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Research article

Multiplicity result to a system of over-determined Fredholm fractional integro-differential equations on time scales

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Abstract: In present paper, several conditions ensuring existence of three distinct solutions of a system of over-determined Fredholm fractional integro-differential equations on time scales are derived. Variational methods are utilized in the proofs.

Keywords: Riemann-Liouville derivatives; fractional boundary value problem; time scales; variational methods; critical points; multiplicity
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1. Introduction

As we all know, the time scale theory can unify discrete and continuous analysis, the study of dynamic equations on time scales can unify the study of difference equations and differential equations At present, time scale theory and its application have attracted more and more attention [1–3]. On the other hand, fractional calculus is a generalization of integer calculus. In recent years, the theory and application of fractional calculus has become a hot field [4–9]. The continuous fractional calculus has been well developed [10–12]. However, the study of discrete fractional calculus [13–16] is more complicated than its continuous counterpart. Therefore, the study of fractional dynamic equations on time scales has important theoretical and practical value. The concept of fractional derivative of Riemann-Liouville type on time scales was introduced by N. Benkhettou, A. Hammoudi and D. F. M. Torres in [17]. What happened then was a craze for the studying of it, such as [18–23]. However, as far as we know, there is almost no research on fractional boundary value problems of fractions on time scales. Therefore, a substantive investigation of the subject seems promising.

Over-determined equations have always come in considerable problems from mathematical physics [24–26]. When we discuss the solution of over-determined linear systems, the least squares method is the most widely used [26–28].

Recently, the boundary value problem of second-order impulsive differential inclusion involving relativistic operator is studied in [29] by using non-smooth critical point theorem for locally Lipschitz functionals. The authors of [30] investigate a class of two-point boundary value problems whose highest-order term is a Caputo fractional derivative. The existence and multiplicity of positive solutions for a nonlinear fractional differential equation boundary value problem is established in [31] by the fixed-point index theory and the Leray-Schauder degree theory. In [32], a class of fuzzy differential equations with variable boundary value conditions is studied by applying the upper and lower solutions method and the monotone iterative technique. In [33], some existence results about first-order fuzzy differential equation with two-point boundary value condition are obtained by the upper and lower solutions method. In [34], some existence results about first-order fuzzy differential equation with two-point boundary value condition are provided by using the contraction mapping principle in a complete metric space. Boundary value problems on time scales are investigated in [35–39].

Through literature search, we found that over-determined boundary value problems on time scales have not been studied yet, therefore, in this paper, we will study the following nonlinear system of over-determined Fredholm fractional integro-differential equations on time scales with periodic boundary condition(FBVP$_{\mathbb{T}}$ for short):

$$\begin{cases} {}_t^{\mathbb{T}}D_b^{\alpha_i}(\kappa_i(t)\,{}_a^{\mathbb{T}}D_t^{\alpha_i}u_i(t)) = \eta G_{u_i}(t, u_1(t), \ldots, u_N(t)) + \displaystyle\int_{[a,b)_{\mathbb{T}}} \xi_i(t,\tau)u_i(\tau)\Delta\tau, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta - a.e.\ t \in [a,b]_{\mathbb{T}},\ i = \overline{1,N}; & \\ u_i(t) = \displaystyle\int_{[a,b)_{\mathbb{T}}} \xi_i(t,\tau)u_i(\tau)\Delta\tau, \qquad\qquad \Delta - a.e.\ t \in [a,b]_{\mathbb{T}},\ i = \overline{1,N}; & \\ u_i(a) = u_i(b) = 0, \qquad\qquad\qquad\qquad\qquad\qquad i = \overline{1,N}, & \end{cases} \tag{1.1}$$

where $\eta > 0$ is a real constant, $0 < \alpha_i \leq 1$, $\kappa_i \in L_\Delta^\infty[a,b]_{\mathbb{T}}$, $\overline{\kappa_i} = ess \inf_{t\in[a,b]_{\mathbb{T}}} \kappa_i(t) > 0$ and $G : [a,b]_{\mathbb{T}}\times\mathbb{R}^N \to \mathbb{R}$ is a function that $G(t, u_1, \ldots, u_n)$ is continuous with respect to t and continuously differentiable with respect to u_i, i.e., $G(\cdot, u_1, \ldots, u_N) \in C([a,b]_{\mathbb{T}})$ and $G(t, \cdot, \ldots, \cdot) \in C^1(\mathbb{R}^N)$, $\xi_i(\cdot,\cdot) \in C([a,b]_{\mathbb{T}}, [a,b]_{\mathbb{T}})$ and so the kernel ξ_i is bounded by M_i, G_s denotes the partial Δ−derivative of G with respect to s, ${}_t^{\mathbb{T}}D_b^\alpha$ and ${}_a^{\mathbb{T}}D_t^\alpha$ are the right and the left Riemann−Liouville fractional derivative operators of order α defined on $\mathbb{T}$ respectively.

When $\mathbb{T} = \mathbb{R}$, FBVP$_{\mathbb{T}}$ (1.1) reduces to the following standard nonlinear system of over-determined Fredholm fractional integro-differential equations

$$\begin{cases} {}_tD_b^{\alpha_i}(\kappa_i(t)\,{}_aD_t^{\alpha_i}u_i(t)) = \eta G_{u_i}(t, u_1(t), \ldots, u_N(t)) + \displaystyle\int_{[a,b]} \xi_i(t,\tau)u_i(\tau)d\tau, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a.e.\ t \in [a,b],\ i = \overline{1,N}; & \\ u_i(t) = \displaystyle\int_{[a,b]} \xi_i(t,\tau)u_i(\tau)d\tau, \qquad\qquad a.e.\ t \in [a,b],\ i = \overline{1,N}; & \\ u_i(a) = u_i(b) = 0, \qquad\qquad\qquad\qquad\qquad i = \overline{1,N}. & \end{cases}$$

which has been studied by E. Shivanian in [40].

2. Preliminaries

In this section, we briefly collect some notations, definitions, and some lemmas, propositions and theorems, which play an important role in the proof of our main results.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. Throughout this paper, we denote by $\mathbb{T}$ a time scale. We will use the following notations: $J_{\mathbb{R}}^0 = [a, b[$, $J_{\mathbb{R}} = [a, b]$, $J^0 = J_{\mathbb{R}}^0 \cap \mathbb{T}$, $J = J_{\mathbb{R}} \cap \mathbb{T}$, $J^k = [a, \rho(b)] \cap \mathbb{T}$.

Definition 2.1. *[41] (Fractional integral on time scales) Suppose h is an integrable function on J. Let $0 < \alpha \le 1$. The left fractional integral of order α of h is defined by*

$$ {}_{a}^{\mathbb{T}}I_t^{\alpha} h(t) := \int_a^t \frac{(t-\sigma(s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s. $$

The right fractional integral of order α of h is defined by

$$ {}_{t}^{\mathbb{T}}I_b^{\alpha} h(t) := \int_t^b \frac{(\sigma(s)-t)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s, $$

where Γ is the gamma function.

