A be a commutative k-algebra, the morphism from the affine k-variety X_A to the affine line k is called a function on X_A, the set of all functions on X_A is written by $V_k(X_A, k)$. When A is reduced, then $V_k(X_k, A) \cong A$. The two correspondences defined by

$$A \mapsto X_A = M_k(A, k), \quad X_A \mapsto V_k(X_A, k)$$

which give an anti-equivalence between the category of finitely generated commutative reduced k-algebras and the category of affine algebraic k-varieties.

Theorem 2.1. [1] If A, B are commutative k-algebras, we get

$$M_k(A, k) \times M_k(B, k) \cong M_k(A \otimes B, k).$$

Therefore, the product $X_A \times X_B$ of X_A and X_B is taken as sets which admits a structure of an affine k-variety, and is called the product of X_A and X_B. If A and B are affine k-algebras, $A \otimes_k B$ is also an affine k-algebra.

Definition 2.5. [1] A finitely generated commutative reduced k-algebra is called an affine k-algebra.

Definition 2.6. [1] Given an affine k-variety $G = M_k(H, k)$ with H is an affine k-algebra, G admits a group structure, if the two maps

$$m : G \times G \to G, \ \ (x, y) \mapsto xy,$$

$$s : G \to G, \ \ x \mapsto x^{-1}, \ \ \text{where } \ xx^{-1} = x^{-1}x = e$$

are morphisms of affine k-varieties, then we call G an affine k-group.

In particular, when G is an affine algebraic k-variety, G is called an affine algebraic k-group. The map m, s and the embedding $e \to G$ of the identity elements e into G, correspond to the k-algebra morphisms

$$\Delta : H \to H \otimes H, \quad S : H \to H, \quad \varepsilon : H \to k.$$

In this situation, to say that the multiplication map $m : G \times G \to G$ satisfies the group axioms is equivalent to say that Δ and ε satisfy the axioms of k-coalgebras and that S is the antipode, in which case, H turns out to be a k-Hopf algebra. Therefore an affine k-group is precisely $G = M_k(H, k)$, where H is commutative reduced k-Hopf algebra, and G is regarded as a group where the multiplication is given via the convolution defined by

$$(x * y)(f) = \Sigma_{(f)} x(f_{(1)}) y(f_{(2)}), \quad f \in H, x, y \in G.$$

In other words, G is the group $G(H^{\circ})$ of all group-like elements of the dual k-Hopf algebra H° of H, where

$$H^{\circ} = \{f \in H^{*} | f(I) = 0, \text{for some ideal } \ I \ \text{ of } \ A \ \text{ such that } \ dimA/I < \infty\}.$$

Then the two correspondences

$$H \mapsto M_k(H, k), \quad G \mapsto V_k(G, k)$$

give an anti-equivalence between the category of affine algebraic k-groups and the category of finitely generated commutative reduced k-Hopf algebras.

Definition 2.7. [1] A k-Hopf algebra whose underlying k-algebra is an affine k-algebra is called an affine k-Hopf algebra.

Definition 2.8. A weak k-Hopf algebra whose underlying k-algebra is an affine k-algebra is called an affine weak k-Hopf algebra.

It is known that for every regular monoid S, its semigroup algebra kS over k is a weak Hopf algebra as generalization of a group algebra. On the other hand, the set of all group-like elements of a weak Hopf algebra is a regular monoid. So, we can give the definition of affine algebra k-regular monoid.

Definition 2.9. Given an affine k-algebra H, the affine k-variety $S = M_k(H, k)$ admits a regular monoid structure, if the two maps

$$m : S \times S \to S, \quad (x, y) \mapsto xy,$$
$$t : S \to S, \quad x \mapsto x^{*}, \quad \text{where } \ x^{*} \ \text{ is a fixed element in } V_x$$

are morphisms of affine k-varieties, then we call S an affine k-regular monoid.

Then, as the result of k-Hopf algebra H and affine k-group $G = M_k(H, k)$, we can ge the main result of this paper:

Theorem 2.2. (Main result) Let k be an algebraically closed field, H be a commutative weak k-Hopf algebra, the two correspondences

$$H \mapsto M_k(H, k), \quad S \mapsto V_k(S, k)$$

give an anti-equivalence between the category of affine algebraic k-regular monoids and the category of finitely generated commutative reduced weak k-Hopf algebras.

