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Research article

Faber polynomial coefficients estimates for certain subclasses of q-Mittag-Leffler-Type analytic and bi-univalent functions

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Abstract: In this paper, we introduce the q-analogus of generalized differential operator involving q-Mittag-Leffler function in open unit disk

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and define new subclass of analytic and bi-univalent functions. By applying the Faber polynomial expansion method, we then determined general coefficient bounds $|a_n|$, for $n \geq 3$. We also highlight some known consequences of our main results.

Keywords: analytic functions; univalent functions; analytic and bi-univalent function; q-derivative; subordination; Faber polynomials; q-Mittag-Leffler function
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1. Introduction, definitions and motivation

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized under the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Furthermore, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in E.

Let $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in E),$$

we define the convolution (or Hadamard product) of f and g as:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in E).$$

Let $f, h \in \mathcal{A}$, f is subordinate to h if there exists a Schwarz function u, where

$$u(0) = 0 \quad \text{and} \quad |u(z)| < 1 \qquad (z \in E),$$

such that

$$f(z) = h(u(z)) \qquad (z \in E).$$

We denote this subordination by

$$f \prec h \ \text{ or } \ f(z) \prec h(z), \quad (z \in E).$$

In particular, if the function h is univalent in E, the above subordination is equivalent to

$$f(0) = h(0) \qquad f(E) \subset h(E).$$

We see that (see [24]) for the Schwarz function $u(z)$, we have

$$|u_n| \leq 1.$$

The Koebe-one quarter Theorem (see [24]) shows that the image of E under every univalent function $f \in \mathcal{A}$ contains a disk $\{w : |w| < \frac{1}{4}\}$ of radius $\frac{1}{4}$. Every univalent function f has an inverse f^{-1} defined on some disk containing the disk $\{w : |w| < \frac{1}{4}\}$ and satisfying:

$$f^{-1}(f(z)) = z \qquad (z \in E)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \ \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \ldots. \tag{1.2}$$

A function f is said to be bi-univalent on E if both f and $g = f^{-1}$ are univalent on E. We denote the class of all such functions by Σ.

Lewin [50] studied the class of bi-univalent functions, in fact he obtained the bound

$$|a_2| \leq 1.51.$$

Netanyahu [53] showed that $\max|a_2| = \frac{4}{3}$. Brannan and Clunie [16] conjectured that $|a_2| \leq \sqrt{2}$. In recent years, the pioneering work of Srivastava *et al.* [67] essentially revived the investigation of various subclasses of the analytic and bi-univalent function class Σ. In fact, in a remarkably large number of sequels to the pioneering work of Srivastava *et al.* [67], several different subclasses of the analytic and bi-univalent function class were introduced and studied analogously by the many authors (see, for example, [11, 17–21, 29, 31, 38, 39, 64, 67].

In Geometric Function Theory (GFT), the quantum (or q-) calculus used as important tools to study different families of analytic function and due to the application in mathematics and some related areas it has inspired a number of well-known mathematicians.

The quantum (or q-) calculus is widely applied in various operators which include the q-difference (q-derivative) operators and these operators plays an important role in GFT as well as in the theory of hypergeometric series, quantum theory, number theory and statistical mechanics. Jackson [35, 36] was among the few researchers who defined the q-derivative and q-integral operator as well as provided some of their applications. Also Ismail *et al.* [34] introduced research work in connection with function theory and q-theory. Letter on, by using q-beta function Gupta [14] introduced q-Baskakov-Durrmeyer operator while q-Picard and q-Gauss-Weierstrass singular integral operators introduced and studied by Aral in [13].

Historically, Srivastava studied univalent function theory by using q-calculus, see for detail [62, 63]. Moreover, Kanas and Raducanu [40] introduced the q-analogue of Ruscheweyh differential operator and Arif *et al.* [15] discussed some of its applications for multivalent functions. For detailed study about q-analogous of operators we may refer to [1, 33, 42–49, 59, 66].