Definition 2.2. *[41] (Riemann-Liouville fractional derivative on time scales) Let $t \in \mathbb{T}$, $0 < \alpha \le 1$, and $h : \mathbb{T} \to \mathbb{R}$. The left Riemann-Liouville fractional derivative of order α of h is defined by*

$$ {}_{a}^{\mathbb{T}}D_t^{\alpha} h(t) := \left({}_{a}^{\mathbb{T}}I_t^{1-\alpha} h(t) \right)^{\Delta} = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-\sigma(s))^{-\alpha} h(s)\Delta s \right)^{\Delta}. $$

The right Riemann-Liouville fractional derivative of order α of h is defined by

$$ {}_{t}^{\mathbb{T}}D_b^{\alpha} h(t) := -\left({}_{t}^{\mathbb{T}}I_b^{1-\alpha} h(t) \right)^{\Delta} = \frac{-1}{\Gamma(1-\alpha)} \left(\int_t^b (\sigma(s)-t)^{-\alpha} h(s)\Delta s \right)^{\Delta}. $$

Theorem 2.1. *[19] Let $\alpha > 0$, $p, q \ge 1$, and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$, where $p \ne 1$ and $q \ne 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Moreover, if*

$$ {}_{a}^{\mathbb{T}}I_t^{\alpha}(L^p) := \left\{ f : f = {}_{a}^{\mathbb{T}}I_t^{\alpha} g, g \in L^p(J) \right\} $$

and

$$ {}_{t}^{\mathbb{T}}I_b^{\alpha}(L^p) := \left\{ f : f = {}_{t}^{\mathbb{T}}I_b^{\alpha} g, g \in L^p(J) \right\}, $$

then the following integration by parts formulas hold:

(a) *If $\varphi \in L^p(J)$ and $\psi \in L^q(J)$, then*

$$ \int_{J^0} \varphi(t) \left({}_{a}^{\mathbb{T}}I_t^{\alpha} \psi \right)(t)\Delta t = \int_{J^0} \psi(t) \left({}_{t}^{\mathbb{T}}I_b^{\alpha} \varphi \right)(t)\Delta t. $$

(b) *If $g \in {}_{t}^{\mathbb{T}}I_b^{\alpha}(L^p)$ and $f \in {}_{a}^{\mathbb{T}}I_t^{\alpha}(L^q)$, then*

$$ \int_{J^0} g(t) \left({}_{a}^{\mathbb{T}}D_t^{\alpha} f \right)(t)\Delta t = \int_{J^0} f(t) \left({}_{t}^{\mathbb{T}}D_b^{\alpha} g \right)(t)\Delta t. $$

Proposition 2.1. *[42] Suppose* $p \in \bar{\mathbb{R}}$ *and* $p \geq 1$. *Let* $p' \in \bar{\mathbb{R}}$ *be such that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then, if* $f \in L^p_\Delta(J^0)$ *and* $g \in L^{p'}_\Delta(J^0)$, *then* $f \cdot g \in L^1_\Delta(J^0)$ *and*

$$\|f \cdot g\|_{L^1_\Delta} \leq \|f\|_{L^p_\Delta} \cdot \|g\|_{L^{p'}_\Delta}.$$

Definition 2.3. *[43] Let* $0 < \alpha \leq 1$ *and let* $1 \leq p < \infty$. *By left Sobolev space of order* α *we will mean the set* $W^{\alpha,p}_{\Delta,a^+} = W^{\alpha,p}_{\Delta,a^+}(J, \mathbb{R}^N)$ *given by*

$$W^{\alpha,p}_{\Delta,a^+} := \left\{ u \in L^p_\Delta;\ \exists\, g \in L^p_\Delta,\ \forall \varphi \in C^\infty_{c,rd} \text{ such that } \int_{J^0} u(t) \cdot {}^{\mathbb{T}}_{t}D^\alpha_b \varphi(t) \Delta t = \int_{J^0} g(t) \cdot \varphi(t) \Delta t \right\}.$$

Remark 2.1. *[43] A function g given in Definition 2.3 will be called the weak left fractional derivative of order* $0 < \alpha \leq 1$ *of u; let us denote it by* ${}^{\mathbb{T}}u^\alpha_{a^+}$.

Theorem 2.2. *[43] If* $0 < \alpha \leq 1$ *and* $1 \leq p < \infty$, *then the weak left fractional derivative* ${}^{\mathbb{T}}u^\alpha_{a^+}$ *of a function* $u \in W^{\alpha,p}_{\Delta,a^+}$ *coincides with its left Riemann-Liouville fractional derivative* ${}^{\mathbb{T}}_{a}D^\alpha_t u$ $\Delta -$ *a.e. on J.*

Theorem 2.3. *[43] Let* $0 < \alpha \leq 1, 1 \leq p < \infty$ *and* $u \in L^p_\Delta$. *Then* $u \in W^{\alpha,p}_{\Delta,a^+}$ *iff there exists a function* $g \in L^p_\Delta$ *such that*

$$\int_{J^0} u(t) {}^{\mathbb{T}}_{t}D^\alpha_b \varphi(t) \Delta t = \int_{J^0} g(t) \varphi(t) \Delta t, \quad \varphi \in C^\infty_{c,rd}.$$

In such a case there exists the left Riemann-Liouville derivative ${}^{\mathbb{T}}_{a}D^\alpha_t u$ *of u and* $g = {}^{\mathbb{T}}_{a}D^\alpha_t u$.

Remark 2.2. *[43] The function g will be called the weak left fractional derivative of* $u \in W^{\alpha,p}_{\Delta,a^+}$ *of order* α. *From the above theorem it follows that it coincides with an appropriate Riemann-Liouville derivative.*

Let us fix $0 < \alpha \leq 1$ and consider in the space $W^{\alpha,p}_{\Delta,a^+}$ a norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ given by

$$\|u\|^p_{W^{\alpha,p}_{\Delta,a^+}} = \|u\|^p_{L^p_\Delta} + \|{}^{\mathbb{T}}_{a}D^\alpha_t u\|^p_{L^p_\Delta}, \quad u \in W^{\alpha,p}_{\Delta,a^+}.$$

Theorem 2.4. *[43] The space* $W^{\alpha,p}_{\Delta,a^+}$ *is complete with respect to each of the norms* $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ *and* $\|\cdot\|_{a,W^{\alpha,p}_{\Delta,a^+}}$ *for any* $0 < \alpha \leq 1$, $1 \leq p < \infty$.

Theorem 2.5. *[43] The space* $W^{\alpha,p}_{\Delta,a^+}$ *is reflexive with respect to the norm* $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ *for any* $0 < \alpha \leq 1$ *and* $1 < p < \infty$.

Theorem 2.6. *[43] The space* $W^{\alpha,p}_{\Delta,a^+}$ *is separable with respect to the norm* $\|\cdot\|_{W^{\alpha,p}_{\Delta,a^+}}$ *for any* $0 < \alpha \leq 1$ *and* $1 \leq p < \infty$.

Proposition 2.2. *[43] Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$. *For all* $u \in W^{\alpha,p}_{\Delta,a^+}$, *if* $1 - \alpha \geq \frac{1}{p}$ *or* $\alpha > \frac{1}{p}$, *then*

$$\|u\|_{L^p_\Delta} \leq \frac{b^\alpha}{\Gamma(\alpha+1)} \|{}^{\mathbb{T}}_{a}D^\alpha_t u\|_{L^p_\Delta}; \tag{2.1}$$

if $\alpha > \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\|u\|_\infty \leq \frac{b^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}^{\mathbb{T}}_{a}D^\alpha_t u\|_{L^p_\Delta}. \tag{2.2}$$

Remark 2.3. *[43] It follows from* (2.1) *and* (2.2) *that* $W^{\alpha,p}_{\Delta,a^+}$ *is continuously immersed into* $C(J,\mathbb{R}^N)$ *with the natural norm* $\|\cdot\|_\infty$.

Proposition 2.3. *[43] Let* $0<\alpha\le 1$ *and* $1<p<\infty$. *Assume that* $\alpha>\frac{1}{p}$ *and the sequence* $\{u_k\}\subset W^{\alpha,p}_{\Delta,a^+}$ *converges weakly to u in* $W^{\alpha,p}_{\Delta,a^+}$. *Then,* $u_k\to u$ *in* $C(J,\mathbb{R}^N)$, *i.e.,* $\|u-u_k\|_\infty=0$, *as* $k\to\infty$.