4. The proof of main result

Theorem 3.1. [10] Let $H = (H, m, \mu, \Delta, \varepsilon, T)$ be a weak k-Hopf algebra, for any $a, b \in H$,$T(ab) = T(b)T(a)$, then $H^\circ = (H^\circ, \Delta^*, \varepsilon^*, m^*, \mu^*, T^*)$ is also a weak k-Hopf algebra, where T^* is weak Hopf antipode of H°.

Proposition 3.1. Let H be a weak k-Hopf algebra, A be a commutative k-algebra, then the set $Alg(H, A)$ of algebra morphisms from H to A is a regular sub-monoid of $Hom(H, A)$.

Proof. Obviously, $\mu_A \varepsilon_H$ is an algebraic morphism form H to A, and it is the identity of $Alg(H, A)$.

Let $f, g \in Alg(H, A)$, T be the weak antipode, for any $x, y \in H$,

$$\begin{aligned}(f * g)(xy) &= \sum_{(xy)} f((xy)_{(1)})g((xy)_{(2)}) = \sum_{(x),(y)} f(x_{(1)})f(y_{(1)})g(x_{(2)})g(y_{(2)}) \\ &= \sum_{(x),(y)} f(x_{(1)})g(x_{(2)})f(y_{(1)})g(y_{(2)}) = (f * g)(x)(f * g)(y)\end{aligned}$$

$$\begin{aligned}(f * f \circ T * f)(x) &= \sum_{(x)} f(x_{(11)})f(T(x_{(12)}))f(x_{(2)}) \\ &= f(\sum_{(x)} x_{(11)}T(x_{(12)})x_{(2)}) = f(x)\end{aligned}$$

So, $f * f \circ T * f = f$, similarly $f \circ T * f * f \circ T = f \circ T$. Hence, $Alg(H, A)$ is a regular sub-monoid of $Hom(H, A)$.

Definition 3.2. [10] Let H be a weak Hopf algebra, I is an ideal of H.

(1) $\Delta(I) \subseteq H \otimes I + I \otimes H$;

(2) $\varepsilon(I) = 0$;

(3) $T(I) \subseteq I$.

Then, I is called a biideal of H if it satisfies (1) and (2). I is called a weak Hopf ideal of H if it satisfies (1)–(3).

Beishang Ren [18] had discussed some properties of quotient Hopf algebra and the Hopf algebra homomorphism by the basic methods of Hopf algebras. Similarly, the following theorem from Theorem 3.2 to Theorem 3.4 will discuss the relative properties about quotient weak Hopf algebra and weak Hopf algebra homomorphism.

Theorem 3.2. Let K, H, N be weak Hopf algebras, $f : K \to H$, $g : K \to N$ are weak Hopf algebra morphisms. If g is epimorphism, and $ker(g) \subseteq ker(f)$, then there exists a unique weak Hopf algebra morphism $h : N \to H$ such that $hg = f$.

Proof. At first, there is a unique k-linear map $h : N \to H$, such that $hg = f$. Since g is epimorphism, for any $n_1, n_2 \in N$, there exist $k_1, k_2 \in K$ such that $n_1 = g(k_1), n_2 = g(k_2)$, hence

$$\begin{aligned}h(n_1 n_2) &= h(g(k_1)g(k_2)) = h(g(k_1 k_2)) \\ &= f(k_1 k_2) = f(k_1)f(k_2) = hg(k_1)hg(k_2).\end{aligned}$$

So, h is an algebra morphism.

On the other hand,

$$\begin{aligned}\Delta_H(hg) &= \Delta_H f = (f \otimes f)\Delta_K = (hg \otimes hg)\Delta_K \\ &= (h \otimes h)(g \otimes g)\Delta_K = (h \otimes h)\Delta_N g.\end{aligned}$$

Since g is epimorphism, there is $\Delta_H h = (h \otimes h)\Delta_N$.

Moreover, $\varepsilon_H(hg) = \varepsilon_H f = \varepsilon_K = \varepsilon_N g$, so $\varepsilon_H h = \varepsilon_N$. Hence, h is a coalgebra morphism.

At last,

$$T_H(hg) = T_H f = f T_K = hgT_K = hT_N g,$$

that is $T_H h = hT_N$.

Hence, h is a weak Hopf algebra morphism, and it is unique by the uniqueness of the linear map.