Definition 1.1. (see [36]) The q-number $[t]_q$ and q-factorial $[n]_q!$ for $q \in (0, 1)$ is defined as:

$$[t]_q = 1 + q + q^2 + ... + q^{n-1}, \quad (t = n \in \mathbb{N})$$

and

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \ (n \in \mathbb{N})$$

where

$$[0]_q! = 1 \quad \text{and} \quad [t]_q = \frac{1 - q^t}{1 - q} \quad (t \in \mathbb{C}).$$

Definition 1.2. (see [36]) The q-generalized Pochhammer symbol $[t]_{n,q}$, $t \in \mathbb{C}$, is defined as:

$$[t]_{n,q} = [t]_q[t + 1]_q[t + 2]_q...[t + n - 1]_q, \quad (n \in \mathbb{N}).$$

and the q-Gamma function be given as:

$$[t]_q = \frac{\Gamma_q(t + 1)}{\Gamma_q(t)} \text{ and } \Gamma_q(1) = 1.$$

Definition 1.3. ([36]) For $f \in \mathcal{A}$, the q-derivative operator or q-difference operator be defined as:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ \ z \in E. \tag{1.3}$$

Combining (1.1) and (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Note that

$$D_q z^n = [n]_q z^{n-1} \quad \text{and} \quad D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

We can observe that

$$\lim_{q \to 1-} D_q f(z) = f'(z).$$

Mittag-Leffler introduced Mittag-Leffler function $\mathcal{H}_\alpha(z)$ in [51, 52] as:

$$\mathcal{H}_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} z^n, \ \ (\alpha \in \mathbb{C}, \ \Re(\alpha)) > 0,$$

and its generalization $\mathcal{H}_{\alpha,\beta}(z)$ introduced by Wiman [70] as:

$$\mathcal{H}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} z^n, \ \ (\alpha, \ \beta \in \mathbb{C}, \ \Re(\alpha), \ \Re(\beta)) > 0.$$

For more study about Mittag-Leffler function see article [**?**, 12, 54, 65, 68].

The q-Mittag-Leffler function is defined by (see [58])

$$\mathcal{H}_{\alpha,\beta}(z, q) = \sum_{n=0}^{\infty} \frac{1}{\Gamma_q(\alpha n + \beta)} z^n \qquad (\alpha, \ \beta \in \mathbb{C}, \ \Re(\alpha), \ \Re(\beta)) > 0. \tag{1.4}$$

Note that q-Mittag-Leffler function is the specialized case of the q-Fox-Wright function ${}_r\Phi_s(z, q)$, (see, for details, [60, 61]). Since the q-Mittag-Leffler function $\mathcal{H}_{\alpha,\beta}(z, q)$ defined by (1.4) does not belong to the normalized analytic function class $\mathcal{A}$.

Now, we define the normalization of this q-Mittag-Leffler function $\mathcal{F}_{\alpha,\beta}(z)$ as:

$$\mathcal{F}_{\alpha,\beta}(z, q) = z\Gamma_q(\beta)\mathcal{H}_{\alpha,\beta}(z)$$

$$\mathcal{F}_{\alpha,\beta}(z, q) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(n-1) + \beta)} z^n,$$

where $z \in E$, $\Re\alpha > 0, \beta \in \mathbb{C} \backslash \{0, -1, -2, ...\})$. Corresponding to $\mathcal{F}_{\alpha,\beta}(z, q)$ and for $f \in \mathcal{A}$, we define the following differential operator $\mathcal{D}_{\delta,\mu}^{m,q}(\alpha, \beta) : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{D}_{\delta,\mu}^{0,q}(\alpha, \beta) f(z) = f(z) * \mathcal{F}_{\alpha,\beta}(z, q),$$

$$\mathcal{D}^{1,q}_{\delta,\mu}(\alpha,\beta)f(z) = (1-\delta+\mu)\left(f(z) * \mathcal{F}_{\alpha,\beta}(z,q)\right)$$
$$+ (\delta-\mu)zD_q\left(f(z) * \mathcal{F}_{\alpha,\beta}(z,q)\right) + \delta\mu z^2 D_q^2\left(f(z) * \mathcal{F}_{\alpha,\beta}(z,q)\right), \tag{1.5}$$