Remark 2.4. *[43] It follows from Proposition 2.3 that* $W^{\alpha,p}_{\Delta,a^+}$ *is compactly immersed into* $C(J,\mathbb{R}^N)$ *with the natural norm* $\|\cdot\|_\infty$.

Theorem 2.7. *[44] Let E be a reflexive real Banach space and* $\Phi: E\to\mathbb{R}$ *a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable in which its Gâteaux derivative has a continuous inverse on* E^*. *Furthermore, suppose that* $\Psi: E\to\mathbb{R}$ *be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional in which its Gâteaux derivative is compact, such that* $\Phi(0)=\Psi(0)=0$. *Suppose also there exist* $r\in\mathbb{R}$ *and* $u_1\in E$ *with* $0<r<\Phi(u_1)$, *satisfying*

(H_1) $\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)<r\frac{\Psi(u_1)}{\Phi(u_1)}$

(H_2) $\forall\eta\in I_r:=\left]\frac{\Phi(u_1)}{\Psi(u_1)},\frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, *the functional* $\Phi-\eta\Psi$ *is coercive.*

Therefore, for each $\eta\in I_r$, *the functional* $\Phi-\eta\Psi$ *admits at least three distinct critical points in E.*

Note that if $\kappa_i(\cdot)\in L^\infty_\Delta(J)$, $\overline{\kappa_i}=ess\inf\limits_{t\in J}\kappa_i(t)>0$, an equivalent norm in $W^{\alpha_i,p}_{\Delta,a^+}$ is

$$\|u\|_{\kappa_i,\alpha_i}=\left(\int_{J^0}\kappa_i(t)|{}^{\mathbb{T}}_aD^{\alpha_i}_tu(t)|^p\Delta t+\int_{J^0}|u(t)|^p\Delta t\right)^{\frac{1}{p}},\quad \forall u\in W^{\alpha_i,p}_{\Delta,a^+},\ i=\overline{1,N}. \tag{2.3}$$

It easily follows from $\kappa_i(\cdot)\in L^\infty_\Delta(J)$, $\overline{\kappa_i}=ess\inf\limits_{t\in J}\kappa_i(t)>0$ and Proposition 2.2 that

$$\|u\|_{L^p_\Delta}\le\frac{b^{\alpha_i}}{\Gamma(\alpha_i+1)(\overline{\kappa_i})^{\frac{1}{p}}}\left(\int_{J^0}\kappa_i(t)|{}^{\mathbb{T}}_aD^{\alpha_i}_tu(t)|^p\Delta t\right)^{\frac{1}{p}}, \tag{2.4}$$

$$\|u\|_\infty\le\frac{b^{\alpha_i-\frac{1}{p}}}{\Gamma(\alpha_i)((\alpha_i-1)q+1)^{\frac{1}{q}}(\overline{\kappa_i})^{\frac{1}{q}}}\left(\int_{J^0}\kappa_i(t)|{}^{\mathbb{T}}_aD^{\alpha_i}_tu(t)|^p\Delta t\right)^{\frac{1}{p}}. \tag{2.5}$$

The equality (2.3) and inequality (2.4) yield that the norm defined by (2.3) is equivalent to the following norm

$$\|u\|_{\alpha_i}=\left(\int_{J^0}\kappa_i(t)|{}^{\mathbb{T}}_aD^{\alpha_i}_tu(t)|^2\Delta t\right)^{\frac{1}{2}},\quad\forall u\in W^{\alpha_i,2}_{\Delta,a^+},\ i=\overline{1,N}, \tag{2.6}$$

which is induced by the following inner product

$$(u,v)_{\alpha_i}=\left(\int_{J^0}\kappa_i(t)\,{}^{\mathbb{T}}_aD^{\alpha_i}_tu(t)\,{}^{\mathbb{T}}_aD^{\alpha_i}_tv(t)\Delta t\right)^{\frac{1}{2}},\quad\forall u,\ v\in W^{\alpha_i,2}_{\Delta,a^+},\ i=\overline{1,N}.$$

In the following analysis, we will work with the norm given by (2.6). Now, let $p = 2$, define $E = \prod_{i=1}^{N} W^{\alpha_i,2}_{\Delta,a^+}$ equipped with the norm

$$\|U\|_E = \sum_{i=1}^{N} \|u_i\|^2_{\alpha_i}, \quad U = (u_1, u_2, \ldots, u_N) \in E,\ i = \overline{1, N}. \tag{2.7}$$

Definition 2.4. *We call $U = (u_1, u_2, \ldots, u_N) \in E$ the weak solution of FBVP$_{\mathbb{T}}$ (1.1) if the following equation holds*

$$\begin{aligned} &\sum_{i=1}^{N} \int_{J^0} \kappa_i(t)\, {}^{\mathbb{T}}_{a}D^{\alpha_i}_{t} u_i(t)\, {}^{\mathbb{T}}_{a}D^{\alpha_i}_{t} v_i(t) \Delta t - \sum_{i=1}^{N} \int_{J^0} \int_{J^0} \xi_i(t,\tau) u_i(\tau) v_i(t) \Delta\tau\Delta t \\ -\eta &\int_{J^0} \sum_{i=1}^{N} G_{u_i}(t, u_1(t), \ldots, u_N(t)) v_i(t) \Delta t = 0, \qquad \forall\, V = (v_1, v_2, \ldots, v_N) \in E. \end{aligned} \tag{2.8}$$

3. Main result

In this section, we present and prove our main result as follows.

Theorem 3.1. *Suppose that $G : J \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $G(\cdot, u_1, \ldots, u_N) \in C(J)$, $G(t, \cdot, \ldots, \cdot) \in C^1(\mathbb{R}^N)$ and $G(t, 0, \ldots, 0) = 0$ for all $t \in J$. Moreover, suppose that there are a positive constant r and a function $Z(t) = (z_1(t), \ldots, z_N(t))$ such that the following suppositions hold:*

(S_0) $\alpha_i \in \left]\frac{1}{2}, 1\right]$;

(S_1) $M_i < \frac{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i - 1)}{b^{2\alpha_i+1}}$;

(S_2) $\sum_{i=1}^{N} \|z_i\|^2_{\alpha_i} \geq 2r + \sum_{i=1}^{N} \int_{J^0} \int_{J^0} \xi_i(t,\tau) z_i(\tau) z_i(t) \Delta\tau\Delta t$;

(S_3) $\dfrac{\int_{J^0} \sup\limits_{(v_1,v_2,\ldots,v_N)\in\Upsilon(Cr)} G(t, v_1, \ldots, v_N) \Delta t}{r} < \dfrac{2\int_{J^0} G(t, z_1(t), \ldots, z_N(t)) \Delta t}{\sum_{i=1}^{N} \|z_i\|^2_{\alpha_i} - \sum_{i=1}^{N} \int_{J^0} \int_{J^0} \xi_i(t,\tau) z_i(\tau) z_i(t) \Delta\tau\Delta t}$;

(S_4) $\liminf\limits_{\forall i : |v_i| \to +\infty} \dfrac{G(t, v_1, \ldots, v_N)}{\sum_{i=1}^{N} |v_i|^2} < \dfrac{\int_{J^0} \sup\limits_{(v_1,v_2,\ldots,v_N)\in\Upsilon(Cr)} G(t, v_1, \ldots, v_N) \Delta t}{2Ar}$.