Theorem 3.3. Let K, H, N be weak Hopf algebras, $f : K \to H$, $h : N \to H$ are weak Hopf algebra morphisms. If h is monomorphism, and $Im(f) \subseteq Im(h)$, then there exists a unique weak Hopf algebra morphism $g : K \to N$ such that $hg = f$.

Proof. At first, there is a unique k-linear map $g : K \to N$, such that $hg = f$. Since f, h are algebra morphisms, so, for any $k_1, k_2 \in K$,

$$h(g(k_1k_2)) = f(k_1k_2) = f(k_1)f(k_2) = hg(k_1)hg(k_2) = h(g(k_1)g(k_2)).$$

Since h is monomorphism, $g(k_1k_2) = g(k_1)g(k_2)$, so h is an algebra morphism.

On the other hand,

$$(h \otimes h)\Delta_N g = \Delta_H hg = \Delta_H f = (f \otimes f)\Delta_K = (h \otimes h)(g \otimes g)\Delta_K.$$

Since h is monomorphism, there is $\Delta_N g = (g \otimes g)\Delta_K$.

Moreover, $\varepsilon_N g = \varepsilon_H hg = \varepsilon_H f = \varepsilon_K$, so $\varepsilon_N g = \varepsilon_K$. Hence, g is a coalgebra morphism.

At last,

$$hgT_K = fT_K = T_H f = T_H hg = hT_N g,$$

that is $gT_K = T_N g$.

Hence, g is a weak Hopf algebra morphism, and it is unique by the uniqueness of the linear map.

Theorem 3.4. Let H, L be weak Hopf algebras, $\varphi : H \to L$ is weak Hopf algebra morphism, I is a weak Hopf ideal of H and $\pi : H \to H/I$ is a regular morphism, then

(1) $Ker\varphi = \{x \in H : \varphi(x) = 0\}$ is a weak Hopf ideal of H.

(2) H/I has unique weak Hopf algebra structure such that π is a weak Hopf algebra morphism.

(3) For any weak Hopf ideal I which satisfies $I \subseteq Ker\varphi$, there is unique weak Hopf algebra morphism $\bar{\varphi} : H/I \to L$ such that $\bar{\varphi} \circ \pi = \varphi$.

Proof. (1) $Ker\varphi = \{x \in H : \varphi(x) = 0\}$. For any $x \in Ker\varphi$, $\varphi(x) = 0$, then $(\varphi \otimes \varphi)\Delta_H(x) = \Delta_L\varphi(x) = 0$. So,

$$\Delta_H(x) \in Ker(\varphi \otimes \varphi) = Ker\varphi \otimes H + H \otimes Ker\varphi,$$

hence, $\Delta_H(Ker\varphi) \subseteq Ker\varphi \otimes H + H \otimes Ker\varphi$.

On the other hand, $\varepsilon_H(x) = \varepsilon_L\varphi(x) = 0$, $\varphi T_H(x) = T_L\varphi(x) = 0$, so, $T_H(x) \in Ker\varphi$, that is $T_H(Ker\varphi) \subseteq Ker\varphi$. In conclusion, $Ker\varphi$ is a weak Hopf ideal of H.

(2) We can suppose $H = (H, m, \mu, \Delta, \varepsilon, T)$, let $H/I = (H/I, m', \mu', \Delta', \varepsilon', T')$. For any $x, y \in H$, we define

$$m'((x + I)(y + I)) = xy + I, \quad \mu'(1) = \mu(1) + I,$$

$$\Delta'(x + I) = \sum_{(x)}(x_{(1)} + I) \otimes (x_{(2)} + I), \quad \varepsilon'(x + I) = \varepsilon(x) + I, \quad T'(x + I) = T(x) + I.$$

So, it can prove that $H/I = (H/I, m', \mu', \Delta', \varepsilon', T')$ has a weak Hopf algebra structure such that π is a weak Hopf algebra morphism.

Suppose H/I has another weak Hopf algebra structure $H/I = (H/I, m_1, \mu_1, \Delta_1, \varepsilon_1, T_1)$ such that π is a weak Hopf algebra morphism. Then

$$m_1((x+I)(y+I)) = m(\pi(x)\pi(y)) = m\pi(xy) = ab + I;$$

$$\mu_1(1) = (\pi \circ \mu)(1) = \mu(1) + I;$$
$$\Delta_1(x+I) = \Delta_1\pi(x) = (\pi \otimes \pi)\Delta(x) = \sum_{(x)}(x_{(1)} + I) \otimes (x_{(2)} + I);$$
$$\varepsilon_1(x+I) = \varepsilon_1\pi(x) = \varepsilon(x);$$
$$T_1(x+I) = T_1\pi(x) = \pi T(x) = T(x) + I.$$

Hence, $m' = m_1, \mu' = \mu_1, \Delta' = \Delta_1, \varepsilon' = \varepsilon_1, T' = T_1$, H/I has the unique weak Hopf algebra structure such that π is a weak Hopf algebra morphism.