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$$\mathcal{D}^{m,q}_{\delta,\mu}(\alpha,\beta)f(z) = \mathcal{D}^{q}_{\delta,\mu}\left(\mathcal{D}^{m-1}_{\delta,\mu}(\alpha,\beta)f(z)\right). \tag{1.6}$$

If $f(z)$ is given by (1.1), then from (1.5) and (1.6), we have

$$\mathcal{D}^{m,q}_{\delta,\mu}(\alpha,\beta)f(z) = z + \sum_{n=2}^{\infty} \Psi(\alpha,\beta,q,n)\left(\varphi(\delta,\mu,q,n)\right)^m a_n z^n,$$

where

$$\varphi(\delta,\mu,q,n) = 1 + \left(\delta\mu\,[n]_q\,[n-1]_q + q(\delta-\mu)[n]_q\right), \tag{1.7}$$
$$\Psi(\alpha,\beta,q,n) = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(n-1)+\beta)}. \tag{1.8}$$

Each of the following special case of the above-mentioned operator $\mathcal{D}^{m,q}_{\delta,\mu}(\alpha,\beta) : \mathcal{A} \to \mathcal{A}$ is worthy of noted.
(i) For $\mu = 0$, $\alpha = 0, \beta = 1$, and $\delta = 1$, we get Salagean q-differential operator introduced by Salagean in [27].
(ii) For $q \to 1-$, $\mu = 0$, $\alpha = 0, \beta = 1$, and $\delta = 1$, we get Salagean differential operator introduced by Salagean in [55].
(iii) For $q \to 1-$, $\mu = 0$, $\alpha = 0$, and $\beta = 1$, we get Al-Oboudi operator [2].
(iv) For $q \to 1-$, and $m = 0$, we have $E_{\alpha,\beta}(z)$ introduced in [65].
(v) For $q \to 1-$, $\alpha = 0$, and $\beta = 1$, we have Raducanu and Orhan differential operator [22] see also [23].

The Faber polynomials introduced by Faber [25] play an important role in various areas of mathematical sciences, especially in Geometric Function Theory see also [28, 56, 57]. Not much is known about the bounds on general coefficients $|a_n|$, for $n \geq 3$ of bi-univalent functions. In the literature only a few work determining the general coefficient $|a_n|$, for $n \geq 3$ for the analytic bi-univalent function given by (1.1). For more study see [3, 4, 30, 32, 37, 69].

Here in this paper we define new subclass of bi-univalent functions and determine estimates for the general coefficient bounds $|a_n|$ for $n \geq 3$, by using Faber polynomial expansions and newly defined q-analogue of differential operator. Throughout in this paper, we assume that

$$0 \leq \mu \leq \delta,\ 0 \leq \delta,\ 0 < q < 1, -1 \leq B < A \leq 1, \lambda \geq 1,\ m \in N_0 = N \cup \{0\}.$$

Definition 1.4. A function $f \in \Sigma$ is said to be in the class $\mathcal{B}^{m,\lambda,\mu,\delta}_{\Sigma}(\alpha,\beta,q,A,B)$ if the following subordinations are satisfied:

$$\frac{(1-\lambda)\,\mathcal{D}^{m,q}_{\delta,\mu}(\alpha,\beta)f(z) + \lambda\mathcal{D}^{m+1,q}_{\delta,\mu}(\alpha,\beta)f(z)}{z} \prec \frac{1+Az}{1+Bz},$$

and

$$\frac{(1-\lambda)\,\mathcal{D}^{m,q}_{\delta,\mu}(\alpha,\beta)g(w)+\lambda\mathcal{D}^{m+1,q}_{\delta,\mu}(\alpha,\beta)g(w)}{w}\prec\frac{1+Aw}{1+Bw},$$

where the function g is given by (1.2).

Remark 1.5. First of all, it ids easy to see that

$$\lim_{q\to 1-}\left(\mathcal{B}^{m,\lambda,0,1}_{\Sigma}(0,1,q,1,-1)\right)=\mathcal{B}_{\Sigma}(m,\lambda,\varphi),$$

where $\mathcal{B}_{\Sigma}(m,\lambda,\varphi)$ is the function class introduced and studied by Altinkaya and Yalcin [11]. Secindly, we have

$$\lim_{q\to 1-}\mathcal{B}^{0,\lambda,0,1}_{\Sigma}(0,1,q,1,-1)=\mathcal{B}_{\Sigma}(\varphi,\lambda)$$

where the class $\mathcal{B}_{\Sigma}(\varphi,\lambda)$ was introduced by Frasin and Aouf [26].