where

$$C = \max_{1\leq i\leq N} \left\{ \frac{b^{2\alpha_i - 1}}{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i - 1) - b^{2\alpha_i+1} M_i} \right\},$$

$$\Upsilon(Cr) = \left\{ F = (v_1, v_2, \ldots, v_N) \in \mathbb{R}^N : \frac{1}{2} \sum_{i=1}^{N} v_i^2 \leq Cr \right\},$$

$$\sigma_i = 1 - \frac{b^{2\alpha_i+1} M_i}{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i - 1)}, \quad \sigma = \min_{1\leq i\leq N} \sigma_i, \quad A = \max_{1\leq i\leq N} \left\{ \frac{b^{2\alpha_i}}{\sigma\Gamma^2(\alpha_i + 1)\overline{\kappa_i}} \right\}.$$

Then FBVP$_{\mathbb{T}}$ (1.1) has at least three distinct weak solutions in E, for those η's belong to the following interval

$$I_r = \left] \frac{\sum_{i=1}^{N} \|z_i\|_{\alpha_i}^2 - \sum_{i=1}^{N} \int_{J^0}\int_{J^0} \xi_i(t,\tau) z_i(\tau) z_i(t) \Delta\tau\Delta t}{2\int_{J^0} G(t, z_1(t), \ldots, z_N(t))\Delta t}, \frac{r}{\int_{J^0} \sup_{(v_1, v_2, \ldots, v_N)\in\Upsilon(Cr)} G(t, v_1, \ldots, v_N)\Delta t} \right[. \tag{3.1}$$

Proof. Theorem 2.7 will be the powerful tool for us to prove Theorem 3.1. It follows from the fact that $W_{\Delta,a^+}^{\alpha_i,p}$ is a reflexive and separable Banach space and that $E = \prod_{i=1}^{N} W_{\Delta,a^+}^{\alpha_i,2}$ equipped with the norm $\|u\|_E$ is also a reflexive and separable Banach space. Next, for any given $U = (u_1(t), \ldots, u_N(t)) \in E$, define Φ, $\Psi : E \to \mathbb{R}$ as follows:

$$\Phi(U) = \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} \int_{J^0}\int_{J^0} \xi_i(t,\tau) u_i(\tau) u_i(t) \Delta\tau\Delta t \tag{3.2}$$

and

$$\Psi(U) = \int_{J^0} G(t, u_1(t), \ldots, u_N(t))\Delta t. \tag{3.3}$$

The functionals Φ and Ψ are well defined, G$\hat{a}$teaux differentiable and for all $V = (v_1, v_2, \ldots, v_N) \in E$, their G$\hat{a}$teaux derivatives are given as

$$\Phi'(U)(V) = \sum_{i=1}^{N} \int_{J^0} \kappa_i(t)\, {}_a^{\mathbb{T}}D_t^{\alpha_i} u_i(t)\, {}_a^{\mathbb{T}}D_t^{\alpha_i} v_i(t)\Delta t - \sum_{i=1}^{N} \int_{J^0}\int_{J^0} \xi_i(t,\tau) u_i(\tau) v_i(t) \Delta\tau\Delta t \tag{3.4}$$

and

$$\Psi'(U)(V) = \int_{J^0} G_{u_i}(t, u_1(t), \ldots, u_N(t)) v_i(t)\Delta t, \tag{3.5}$$

respectively. In fact, $\Phi(U), \Psi(U) \in E^*$, where E^* is dual space of E. It is easy to see that the functional Φ is sequentially weakly lower semicontinuous and its G$\hat{a}$teaux derivative admits a continuous inverse on E^*. Besides, in view of (3.2), $|\xi_i(t,\tau)| \le M_i$ and by the definition of σ, one gets

$$\begin{aligned}
\Phi(U) =& \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} \int_{J^0}\int_{J^0} \xi_i(t,\tau) u_i(\tau) u_i(t) \Delta\tau\Delta t \\
\ge& \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} M_i b \|u_i\|_\infty \int_{J^0} u_i(t)\Delta t \\
\ge& \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} M_i b^2 \|u_i\|_\infty^2 \\
\ge& \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} M_i b^2 \frac{b^{2\alpha_i+1}}{\Gamma(\alpha_i)(2\alpha_i-1)^{\frac{1}{q}}\overline{\kappa_i}} \|u_i\|_{\alpha_i}^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2\left(1 - M_i b^2 \frac{b^{2\alpha_i+1}}{\Gamma(\alpha_i)(2\alpha_i-1)^{\frac{1}{q}}\kappa_i}\right)\\
&\geq \frac{\sigma}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2\\
&= \frac{\sigma}{2}\|U\|_E. \qquad (3.6)
\end{aligned}
$$

Because of $\sigma > 0$ and ($\mathbf{S_1}$), so it follows from (3.6) that $\lim_{\|U\|_E\to+\infty}\Phi(U) = +\infty$. That is to say, Φ is coercive.

Suppose that $\lim_{m\to+\infty} U_m \rightharpoonup U$ in E, where $U_m(t) = (u_{m,1}(t), \ldots, u_{m,N}(t))$, then U_m converges uniformly to U on J by Proposition 2.3. Therefore, we have

$$
\begin{aligned}
\limsup_{m\to+\infty}\Psi(U_m) &= \limsup_{m\to+\infty}\int_{J^0} G(t, u_{m,1}(t), \ldots, u_{m,N}(t))\Delta t\\
&\leq \int_{J^0} G(t, u_1(t), \ldots, u_N(t))\Delta t\\
&= \Psi(U), \qquad (3.7)
\end{aligned}
$$

which yields that Ψ is sequentially weakly upper semicontinuous. In addition, taking $G(t, \cdot, \ldots, \cdot) \in C^1(\mathbb{R}^N)$ into account, one obtains

$$
\lim_{m\to+\infty} G(t, u_{m,1}(t), \ldots, u_{m,N}(t)) = G(t, u_1(t), \ldots, u_N(t)), \quad \forall t \in J.
$$

Consequently, the Lebesgue control convergence theorem on time scales implies that $\Psi'(U_m) \to \Psi'(U)$ strongly, as a result, we conclude that Ψ' is strongly continuous on E. Hence, $\Psi' : E \to E^*$ is a compact operator.

Assume $U_0(t) = (0, \ldots, 0)$ and $U_1(t) = Z(t)$, in consideration of ($\mathbf{S_2}$) , one derives that

$$
\begin{aligned}
0 < r &\leq \frac{1}{2}\sum_{i=1}^{N}\|z_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N}\int_{J^0}\int_{J^0}\xi_i(t,\tau)u_i(\tau)u_i(t)\Delta\tau\Delta t\\
&= \frac{1}{2}\sum_{i=1}^{N}\|z_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N}\int_{J^0}\int_{J^0}\xi_i(t,\tau)u_i(\tau)u_i(t)\Delta\tau\Delta t\\
&= \Psi(U_1). \qquad (3.8)
\end{aligned}
$$

It is obvious for us to deduce that $\Phi(U_0(t)) = \Psi(U_0(t)) = 0$ by (3.2) and (3.3).

Now, we are in a position to demonstrate that Φ and Ψ satisfy the conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) of Theorem 2.7.