(3) Since π is epimorphism, and $Ker\pi = I \subseteq Ker\varphi$, then we can prove the conclusion by the result of Theorem 3.2.

Theorem 3.5. (1) Let A, B be weak Hopf algebras, and the weak antipodes T_A, T_B of A and B are anti-algebra morphisms respectively. $\varphi : A \to B$ is a weak Hopf algebra morphism over the field k, then $\varphi^\circ : B^\circ \to A^\circ$ is also a weak Hopf algebra morphism.

(2) Let C be the category of the weak Hopf algebras which weak antipodes are anti-algebra morphisms, then, the anti-function from C to itself

$$\begin{aligned} \phi : & C \to C \\ & A \mapsto A^\circ \end{aligned}$$

is adjoint, that is to say that for any $A, B \in C$, there is a natural isomorphism:

$$\theta : Hom_{weakHopf}(A, B^\circ) \to Hom_{weakHopf}(B, A^\circ).$$

Proof. (1) At first, φ° is a bialgebra morphism on k. And, $\varphi T_A = T_B\varphi$, so $T_A^*\varphi^\circ = \varphi^\circ T_B^*$, that is $T_{A^\circ}\varphi^\circ = \varphi^\circ T_{B^\circ}$. Hence, $\varphi^\circ : B^\circ \to A^\circ$ is a weak Hopf algebra morphism.

(2) Let $\varphi : A \to B$ be a weak Hopf algebra morphism, then $\varphi^\circ : B^\circ \to A^\circ$ is a weak Hopf algebra morphism from the result of (1). Base on the inclusion relation $B^\circ \subset B^*$, we can deduce a k-algebra morphism $(B^*)^\circ \to (B^\circ)^\circ$. On the other hand, $B \subset (B^*)^\circ$, so there is a coalgebra morphism

$$\begin{aligned} \xi_B : & B \to (B^\circ)^\circ \\ & b \mapsto \xi_B(b) : f \mapsto f(b), \forall b \in B, f \in B^\circ. \end{aligned}$$

Since the identity of $(B^\circ)^\circ$ is $\varepsilon(B^\circ)$, then for any morphism $f \in B^\circ$, $f \mapsto f(1)$, this morphism is $\xi_B(1)$ exactly.

For any $a, b \in B$, $f \in B^\circ$,

$$(\xi_B(a)\xi_B(b))(f) = (\xi_B(a) \otimes \xi_B(b))\Delta_{B^\circ}(f) = fm_B(a \otimes b) = f(ab) = \xi_B(ab)(f);$$

$$(T_{(B^\circ)^\circ} \circ \xi_B)(b)(f) = (\xi_B(b))T_{B^\circ}(f) = T_{B^\circ}(f)(b) = fT_B(b) = (\xi_B T_B)(b)(f).$$

That is $T_{(B^\circ)^\circ} \circ \xi_B = \xi_B T_B$. Hence, ξ_B is a weak Hopf algebra morphism.

At last, let $\varphi \in Hom_{weakHopf}(A, B^\circ)$, $\psi \in Hom_{weakHopf}(B, A^\circ)$, define

$$\theta : Hom_{weakHopf}(A, B^\circ) \to Hom_{weakHopf}(B, A^\circ)$$
$$\varphi \mapsto \varphi^\circ \circ \xi_B;$$

$$\Phi : Hom_{weakHopf}(B, A^\circ) \to Hom_{weakHopf}(A, B^\circ)$$
$$\psi \mapsto \psi^\circ \circ \xi_A.$$

Obviously $\Phi(\theta(\varphi)) = \varphi$, $\theta(\Phi(\psi)) = \psi$, so θ is an isomorphism.