In this article, we defined certain new subclasses of analytic and bi-univalent functions which involve the differential operator of q-Mittag-Leffer functions. Then by applying the method of Faber polynomial expansions, we determined general coefficients bond $|a_n|$, for $n\geq 3$. We also highlight some known consequences of our main results.

2. Main results

By using the Faber polynomial expansion of functions f of the form (1.1), the coefficients of its inverse map $g=f^{-1}$ are given by,

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K^{-n}_{n-1}(a_2,a_3,...)w^n,$$

where

$$\begin{aligned}K^{-n}_{n-1}=&\frac{(-n)!}{(-2n+1)!(n-5)!}a_2^{n-1}+\frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4\\&+\frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]\\&+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]\\&+\sum_{j\geq 7}a_2^{n-j}V_j\end{aligned}$$

and $g=f^{-1}$ given by (1.2), V_j with $7\leq j\leq n$ is a homogeneous polynomial in the variables $|a_2|,|a_3|,...,|a_n|$ (see [5]). In particular, the first three terms of K^{-n}_{n-1} are

$$\frac{1}{2}K^{-2}_{1}=-a_2$$

$$\frac{1}{3}K_2^{-3} = 2a_2^2 - a_3$$
$$\frac{1}{4}K_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^p (see [4]) is,

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1},$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, ...)$ (see [6]) given by

$$E_{n-1}^m(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_1!...\mu_{n-1}!}, \quad \text{for } m \leq n.$$

While $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

and

$$\mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n - 1.$$

Evidently, (see [3])

$$E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1},$$

or equivalently,

$$E_n^m(a_1, a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_1!...\mu_n!}, \qquad \text{for } m \leq n,$$

again $a_1 = 1$, and the taking the sum over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$
$$\mu_1 + 2\mu_2 + ... + (n)\mu_n = n.$$

It is clear that

$$E_n^n(a_1, ..., a_n) = E_1^n$$

the first and last polynomials are

$$E_n^n = a_1^n \quad \text{and} \quad E_n^1 = a_n.$$

Theorem 2.1. *Let $f \in \mathcal{B}_{\Sigma}^{m,\lambda,\mu,\delta}(\alpha, \beta, q, A, B)$. If $a_i = 0$; $2 \leq i \leq n-1$, then*

$$|a_n| \leq \frac{A - B}{\{1 + (\varphi - 1)\lambda\}\Psi(\alpha, \beta, q, n)(\varphi(\delta, \mu, q, n))^m}, \qquad n \geq 3,$$

where φ is given by (1.7).

Proof. Let f be given by (1.1), we have

$$\frac{(1-\lambda)\,\mathcal{D}_{\delta,\mu}^{m,q}(\alpha,\beta)f(z)+\lambda\mathcal{D}_{\delta,\mu}^{m+1,q}(\alpha,\beta)f(z)}{z}$$
$$=1+\sum_{n=2}^{\infty}\{1+(\varphi-1)\,\lambda\}\,\Psi\,(\alpha,\beta,q,n)\,(\varphi\,(\delta,\mu,q,n))^{m}\,a_nz^{n-1}$$

and for its inverse map $g=f^{-1}$, we have

$$\frac{(1-\lambda)\,\mathcal{D}_{\delta,\mu}^{m,q}(\alpha,\beta)g(w)+\lambda\mathcal{D}_{\delta,\mu}^{m+1,q}(\alpha,\beta)g(w)}{w}$$
$$=1+\sum_{n=2}^{\infty}\{1+(\varphi-1)\,\lambda\}\,\Psi\,(\alpha,\beta,q,n)\,(\varphi\,(\delta,\mu,q,n))^{m}$$
$$\cdot\frac{1}{n}K_{n-1}^{-n}(a_2,a_3....,a_n)w^{n-1}$$
$$=1+\sum_{n=2}^{\infty}\{1+(\varphi-1)\,\lambda\}\,\Psi\,(\alpha,\beta,q,n)\,(\varphi\,(\delta,\mu,q,n))^{m}\,b_nw^{n-1},$$