In view of (3.2), $|\xi_i(t,\tau)| \leq M_i$ and (2.5), we get

$$
\begin{aligned}
&\Phi^{-1}(]-\infty, r])\\
&= \{U \in E : \Phi(U) \leq r\}\\
&= \left\{U \in E : \frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N}\int_{J^0}\int_{J^0}\xi_i(t,\tau)u_i(\tau)u_i(t)\Delta\tau\Delta t \leq r\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\subseteq \left\{ U \in E : \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} M_i b \|u_i\|_\infty \int_{J^0} u_i(t)\Delta t \le r \right\} \\
&\subseteq \left\{ U \in E : \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \frac{1}{2}\sum_{i=1}^{N} M_i b^2 \|u_i\|_\infty^2 \le r \right\} \\
&\subseteq \left\{ U \in E : \sum_{i=1}^{N} \frac{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i - 1)}{2b^{2\alpha_i-1}} \|u_i\|_\infty^2 - \frac{1}{2}\sum_{i=1}^{N} M_i b^2 \|u_i\|_\infty^2 \le r \right\} \\
&= \left\{ U \in E : \sum_{i=1}^{N} \frac{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i - 1) - b^{2\alpha_i+1}M_i}{2b^{2\alpha_i-1}} \|u_i\|_\infty^2 \le r \right\} \\
&\subseteq \left\{ U \in E : \frac{1}{2C}\sum_{i=1}^{N} \|u_i\|_\infty^2 \le r \right\} \\
&\subseteq \left\{ U \in E : \frac{1}{2}\sum_{i=1}^{N} |u_i(t)|^2 \le Cr, \forall t \in J \right\} \\
&\subseteq \Upsilon(Cr),
\end{aligned} \tag{3.9}
$$

which implies that

$$
\begin{aligned}
\sup_{U\in\Phi^{-1}(]-\infty,r])} \Psi(U) &= \sup_{U\in\Phi^{-1}(]-\infty,r])} \int_{J^0} G(t, u_1(t), \ldots, u_N(t))\Delta t \\
&\le \sup_{F\in\Upsilon(Cr)} \int_{J^0} G(t, v_1(t), \ldots, v_N(t))\Delta t \\
&= \int_{J^0} \sup_{F\in\Upsilon(Cr)} G(t, v_1(t), \ldots, v_N(t))\Delta t \\
&= \Psi(U_1),
\end{aligned} \tag{3.10}
$$

together with ($\mathbf{S_3}$), one can obtain

$$
\begin{aligned}
\frac{\sup_{U\in\Phi^{-1}(]-\infty,r])} \Psi(U)}{r} &= \frac{\sup_{U\in\Phi^{-1}(]-\infty,r])} \int_{J^0} G(t, u_1(t), \ldots, u_N(t))\Delta t}{r} \\
&\le \frac{\sup_{F\in\Upsilon(Cr)} \int_{J^0} G(t, v_1(t), \ldots, v_N(t))\Delta t}{r} \\
&< \frac{2\int_{J^0} G(t, z_1(t), \ldots, z_N(t))\Delta t}{\sum_{i=1}^{N} \|z_i\|_{\alpha_i}^2 - \sum_{i=1}^{N} \int_{J^0}\int_{J^0} \xi_i(t,\tau) z_i(\tau) z_i(t)\Delta\tau\Delta t} \\
&= \frac{\Psi(Z(t))}{\Phi(Z(t))} \\
&= \frac{\Psi(U_1)}{\Phi(U_1)}.
\end{aligned} \tag{3.11}
$$

Consequently, $\frac{\sup\limits_{U\in\Phi^{-1}(]-\infty,r])}\Psi(U)}{r} < \frac{\Psi(U_1)}{\Phi(U_1)}$. That is to say, $(\mathbf{H_1})$ of Theorem 2.7 is verified.

In addition, with an eye to $(\mathbf{S_4})$, there exist two real constants μ and ε such that

$$\frac{\mu}{\sigma} < \frac{\sup\limits_{F\in\Upsilon(Cr)}\int_{J^0} G(t,v_1(t),\ldots,v_N(t))\Delta t}{r} \tag{3.12}$$

and

$$G(t,v_1(t),\ldots,v_N(t)) \le \frac{\mu}{2A\sigma}\sum_{i=1}^{N}|v_i|^2+\varepsilon, \quad \forall F=(v_1,v_2,\ldots,v_N)\in\mathbb{R}^N,\ t\in J. \tag{3.13}$$

Hence, for fixed $U=(u_1(t),\ldots,u_N(t))\in E$, one has

$$G(t,u_1(t),\ldots,u_N(t)) \le \frac{\mu}{2A\sigma}\sum_{i=1}^{N}|u_i|^2+\varepsilon, \quad \forall t\in J. \tag{3.14}$$

According to (3.2), (3.3), $|\xi_i(t,\tau)|\le M_i$, the expression of σ_i, (2.4), (2.5), (3.1) and (3.12), one gets

$$\begin{aligned}
&\Phi(U)-\eta\Psi(U)\\
=\ &\frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\frac{1}{2}\sum_{i=1}^{N}\int_{J^0}\int_{J^0}\xi_i(t,\tau)u_i(\tau)u_i(t)\Delta\tau\Delta t-\eta\int_{J^0}G(t,u_1(t),\ldots,u_N(t))\Delta t\\
\ge\ &\frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\frac{1}{2}\sum_{i=1}^{N}M_ib^2\|u_i\|_\infty^2-\eta\int_{J^0}G(t,u_1(t),\ldots,u_N(t))\Delta t\\
\ge\ &\frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\frac{1}{2}\sum_{i=1}^{N}M_ib^2\|u_i\|_\infty^2-\frac{\eta\mu}{2A\sigma}\int_{J^0}\left(\sum_{i=1}^{N}|u_i(t)|^2\right)\Delta t-\eta\varepsilon b\\
=\ &\frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\frac{1}{2}\sum_{i=1}^{N}M_ib^2\|u_i\|_\infty^2-\frac{\eta\mu}{2A\sigma}\sum_{i=1}^{N}\|u_i\|_{L^2_\Delta}^2-\eta\varepsilon b\\
\ge\ &\frac{1}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\frac{1}{2}\sum_{i=1}^{N}\frac{b^{2\alpha_i+1}M_i}{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i-1)}\|u_i\|_{\alpha_i}^2-\frac{\eta\mu}{2A\sigma}\sum_{i=1}^{N}\frac{b^{2\alpha_i}}{\Gamma^2(\alpha_i+1)\overline{\kappa_i}}\|u_i\|_{\alpha_i}^2-\eta\varepsilon b\\
=\ &\frac{1}{2}\sum_{i=1}^{N}\sigma_i\|u_i\|_{\alpha_i}^2-\frac{\eta\mu}{2A\sigma}\sum_{i=1}^{N}\frac{b^{2\alpha_i}}{\Gamma^2(\alpha_i+1)\overline{\kappa_i}}\|u_i\|_{\alpha_i}^2-\eta\varepsilon b\\
\ge\ &\frac{\sigma}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\frac{\eta\mu}{2}\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\eta\varepsilon b\\
\ge\ &\frac{1}{2}\left(\sigma-\frac{\mu r}{\sup\limits_{F\in\Upsilon(Cr)}\int_{J^0}G(t,v_1(t),\ldots,v_N(t))\Delta t}\right)\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2-\eta\varepsilon b\\
\to\ &+\infty, \quad \text{as } \|U\|\to+\infty.
\end{aligned} \tag{3.15}$$

In other words, the functional $\Phi - \eta\Psi$ is coercive and so, ($\mathbf{H_2}$) of Theorem 2.7 is also testified.

Since the weak solutions of FBVP$_{\mathbb{T}}$ (1.1) are exactly the solutions of the equation $\Phi'(U) - \eta\Psi'(U) = 0$, by Theorem 2.7, we conclude that for $\eta \in I_r$, FBVP$_{\mathbb{T}}$ (1.1) has at least three distinct points in E. The proof is complete. $\square$

4. Example

In order to illustrate our theoretical results, we give a example as follows.