Let S be a monoid with identity element e, then kS is a weak Hopf algebra with $\Delta(x) = x \otimes x$, $\varepsilon(x) = e$, $T(x) = x^*$, $x^* \in V(x)$. The dual k-linear space $(kS)^*$ of kS has $(kS)^* \cong Map(S, k)$, we denote $M_k(S) = Map(S, k)$. The following action makes $M_k(S)$ a two-sided kS-module.

$$(xfy)(z) = f(yzx), \quad f \in M_k(S), \quad x, y, z \in S.$$

For $f \in M_k(S)$, we denote the left kS-module, the right kS-module and the two-sided kS-module generated by f respectively by kSf, kfS, and $kSfS$.

Theorem 3.6. [1] The k-linear map

$$\pi : M_k(S) \otimes M_k(S) \to M_k(S \times S)$$
$$f \otimes g \mapsto \pi(f \otimes g) : (x, y) \mapsto f(x)g(y), \forall x, y \in S, f, g \in M_k(S)$$

is injective; the k-algebra morphism

$$\delta : M_k(S) \to M_k(S \times S)$$
$$f \mapsto \delta(f) : (x, y) \mapsto f(xy), \forall x, y \in S, f \in M_k(S)$$

has the following result:

$$\delta(f) \in \pi(M_k(S) \otimes M_k(S)) \Leftrightarrow \dim \ kSf < \infty.$$

Definition 3.3. [1] When $f \in M_k(S)$ satisfies the condition that $\dim \ kSf < \infty$, f is said to be a representative function on S. The set of all representative functions on S is denoted by $R_k(S)$. $R_k(S)$ is a k-subalgebra of $M_k(S)$.

Now we have

Theorem 3.7. [1] $\delta(R_k(S)) \subset \pi(R_k(S) \otimes R_k(S))$.

Thus, $\Delta = \pi^{-1} \circ \delta : R_k(S) \to R_k(S) \otimes R_k(S)$ is a k-algebra morphism that define a comultiplication on $R_k(S)$. Moreover, by defining a k-algebra morphism $\varepsilon : R_k(S) \to k$ by $\varepsilon(f) = f(e)$, $(R_k(S), \Delta, \varepsilon)$ becomes a k-coalgebra, and the fact that Δ, ε are k-algebra morphisms implies that $R_k(S)$ is a k-bialgebra. This is called the representative k-bialgebra of S.

Proposition 3.4. Let S be a regular monoid, and $V(y)V(x) \subseteq V(xy)$ for any $x, y \in S$, then $R_k(S)$ becomes a commutative weak Hopf algebra.

Proof. Since S is a regualr monoid, we can define

$$T : R_k(S) \to R_k(S)$$
$$f \mapsto T(f) : x \mapsto f(x^*), \text{ where } x^* \text{ is a fixed elment in}$$
$$V(x) \text{ which satisfies } T(f)(xy) = T(f)(y)T(f)(x).$$

Then, we can prove $T(f) \in R_k(S)$ directly. Moreover,

$$(id * T * id)(f)(x) = \sum_{(f)}[f_{(11)}T(f_{12})f_{(2)}](x) = \sum_{(f)} f_{(11)}(x)f_{(12)}(x^*)f_{(2)}(x) = f(x).$$

Hence, $id * T * id = id$. Similarly, $T * id * T = T$. At the same time, $R_k(S) \subseteq M_k(S)$, so, $R_k(S)$ is a weak Hopf algebra.

Remark. When S is an inverse monoid or commutative regular monoid, $V(y)V(x) \subseteq V(xy)$ is always right for any $x, y \in S$, then $R_k(S)$ will be a weak Hopf algebra from the above proposition.

Theorem 3.8. (1) Let H be an affine k-algebra, if $S = M_k(H, k)$ is an affine algebra k-regular monoid, then H has weak Hopf algebra structure.

(2) Let H be a weak Hopf algebra, and an affine k-algebra as algebra, then $S = M_k(H, k)$ is an affine algebra k-regular monoid.