where

$$b_n=\frac{1}{n}K_{n-1}^{-n}(a_2,a_3....,a_n).$$

Since, both the functions f and its inverse map $g=f^{-1}$ are in $\mathcal{B}_{\Sigma}^{m,\lambda,\mu,\delta}(\alpha,\beta,q,A,B)$, by the definition of subordination, for $z,w\in E$, there exist two Schwarz functions

$$\psi(z)=\sum_{n=1}^{\infty}c_nz^n$$

and

$$\phi(w)=\sum_{n=1}^{\infty}d_nw^n,$$

such that

$$\frac{(1-\lambda)\,\mathcal{D}_{\delta,\mu}^{m,q}(\alpha,\beta)f(z)+\lambda\mathcal{D}_{\delta,\mu}^{m+1,q}(\alpha,\beta)f(z)}{z}=\frac{1+A(\psi(z))}{1+B(\psi(z))},\tag{2.1}$$

and

$$\frac{(1-\lambda)\,\mathcal{D}_{\delta,\mu}^{m,q}(\alpha,\beta)g(w)+\lambda\mathcal{D}_{\delta,\mu}^{m+1,q}(\alpha,\beta)g(w)}{w}=\frac{1+A(\phi(w))}{1+B(\phi(w))},\tag{2.2}$$

where

$$\frac{1+A(\psi(z))}{1+B(\psi(z))}=1-\sum_{n=1}^{\infty}(A-B)K_n^{-1}(c_{1,}c_2,...,c_n,B)z^n,\tag{2.3}$$

and

$$\frac{1+A(\phi(w))}{1+B(\phi(w))}=1-\sum_{n=1}^{\infty}(A-B)K_n^{-1}(d_{1,}d_2,...,d_n,B)w^n.\tag{2.4}$$

In general [3, 4] for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of $K_n^p(k_1, k_2, ..., k_n, B)$,

$$\begin{aligned} K_n^p(k_1, k_2, ..., k_n, B) &= \frac{p!}{(p-n)!n!}k_1^n B^{n-1} + \frac{p!}{(p-n+1)!(n-2)!}k_1^{n-2}k_2 B^{n-2} \\ &+ \frac{p!}{(p-n+2)!(n-3)!} \times k_1^{n-3}k_3 B^{n-3} \\ &+ \frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}\left[k_4 B^{n-4} + \frac{p-n+3}{2}k_3^2 B\right] \\ &+ \frac{p!}{(p-n+4)!(n-5)!}k_1^{n-5}\left[k_5 B^{n-5} + (p-n+4)k_3 k_4 B\right] \\ &+ \sum_{j \geq 6} k_1^{n-1} X_j, \end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, ..., k_n$.

Comparing the corresponding coefficients of (2.1) and (2.3) yields

$$\begin{aligned} &\{1 + (\varphi(\delta, \mu, q, n) - 1)\lambda\}\Psi(\alpha, \beta, q, n)(\varphi(\delta, \mu, q, n))^m a_n \\ &= -(A-B)K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, B) \end{aligned} \tag{2.5}$$

and similarly, from (2.2) and (2.4) yields

$$\begin{aligned} &\{1 + (\varphi(\delta, \mu, q, n) - 1)\lambda\}\Psi(\alpha, \beta, q, n)(\varphi(\delta, \mu, q, n))^m b_n \\ &= -(A-B)K_{n-1}^{-1}(d_1, d_2, ..., d_{n-1}, B). \end{aligned} \tag{2.6}$$

Note that for $a_i = 0; 2 \leq i \leq n-1$, we have

$$b_n = -a_n$$

and so

$$\begin{aligned} &\{1 + (\varphi(\delta, \mu, q, n) - 1)\lambda\}\Psi(\alpha, \beta, q, n)(\varphi(\delta, \mu, q, n))^m a_n \\ &= -(A-B)c_{n-1}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\{1 + (\varphi(\delta, \mu, q, n) - 1)\lambda\}\Psi(\alpha, \beta, q, n)(\varphi(\delta, \mu, q, n))^m a_n \\ &= (A-B)d_{n-1}. \end{aligned} \tag{2.8}$$