Example 4.1. *Let $\mathbb{T} = \mathbb{Z}$, $N = 3$, $\alpha_1 = 0.75$, $\alpha_2 = 0.8$, $\alpha_3 = 0.9$, $\kappa_1(t) = 1 + t^2$, $\kappa_2(t) = 0.5 + t$, $\kappa_3(t) = 1 + t$, $\xi_1(t,\tau) = \frac{1}{3\times 10^{10}} t\tau$, $\xi_2(t,\tau) = \frac{1}{8\times 10^{10}} t\tau$, $\xi_3(t,\tau) = \frac{1}{4\times 10^{10}} t\tau$, $a = 2.5$, $b = 50$, so we can consider the following nonlinear system of over-determined Fredholm fractional integro-differential equations on time scales with periodic boundary condition:*

$$
\begin{cases}
{}_t\Delta^{0.75}\left((1+t^2)\,\Delta_{2.5}^{0.75} u(t)\right) = \eta G_u(t, u(t), v(t), w(t)) + \sum\limits_{\tau=3}^{49} \frac{1}{3} t\tau u(\tau), & \Delta - a.e.\ t \in [2.5, 50]_{\mathbb{Z}};\\
u(t) = \sum\limits_{\tau=3}^{99} \frac{1}{3} t\tau u(\tau), & \Delta - a.e.\ t \in [2.5, 50]_{\mathbb{Z}};\\
{}_t\Delta^{0.8}\left((0.5+t)\,\Delta_{2.5}^{0.8} v(t)\right) = \eta G_v(t, u(t), v(t), w(t)) + \sum\limits_{\tau=3}^{49} \frac{1}{8} t\tau v(\tau), & \Delta - a.e.\ t \in [2.5, 50]_{\mathbb{Z}};\\
v(t) = \sum\limits_{\tau=3}^{49} \frac{1}{8} t\tau v(\tau), & \Delta - a.e.\ t \in [2.5, 50]_{\mathbb{Z}};\\
{}_t\Delta^{0.9}\left((1+t)\,\Delta_{2.5}^{0.9} w(t)\right) = \eta G_w(t, u(t), v(t), w(t)) + \sum\limits_{\tau=3}^{49} \frac{1}{4} t\tau w(\tau)\Delta\tau, & \Delta - a.e.\ t \in [2.5, 50]_{\mathbb{Z}};\\
w(t) = \sum\limits_{\tau=3}^{49} \frac{1}{4} t\tau w(\tau)\Delta\tau, & \Delta - a.e.\ t \in [2.5, 50]_{\mathbb{Z}};\\
u(3) = u(50) = 0, \quad v(3) = v(50) = 0, \quad w(3) = w(50) = 0,
\end{cases}
$$

where $\Delta_{\varrho}^{\gamma}$ and ${}_t\Delta^{\gamma}$ are the left and right Riemann-Liouville delta fractional difference of order $0 < \gamma \le 1$ respectively.

$$
G(t, u, v, w) := (1+t^2)\begin{cases}
(u^2+v^2+w^2)^2, & u^2+v^2+w^2 \le 2.20550\times 10^5;\\
2\sqrt{u^2+v^2+w^2} - (u^2+v^2+w^2), & u^2+v^2+w^2 > 2.20550\times 10^5.
\end{cases}
$$

It is easy for us to know that G is continuous with respect to t and continuously differentiable with respect to u, v, w. Moreover, $G(t,0,0,0) = 0$ and by simple calculations, we get $M_1 \approx 8.33333\times 10^{-8}$, $M_2 = 3.12500\times 10^{-8}$, $M_3 = 6.25000\times 10^{-8}$,

$$
\overline{\kappa_1} = ess \inf_{t\in[2.5,50]_{\mathbb{Z}}} \kappa_1(t) = ess \inf_{t\in[2.5,50]_{\mathbb{Z}}} (1+t^2) = 1 + 3^2 = 10,
$$

$$
\overline{\kappa_2} = ess \inf_{t\in[2.5,50]_{\mathbb{Z}}} \kappa_2(t) = ess \inf_{t\in[2.5,50]_{\mathbb{Z}}} (0.5+t) = 0.5 + 3 = 3.5,
$$

$$
\overline{\kappa_3} = ess \inf_{t\in[2.5,50]_{\mathbb{Z}}} \kappa_3(t) = ess \inf_{t\in[2.5,50]_{\mathbb{Z}}} (1+t) = 1 + 3 = 4,
$$

$$\sigma_1 = 1 - \frac{b^{2\alpha_1+1}M_1}{\Gamma^2(\alpha_1)\overline{\kappa_1}(2\alpha_1 - 1)} = 1 - \frac{50^{2\times 0.75+1} \times 8.33333 \times 10^{-8}}{\Gamma^2(0.75) \times 10 \times (2 \times 0.75 - 1)} \approx 0.99980,$$

$$\sigma_2 = 1 - \frac{b^{2\alpha_2+1}M_1}{\Gamma^2(\alpha_2)\overline{\kappa_3}(2\alpha_2 - 1)} = 1 - \frac{50^{2\times 0.8+1} \times 3.12500 \times 10^{-8}}{\Gamma^2(0.8) \times 3.5 \times (2 \times 0.8 - 1)} \approx 0.99942,$$

$$\sigma_3 = 1 - \frac{b^{2\alpha_3+1}M_1}{\Gamma^2(\alpha_3)\overline{\kappa_3}(2\alpha_3 - 1)} = 1 - \frac{50^{2\times 0.9+1} \times 6.25000 \times 10^{-8}}{\Gamma^2(0.9) \times 4 \times (2 \times 0.9 - 1)} \approx 0.99902,$$

$$\sigma = \min_{1\le i\le 3} \sigma_i = \min\{\sigma_1, \sigma_2, \sigma_3\} = \min\{0.99980, 0.99942, 0.99902\} = 0.99902,$$

$$\begin{aligned}
C &= \max_{1\le i\le 3}\left\{\frac{b^{2\alpha_i-1}}{\Gamma^2(\alpha_i)\overline{\kappa_i}(2\alpha_i - 1) - b^{2\alpha_i+1}M_i}\right\} \\
&= \max\left\{\frac{50^{2\times 0.75-1}}{\Gamma^2(0.75) \times 10 \times (2\times 0.75 - 1) - 50^{2\times 0.75+1} \times 8.33333 \times 10^{-8}},\right. \\
&\qquad \frac{50^{2\times 0.8-1}}{\Gamma^2(0.8) \times 3.5 \times (2\times 0.8 - 1) - 50^{2\times 0.8+1} \times 3.12500 \times 10^{-8}}, \\
&\qquad \left.\frac{50^{2\times 0.9-1}}{\Gamma^2(0.9) \times 4 \times (2\times 0.9 - 1) - 50^{2\times 0.9+1} \times 6.25000 \times 10^{-8}}\right\} \\
&\approx \max\{0.94196, 3.67460, 6.26321\} \\
&= 6.26321,
\end{aligned}$$

$$\begin{aligned}
A &= \max_{1\le i\le 3}\left\{\frac{b^{2\alpha_i}}{\sigma\Gamma^2(\alpha_i + 1)\overline{\kappa_i}}\right\} \\
&= \max\left\{\frac{50^{2\times 0.75}}{\sigma\Gamma^2(0.75+1) \times 10}, \frac{50^{2\times 0.8}}{\sigma\Gamma^2(0.8+1) \times 3.5}, \frac{50^{2\times 0.9}}{\sigma\Gamma^2(0.9+1) \times 4}\right\} \\
&\approx \max\{41.89765, 172.36593, 309.29445\} \\
&= 309.29445.
\end{aligned}$$

Furthermore, we can define

$$H_1(u(t)) = \frac{1}{6\times 10^{10}}\sum_{\tau=3}^{49} t\tau u(\tau), \quad H_2(v(t)) = \frac{1}{1.6\times 10^{11}}\sum_{\tau=3}^{49} t\tau v(\tau), \quad H_3(w(t)) = \frac{1}{8\times 10^{10}}\sum_{\tau=3}^{49} t\tau w(\tau).$$