Proof. (1) Let $S = (M_k(H, k), w, m, t)$ be the affine k-regular monoid, then

$$\begin{aligned} &m : M_k(H, k) \times M_k(H, k) \to M_k(H, k), \\ &t : M_k(H, k) \to M_k(H, k), \\ &l : e \to M_k(H, k) \end{aligned}$$

are affine algebraic k-variety morphisms. Then, there are k-algebra morphisms $\Delta : H \to H \otimes H$, $T : H \to H$, $\varepsilon : H \to k$ such that

$$m = \Delta^* \circ \lambda, \quad T = t^*, \quad e = \varepsilon,$$

where $\lambda : M_k(H, k) \times M_k(H, k) \to M_k(H \otimes H)$ is an isomorphism, and $\lambda(\lambda \times id) = \lambda(id \times \lambda)$. Then for any $f, g, h \in M_k(H, k)$

$$\begin{aligned} m(m \times id)(f, g, h)(a) &= \Delta^*\lambda(\Delta^*\lambda \times id)(f, g, h)(a) \\ &= \lambda(\lambda \times id)(f, g, h)(\Delta \otimes id)\Delta(a) \end{aligned}$$

Since $m(m \times id) = m(id \times m)$, $\lambda(\lambda \times id) = \lambda(id \times \lambda)$, and the arbitrariness of f, g, h, then $\Delta(\Delta \times id) = \Delta(id \times \Delta)$.

Since $m(f, e)(a) = f(a)$, hence for any $f \in M_k(H, k)$,

$$\Delta^*\lambda(f, e)(a) = \lambda(f, e)(\sum_{(a)} a_{(1)} \otimes a_{(2)}) = \sum_{(a)} f(a_{(1)})\varepsilon(a_{(2)}) = f(\sum_{(a)} a_{(1)}\varepsilon(a_{(2)})) = f(a).$$

From the arbitrariness of f, $\sum_{(a)} a_{(1)}\varepsilon(a_{(2)}) = a$. Similarly, $\sum_{(a)} \varepsilon(a_{(1)})a_{(2)} = a$.

Moreover, $m(m \times id)(f, t(f), f)(a) = f(a)$, that is

$$\sum_{(a)} f(a_{(11)})f(T(a_{12}))f(a_{(2)}) = f(a),$$

hence, $\sum_{(a)} a_{(11)}T(a_{(12)})a_{(2)} = a$, that is to say $id * T * id = id$. Similarly, $T * id * T = T$.

In conclusion, H is a weak Hopf algebra.

(2) Let $H = (H, \mu, \eta, \Delta, \varepsilon, T)$ be a weak Hopf algebra,

$$\begin{aligned} \lambda : M_k(H, k) \times M_k(H, k) &\to M_k(H \otimes H) \\ (f, g) &\mapsto \lambda(f, g), \quad \lambda(f, g)(x \otimes y) = f(x)g(y). \end{aligned}$$

Then, λ is an isomorphism.

Let

$$\begin{aligned}
&m = \Delta^* \circ \lambda : M_k(H,k) \times M_k(H,k) \to M_k(H \otimes H, k) \to M_k(H,k) \\
&t = T^* : M_k(H,k) \to M_k(H,k), \\
&e = \varepsilon.
\end{aligned}$$

Then, for any $f, g, h \in M_k(H,k)$, $x \in H$,

$$\begin{aligned}
m(m \times id)(f,g,h)(x) &= \Delta^* \lambda(\Delta^* \lambda \times id)(f,g,h)(x) \\
&= \sum_{(x)} \lambda(\lambda \times id)(f,g,h)(x_{(11)} \otimes x_{(12)} \otimes x_{(2)}) \\
&= \sum_{(x)} f(x_{(11)}) g(x_{(12)}) h(x_{(2)}).
\end{aligned}$$

Similarly,

$$m(id \times m)(f,g,h)(x) = \sum_{(x)} f(x_1) g(x_{(21)}) h(x_{(22)}).$$

Since $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, so $m(m \times id) = m(id \times m)$.

$$\begin{aligned}
m(m \times id)(f, t(f), f)(x) &= \Delta^* \lambda(\Delta^* \lambda \times id)(f, t(f), f)(x) \\
&= \sum_{(x)} f(x_{(11)}) t(f)(x_{(12)}) f(x_{(2)}) = \sum_{(x)} f(x_{(11)}) f(T(x_{(12)})) f(x_{(2)}) \\
&= f(\sum_{(x)} x_{(11)} T(x_{(12)}) x_{(2)}) = f(x).
\end{aligned}$$

So, $m(m \times id)(f, t(f), f) = f$. Similarly, $m(m \times id)(t(f), f, t(f)) = t(f)$.

At last,

$$m(f,e)(x) = \sum_{(x)} f(x_{(1)}) e(x_{(2)}) = \sum_{(x)} f(x_{(1)}) \varepsilon(x_{(2)}) = f(x),$$

that is $m(f,e) = f$. Similarly, $m(e,f) = f$. So, $S = M_k(H,k)$ is a regular monoid. Moreover, H is an affine k-algebra, then $S = M_k(H,k)$ is an affine algebraic k-regular monoid.