Now taking the absolute values of (2.7) and (2.8) and using the fact that

$$|c_{n-1}| \leq 1 \quad \text{and} \quad |d_{n-1}| \leq 1,$$

we obtain

$$\begin{aligned} |a_n| &= \frac{|-(A-B)c_{n-1}|}{|\{1+(\varphi(\delta,\mu,q,n)-1)\lambda\}\Psi(\alpha,\beta,q,n)(\varphi(\delta,\mu,q,n))^m|} \\ &= \frac{|(A-B)d_{n-1}|}{|\{1+(\varphi(\delta,\mu,q,n)-1)\lambda\}\Psi(\alpha,\beta,q,n)(\varphi(\delta,\mu,q,n))^m|} \\ &\leq \frac{A-B}{\{1 + (\varphi(\delta, \mu, q, n) - 1)\lambda\}\Psi(\alpha, \beta, q, n)(\varphi(\delta, \mu, q, n))^m}. \end{aligned}$$

□

If in Theorem 2.1, we take

$$\mu = 0 = \alpha \quad \text{and} \quad \beta = \delta = 1 = A = -B$$

and let $q \to 1-$, we have the following known result.

Corollary 2.2. *([11]). Let $f \in \mathcal{B}_\Sigma(m, \lambda, \varphi)$. If $a_i = 0$; $2 \le i \le n-1$, then*

$$|a_n| \le \frac{2}{n^m \{1 + (n-1)\lambda\}}; \qquad n \ge 3.$$

Theorem 2.3. *Let $f \in \mathcal{B}_\Sigma^{m,\lambda,\mu,\delta}(\alpha, \beta, q, A, B)$. Then*

$$|a_2| \le \min \left\{ \begin{array}{c} \frac{A-B}{\{1+(\varphi(\delta,\mu,q,2)-1)\lambda\}\Psi(\alpha,\beta,q,2)(\varphi(\delta,\mu,q,2))^m}, \\ \sqrt{\frac{(A-B)\{1+|B|\}}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}}, \end{array} \right\},$$

$$|a_3| \le \min \left\{ \begin{array}{c} \frac{(A-B)^2}{\{\{1+(\varphi(\delta,\mu,q,2)-1)\lambda\}\Psi(\alpha,\beta,q,2)(\varphi(\delta,\mu,q,2))^m\}^2} \\ +\frac{A-B}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}, \\ \frac{(A-B)\{2+|B|\}}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m} \end{array} \right\}$$

$$\left|a_3 - a_2^2\right| \le \frac{A-B}{\{1 + (\varphi(\delta, \mu, q, 3) - 1)\lambda\} \Psi(\alpha, \beta, q, 3) (\varphi(\delta, \mu, q, 3))^m},$$

and

$$\left|a_3 - 2a_2^2\right| \le \frac{|A-B|\{1 + |B|\}}{\{1 + (\varphi(\delta, \mu, q, 3) - 1)\lambda\} \Psi(\alpha, \beta, q, 3) (\varphi(\delta, \mu, q, 3))^m}.$$

Proof. Replacing n by 2 and 3 in (2.5) and (2.6), respectively, we find that

$$\{1 + (\varphi(\delta, \mu, q, 2) - 1)\lambda\} \Psi(\alpha, \beta, q, 2) (\varphi(\delta, \mu, q, 2))^m a_2 = -(A-B) c_1, \tag{2.9}$$

$$\begin{aligned} &\{1 + (\varphi(\delta, \mu, q, 3) - 1)\lambda\} \Psi(\alpha, \beta, q, 3) (\varphi(\delta, \mu, q, 3))^m a_3 \\ &= (A-B) c_2 + B(B-A)c_1^2, \end{aligned} \tag{2.10}$$

$$\{1 + (\varphi(\delta, \mu, q, 2) - 1)\lambda\} \Psi(\alpha, \beta, q, 2) (\varphi(\delta, \mu, q, 2))^m a_2 = (A-B) d_1 \tag{2.11}$$

and

$$\begin{aligned} &\{1 + (\varphi(\delta, \mu, q, 3) - 1)\lambda\} \Psi(\alpha, \beta, q, 3) (\varphi(\delta, \mu, q, 3))^m (2a_2^2 - a_3) \\ &= (A-B) d_2 + B(B-A)d_1^2. \end{aligned} \tag{2.12}$$