Hence, in view of

$$8.33333\times 10^{-8} \approx M_1 < \frac{\Gamma^2(\alpha_1)\overline{\kappa_1}(2\alpha_1 - 1)}{b^{2\alpha_1+1}} = \frac{\Gamma^2(0.75) \times 10 \times (2\times 0.75 - 1)}{50^{2\times 0.75+1}} \approx 4.24710\times 10^{-4},$$

$$3.12500 \times 10^{-8} = M_2 < \frac{\Gamma^2(\alpha_2)\overline{\kappa_2}(2\alpha_2 - 1)}{b^{2\alpha_2+1}} = \frac{\Gamma^2(0.8) \times 3.5 \times (2 \times 0.8 - 1)}{50^{2\times 0.8+1}} \approx 1.08887 \times 10^{-4},$$

and

$$6.25000 \times 10^{-8} = M_3 < \frac{\Gamma^2(\alpha_3)\overline{\kappa_3}(2\alpha_3 - 1)}{b^{2\alpha_3+1}} = \frac{\Gamma^2(0.9) \times 4 \times (2 \times 0.9 - 1)}{50^{2\times 0.9+1}} \approx 6.39275 \times 10^{-5},$$

we see that the hypothesis ($\mathbf{S_1}$) *of Theorem 3.1 holds.*

Consider $z_1(t) = (t-1)^{\underline{1.5}}$, $z_2(t) = (t-1.5)^{\underline{1}}$, $z_3(t) = (t-0.5)^{\underline{2}}$ *and* $r = 0.0001$ *to use Theorem 3.1, one obtains that* $z_1(3) = z_1(50) = z_2(3) = z_2(50) = z_3(3) = z_3(50) = 0$ *and more*

$$\begin{aligned}
\Delta_{2.5}^{0.75} z_1(t) &= \Delta_{2.5}^{0.75}(t-1)^{\underline{1.5}} \\
&= \frac{\Gamma(1.5+1)}{\Gamma(1.5-0.75+1)}(t-1)^{\underline{1.5-0.75}} \\
&= \frac{1.5\Gamma(1.5)}{\Gamma(1.75)}(t-1)^{\underline{0.75}} \\
&= \frac{1.5 \times 0.5\Gamma(0.5)}{0.75\Gamma(0.75)} \frac{\Gamma(t-1+1)}{\Gamma(t-1-0.75+1)} \\
&= \frac{1.5 \times 0.5\Gamma(0.5)}{0.75\Gamma(0.75)} \frac{\Gamma(t)}{\Gamma(t-0.75)} \\
&= \frac{0.75\Gamma(0.5)}{0.75\Gamma(0.75)} \frac{\Gamma(t)}{\Gamma(t-0.75)} \\
&= \frac{\sqrt{\pi}}{\Gamma(0.75)} \frac{\Gamma(t)}{\Gamma(t-0.75)},
\end{aligned}$$

$$\begin{aligned}
\Delta_{2.5}^{0.8} z_2(t) &= \Delta_{2.5}^{0.8}(t-1.5)^{\underline{1}} \\
&= \frac{\Gamma(1+1)}{\Gamma(1-0.8+1)}(t-1.5)^{\underline{1-0.8}} \\
&= \frac{\Gamma(2)}{\Gamma(1.2)}(t-1.5)^{\underline{0.2}} \\
&= \frac{1}{0.2} \frac{\Gamma(t-1.5+1)}{\Gamma(t-1.5-0.2+1)} \\
&= 5\frac{\Gamma(t-0.5)}{\Gamma(t-0.7)},
\end{aligned}$$

$$\begin{aligned}
\Delta_{2.5}^{0.9} z_3(t) &= \Delta_{2.5}^{0.9}(t-0.5)^{\underline{2}} \\
&= \frac{\Gamma(0.5+1)}{\Gamma(0.5-0.9+1)}(t-0.5)^{\underline{2-0.9}} \\
&= \frac{0.5}{\Gamma(0.6)}(t-0.5)^{\underline{1.1}} \\
&= \frac{0.5}{\Gamma(0.6)} \frac{\Gamma(t-0.5+1)}{\Gamma(t-0.5-1.1+1)}
\end{aligned}$$

$$= \frac{0.5}{\Gamma(0.6)}\frac{\Gamma(t+0.5)}{\Gamma(t-0.6)},$$

then, some simple calculations yield that

$$\begin{aligned}
\|z_1\|_{0.75}^2 &= \sum_{t=3}^{49}\kappa_1(t)|\,\Delta_{2.5}^{0.75}z_1(t)|^2\\
&= \sum_{t=3}^{49}(1+t^2)\times\left[\frac{\sqrt{\pi}}{\Gamma(0.75)}\frac{\Gamma(t)}{\Gamma(t-0.75)}\right]^2\\
&\approx 1.89815\times 10^7,
\end{aligned}$$

$$\begin{aligned}
\|z_2\|_{0.8}^2 &= \sum_{t=3}^{49}\kappa_2(t)|\,\Delta_{2.5}^{0.8}z_2(t)|^2\\
&= \sum_{t=3}^{49}(0.5+t)\times\left[5\frac{\Gamma(t-0.5)}{\Gamma(t-0.7)}\right]^2\\
&\approx 1.21632\times 10^5,
\end{aligned}$$

$$\begin{aligned}
\|z_3\|_{0.9}^2 &= \sum_{t=3}^{49}\kappa_3(t)|\,\Delta_{2.5}^{0.9}z_3(t)|^2\\
&= \sum_{t=3}^{49}(1+t)\times\left[\frac{0.5}{\Gamma(0.6)}\frac{\Gamma(t+0.5)}{\Gamma(t-0.6)}\right]^2\\
&\approx 3.49403\times 10^5,
\end{aligned}$$

As a result, $\|z_1\|_{0.75}^2+\|z_2\|_{0.8}^2+\|z_3\|_{0.9}^2 \approx 1.89815\times10^7+1.21632\times10^5+3.49403\times10^5 = 1.94526\times10^7$, *also one gets*

$$\begin{aligned}
H_1(z_1(t)) &= \frac{1}{6\times 10^{10}}\sum_{\tau=3}^{49}t\tau z_1(\tau)\\
&= \frac{1}{6\times 10^{10}}\sum_{\tau=3}^{49}t\tau(\tau-1)^{\underline{1.5}}\\
&= \frac{1}{6\times 10^{10}}\sum_{\tau=3}^{49}t\tau\frac{\Gamma(\tau-1+1)}{\Gamma(\tau-1-1.5+1)}\\
&= \frac{1}{6\times 10^{10}}\sum_{\tau=3}^{49}t\tau\frac{\Gamma(\tau)}{\Gamma(\tau-1.5)}\\
&\approx 3.84937\times 10^{-6}t,
\end{aligned}$$

$$H_2(z_2(t)) = \frac{1}{1.6\times 10^{11}}\sum_{\tau=3}^{49}t\tau z_2(\tau)$$

$$
\begin{aligned}
&= \frac{1}{1.6\times 10^{11}}\sum_{\tau=3}^{49} t\tau(\tau-1.5)^{\underline{1}}\\
&= \frac{1}{1.6\times 10^{11}}\sum_{\tau=3}^{49} t\tau(\tau-1.5)\\
&\approx 2.41168\times 10^{-7}t,
\end{aligned}
$$

$$
\begin{aligned}
H_3(z_3(t)) &= \frac{1}{8\times 10^{10}}\sum_{\tau=3}^{49} t\tau z_3(\tau)\\
&= \frac{1}{8\times 10^{10}}\sum_{\tau=3}^{49} t\tau(\tau-0.5)^{\underline{2}}\\
&= \frac{1}{8\times 10^{10}}\sum_{\tau=3}^{49} t\tau(\tau-0.5)(\tau-0.5-1)\\
&= \frac{1}{8\times 10^{10}}\sum_{\tau=3}^{49} t\tau(\tau-0.5)(\tau-1.5)\\
&\approx 1.77586\times 10^{-5}t.
\end{aligned}
$$