Proof of Theorem 2.2. The theorem can get by the result of Proposition 2.4 and Theorem 3.8.

Example 3.1. Let $H = k[T, T']$ be a weak Hopf algebra which satisfies $TT'T = T$, $T'TT' = T'$, and

$$\Delta(T) = T \otimes T, \ \ \Delta(T') = T' \otimes T', \ \ S(T) = T', \ \ S(T') = T, \ \ \varepsilon(T) = \varepsilon(T') = 1,$$

then $M_k(H,k)$ is an affine algebraic k-regular monoid.

Theorem 3.9. (1) Let WH_k be the category of weak Hopf algebras, MG_r be the category of regular monoids, for any $H, H_1, H_2 \in ObjWH_k$, $g \in Hom_{WH_k}(H_1, H_2)$, we can define

$$\begin{aligned}
\psi : WH_k &\to MG_r & \psi(g) : M_k(H_2,k) &\to M_k(H_1,k) \\
H &\mapsto M_k(H,k) & \varphi_2 &\mapsto \varphi_2 \circ g.
\end{aligned}$$

Then ψ is an anti-function.

(2) Let IG_r be the category of inverse mononids, for any $G, G_1, G_2 \in ObjIG_r$, $f \in Hom_{IG_r}(G_1, G_2)$, $\varphi_2 \in R_k(G_2)$, we can define

$$\begin{aligned} \phi : IG_r \to WH_k \qquad & \phi(f) : R_k(G_2) \to R_k(G_1) \\ G \mapsto R_k(G) \qquad & \qquad \varphi_2 \mapsto \varphi_2 \circ f. \end{aligned}$$

Then, ϕ is an anti-function too.

Proof. (1) For any $H_1, H_2, H_3 \in ObjWH_k$, $g_1 \in Hom_{WH_k}(H_1, H_2)$, $g_2 \in Hom_{WH_k}(H_2, H_3)$. Based on the proof procedure of Theorem 3.8 (2), $\psi(H_1) \in MG_r$. Next, We will prove that $\psi(g_1)$ is a regular monoid morphism. Given $h \in H_1$, $\varphi_1, \varphi_2 \in M_k(H_1, k)$,

$$\phi(g_1)(\varphi_2 * \varphi_1)(h) = (\varphi_2 * \varphi_1)(g_1(h)) = \sum_{(g_1(h))} \varphi_2(g_1(h)_{(1)})\varphi_1(g_1(h)_{(2)})$$

On the other hand,

$$[\psi(g_1)(\varphi_2) * \psi(g_1)(\varphi_1)](h) = \sum_{(h)} \varphi_2(g_1(h_{(1)})\varphi_1(g_1(h_{(2)}).$$

That is $\psi(g_1)(\varphi_2) * \psi(g_1)(\varphi_1) = \psi(g_1)(\varphi_2 * \varphi_1)$.

Moreover,

$$\psi(g_1)(e_2)(h) = e_2(g_1(h)) = \varepsilon_2 g_1(h) = \varepsilon_1(h) = e_1(h),$$

that is $\psi(g_1)(e_2) = e_1$. Hence, $\psi(g_1)$ is a regular monoid morphism.

Next, for any $h \in H_1$,$\varphi \in M_k(H_1, k)$,

$$\psi(g_1 \circ g_2)(\varphi)(h) = \varphi(g_1 \circ g_2)(h),$$

$$(\psi(g_2) \circ \psi(g_1))(\varphi)(h) = \varphi(\sum_{(h)} g_2(h_{(1)})g_1(h_{(2)})) = \varphi(g_1 \circ g_2)(h).$$

Hence, $\psi(g_1 \circ g_2) = \psi(g_2) \circ \psi(g_1)$. Similarly, we can prove $\psi(id_{H_1}) = id_{M_k(H_1,k)}$.

In conclusion, ψ is an anti-function.

(2) Based on the remark of Proposition 3.4, $\phi(G_1) \in ObjWH_k$. Next, we will prove that $\phi(f)$ is a weak Hopf algebra morphism. For any $a \in G_1$,

$$\phi(f)(\varphi_2 \cdot \varphi_2')(a) = (\varphi_2 \cdot \varphi_2')(f(a)) = \varphi_2(f(a))\varphi_2'(f(a)),$$

and

$$(\phi(f)(\varphi_2) * \phi(f)(\varphi_2'))(a) = [\phi(f)(\varphi_2)(a)][\phi(f)(\varphi_2')(a)] = \varphi_2(f(a))\varphi_2'(f(a)).$$

That is

$$\phi(f)(\varphi_2 \cdot \varphi_2') = \phi(f)(\varphi_2) * \phi(f)(\varphi_2').$$

Hence, $\phi(f)$ is an algebra morphism.

For any $a, a' \in G_1$,

$$\Delta_{R_k(G_1)}\phi(f)\varphi_2(a \otimes a') = \phi(f)\varphi_2(aa') = \varphi_2 \circ f(aa'),$$

$$\begin{aligned}(\phi(f)\otimes\phi(f))\Delta_{R_k(G_2)}\varphi_2(a\otimes a') &= \sum_{(i)}(\phi(f)\otimes\phi(f))(r_i\otimes s_i)(a\otimes a')\\ &= \sum_{(i)}\phi(f)(r_i)(a)\phi(f)(s_i)(a')\\ &= \sum_{(i)} r_i(f(a))s_i(f(a'))\\ &= \varphi_2(f(a)f(a')) = (\varphi_2\circ f)(aa'),\end{aligned}$$

where $\Delta_{R_k(G_2)}\varphi_2 = \sum_{(i)} r_i \otimes s_i$. So,

$$\Delta_{R_k(G_1)}\phi(f)\varphi_2 = (\phi(f)\otimes\phi(f))\Delta_{R_k}(G_2)\varphi_2.$$

Moreover,

$$\begin{aligned}(\varepsilon_{R_k}(G_1)\circ\phi(f))(\varphi_2) &= \phi(f)(\varphi_2)(e_1) = \varphi_2(f(e_1))\\ &= \varphi_2(e_2) = \varepsilon_{R_k}(G_2)(\varphi_2),\end{aligned}$$

so, $\varepsilon_{R_k}(G_1)\circ\phi(f) = \varepsilon_{R_k}(G_2)$, that is $\phi(f)$ is a co-algebra morphism.

At last,

$$\phi(f)T_2(\varphi_2)(a) = T_2(\varphi_2)(f(a)) = \varphi_2 t_2(f(a)),$$

$$T_1\phi(f)(\varphi_2)(a) = T_1(\varphi_2 f)(a) = \varphi_2 f(t_1(a)) = \varphi_2 t_2(f(a)).$$

Hence, $\phi(f)T_2 = T_1\phi(f)$.

In conclusion, $\phi(f)$ is a weak Hopf algebra morphism.

On the other side, for any $f_1 \in Hom_{IG_r}(G_1, G_2)$, $f_2 \in Hom_{IG_r}(G_2, G_3)$, $\varphi_3 \in R_k(G_3)$, $\varphi_1 \in R_k(G_1)$, we have

$$\phi(f_2\circ f_1)(\varphi_3)(a) = \varphi_3\circ(f_2\circ f_1)(a),$$

$$(\phi(f_1)\otimes\phi(f_2))(\varphi_3)(a) = \phi(f_1)(\varphi_3\circ f_2)(a) = (\varphi_3\circ f_2\circ f_1)(a).$$

So, $\phi(f_2\circ f_1) = \phi(f_1)\circ\phi(f_2)$. Moreover,

$$\begin{aligned}\phi(id_{G_1})(\varphi_1)(a) &= (\varphi_1\circ id_{G_1})(a) = \varphi_1(a)\\ &= id_{R_k(G_1)}(\varphi_1(a)),\end{aligned}$$

that is $\phi(id_{G_1}) = id_{R_k(G_1)}$.

In summary, ϕ is an anti-function.

5. Conclusions

First of all, we have defined the affine algebraic regular monoid comparing with the concept of affine algebraic group and proved that the anti-function from the category C of weak Hopf algebras whose weak antipodes are anti-algebra morphisms is adjoint at section 2. After that, we have characterized the affine weak k-algebra H whose affine k-variety S = Mk(H,k) admits a regular monoid structure: the category of affine algebraic regular monoids is anti-equivalent to the category of finitely generated commutative reduced weak Hopf algebras at section 3.

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Conflict of interest

We declare that we have no conflict of interest.

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