Therefore, one has

$$
2\sum_{t=3}^{49}(H_1(z_1(t))+H_2(z_2(t))+H_3(z_3(t))) \approx 2(0.00470, 0.00029, 0.021701) = 0.05340.
$$

And we see that

$$
\begin{aligned}
& 1.94526\times 10^{7}\\
\approx\ & \sum_{i=1}^{3}\|z_i\|_{\alpha_i}^2\\
\geq\ & 2r+\sum_{i=1}^{3}\sum_{t=3}^{49}\sum_{\tau=3}^{49}\xi_i(t,\tau)z_i(\tau)z_i(t)\\
=\ & 2r+\sum_{t=3}^{49}\sum_{\tau=3}^{49}\xi_1(t,\tau)z_1(\tau)z_1(t)+\sum_{t=3}^{49}\sum_{\tau=3}^{49}\xi_2(t,\tau)z_2(\tau)z_2(t)+\sum_{t=3}^{49}\sum_{\tau=3}^{49}\xi_3(t,\tau)z_3(\tau)z_3(t)\\
=\ & 2r+\sum_{t=3}^{49}\sum_{\tau=3}^{49}\frac{1}{3\times 10^{10}}t\tau\frac{\Gamma(\tau)}{\Gamma(\tau-1.5)}\frac{\Gamma(t)}{\Gamma(t-1.5)}+\sum_{t=3}^{49}\sum_{\tau=3}^{49}\frac{1}{8\times 10^{10}}t\tau(\tau-1.5)(t-1.5)\\
& +\sum_{t=3}^{49}\sum_{\tau=3}^{49}\frac{1}{4\times 10^{10}}t\tau(\tau-0.5)(\tau-1.5)(t-0.5)(t-1.5)\\
\approx\ & 2\times 0.0001+52.23916\\
=\ & 52.23936.
\end{aligned}
$$

Hence, clearly the hypothesis ($\mathbf{S_2}$) *of Theorem 3.1 holds.*

In light of the expression of G and the fact that $z_1^2(t)+z_2^2(t)+z_3^2(t)=\left[\frac{\Gamma(t)}{\Gamma(t-1.5)}\right]^2+[(t-1.5)]^2+[(t-0.5)(t-1.5)]^2<2.20550\times10^5$ *for all* $t\in[2.5,50]_{\mathbb{Z}}$, *we obtain the following inequality*

$$\begin{aligned}
&\frac{\sum\limits_{t=3}^{49}\sup\limits_{(v_1,v_2,v_3)\in\Upsilon(Cr)}G(t,v_1,v_2,v_3)}{r}\\
=&\frac{\sum\limits_{t=3}^{49}\sup\limits_{(v_1,v_2,v_3)\in\Upsilon(Cr)}(1+t^2)(v_1^2+v_2^2+v_1^2)^2}{r}\\
\le&\frac{\sum\limits_{t=3}^{49}(1+t^2)(2Cr)^2}{r}\\
=&4C^2r\sum_{t=3}^{49}(1+t^2)\\
\approx&634.97309\\
<&3.56496\times10^{10}\\
\approx&\frac{2\sum\limits_{t=3}^{49}(1+t^2)(z_1^2(t)+z_2^2(t)+z_3^2(t))^2}{\sum\limits_{i=1}^{3}\|z_i\|_{\alpha_i}^2-2\sum\limits_{t=3}^{49}(H_1(z_1(t))+H_2(z_2(t))+H_3(z_3(t)))}\\
=&\frac{2\sum\limits_{t=3}^{49}G(t,z_1(t),z_2(t),z_3(t))}{\sum\limits_{i=1}^{3}\|z_i\|_{\alpha_i}^2-2\sum\limits_{t=3}^{49}(H_1(z_1(t))+H_2(z_2(t))+H_3(z_3(t)))}\\
=&\frac{2\sum\limits_{t=3}^{49}G(t,z_1(t),z_2(t),z_3(t))}{\sum\limits_{i=1}^{3}\|z_i\|_{\alpha_i}^2-\sum\limits_{i=1}^{3}\sum\limits_{t=3}^{49}\sum\limits_{\tau=3}^{49}\xi_i(t,\tau)z_i(\tau)z_i(t)}.
\end{aligned}$$

Consequently, the hypothesis ($\mathbf{S_3}$) *of Theorem 3.1 is satisfied as well.*

Actually,

$$\begin{aligned}
&\liminf_{\forall i:|v_i|\to+\infty}\frac{G(t,v_1,v_2,v_3)}{\sum\limits_{i=1}^{3}|v_i|^2}\\
=&\liminf_{|v_1|\to+\infty,\,|v_2|\to+\infty,\,|v_3|\to+\infty}\frac{G(t,v_1,v_2,v_3)}{|v_1|^2+|v_2|^2+|v_3|^2}\\
=&\liminf_{|v_1|\to+\infty,\,|v_2|\to+\infty,\,|v_3|\to+\infty}\frac{(1+t^2)\left[2\sqrt{v_1^2+v_2^2+v_3^2}-(v_1^2+v_2^2+v_3^2)\right]}{|v_1|^2+|v_2|^2+|v_3|^2}
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{|v_1|\to+\infty,\,|v_2|\to+\infty,\,|v_3|\to+\infty} \frac{2(1+t^2)}{\sqrt{v_1^2+v_2^2+v_3^2}} - \liminf_{|v_1|\to+\infty,\,|v_2|\to+\infty,\,|v_3|\to+\infty} (1+t^2) \\
&= -10 \\
&< 1.02649 \\
&\approx \frac{2C^2 r \sum_{t=3}^{49}(1+t^2)}{A} \\
&= \frac{4C^2 r^2 \sum_{t=3}^{49}(1+t^2)}{2Ar} \\
&= \frac{\sum_{t=3}^{49}(1+t^2)(2Cr)^2}{2Ar} \\
&= \frac{\sum_{t=3}^{49} \sup_{(v_1,v_2,v_3)\in\Upsilon(Cr)} (1+t^2)(v_1^2+v_2^2+v_1^2)^2}{2Ar} \\
&= \frac{\sum_{t=3}^{49} \sup_{(v_1,v_2,v_3)\in\Upsilon(Cr)} G(t,v_1,v_2,v_3)}{2Ar}.
\end{aligned}
$$

Therefore, the hypothesis ($\mathbf{S_4}$) *of Theorem 3.1 holds.*

Based on Theorem 3.1, the above nonlinear system has at least three distinct weak solutions in the space $W^{0.75,2}_{\Delta,2.5^+} \times W^{0.8,2}_{\Delta,2.5^+} \times W^{0.9,2}_{\Delta,2.5^+}$ *for any* $\eta \in \left] \frac{\sum_{i=1}^{3} \|z_i\|^2_{\alpha_i} - \sum_{i=1}^{3}\sum_{t=3}^{49}\sum_{\tau=3}^{49} \xi_i(t,\tau) z_i(\tau) z_i(t)}{2\sum_{t=3}^{49} G(t,z_1(t),z_2(t),z_3(t))}, \frac{r}{\sum_{t=3}^{49} \sup_{(v_1,v_2,v_3)\in\Upsilon(Cr)} G(t,v_1,v_2,v_3)} \right[\approx \left]2.80508 \times 10^{-11}, 1.574870 \times 10^{-3}\right[$.

5. Conclusions

In this work, several sufficient conditions ensuring the existence of three distinct solutions of a system of over-determined Fredholm fractional integro-differential equations on time scales are derived by variational methods, which shows that variational methods are powerful and effective methods for studying fractional boundary value problems on time scales.

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Conflict of interest

The authors declare no conflict of interest.

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