From (2.9) and (2.11) we obtain

$$\begin{aligned} |a_2| &= \frac{|-(A-B) c_1|}{\{1 + (\varphi(\delta, \mu, q, 2) - 1)\lambda\} \Psi(\alpha, \beta, q, 2) (\varphi(\delta, \mu, q, 2))^m} \\ &= \frac{|(A-B) d_1|}{\{1 + (\varphi(\delta, \mu, q, 2) - 1)\lambda\} \Psi(\alpha, \beta, q, 2) (\varphi(\delta, \mu, q, 2))^m} \end{aligned}$$

$$\leq \frac{A-B}{\{1+(\varphi(\delta,\mu,q,2)-1)\lambda\}\Psi(\alpha,\beta,q,2)(\varphi(\delta,\mu,q,2))^m}. \tag{2.13}$$

Adding (2.10) and (2.12) implies

$$2\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m a_2^2$$
$$=(A-B)(c_2+d_2)+B(B-A)(c_1^2+d_1^2),$$

or equivalently,

$$|a_2| \leq \sqrt{\frac{(A-B)\{1+|B|\}}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}}. \tag{2.14}$$

From (2.13) and (2.14) we get required assertion.

Now from (2.10), one can easily see that

$$|a_3| = \frac{\left|(A-B)c_2+B(B-A)c_1^2\right|}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}$$
$$\leq \frac{(A-B)\{1+|B|\}}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}.$$

Next in order to find the bound on the coefficient $|a_3|$,we subtract (2.12) from (2.10), we thus obtain

$$2\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m(a_3-a_2^2)$$
$$=(A-B)(c_2-d_2)+B(B-A)(c_1^2-d_1^2). \tag{2.15}$$

Using the fact that $c_1^2 = d_1^2$ and taking the absolute values of both sides of (2.15), we obtain the desired inequality

$$|a_3| \leq |a_2|^2 + \frac{|(A-B)(c_2-d_2)|}{2\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}$$
$$\leq |a_2|^2 + \frac{A-B}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}. \tag{2.16}$$

Substituting the value of a_2^2 from (2.13) into (2.16), we obtain

$$|a_3| \leq \frac{(A-B)^2}{\{\{1+(\varphi(\delta,\mu,q,2)-1)\lambda\}\Psi(\alpha,\beta,q,2)(\varphi(\delta,\mu,q,2))^m\}^2}$$
$$+\frac{A-B}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}.$$

Additionally, substituting the value of a_2^2 from (2.14) into (2.16), we obtain

$$|a_3| \leq \frac{(A-B)\{2+|B|\}}{\{1+(\varphi(\delta,\mu,q,3)-1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}.$$

Solving the equation (2.15) for $a_3 - a_2^2$, we get the desired inequality as:

$$\left|a_3 - a_2^2\right| = \left|\frac{(A-B)(c_2 - d_2) + B(B-A)(c_1^2 - d_1^2)}{2\{1 + (\varphi(\delta,\mu,q,3) - 1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}\right|$$
$$\leq \frac{A-B}{\{1 + (\varphi(\delta,\mu,q,3) - 1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}.$$

Finally we rewrite (2.12) as

$$\{1 + (\varphi(\delta,\mu,q,3) - 1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m(a_3 - 2a_2^2)$$
$$= -\left\{(A-B)d_2 + B(B-A)d_1^2\right\},$$

and therefore

$$\left|a_3 - 2a_2^2\right| = \left|\frac{-\left\{(A-B)d_2 + B(B-A)d_1^2\right\}}{\{1 + (\varphi(\delta,\mu,q,3) - 1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}\right|$$
$$\leq \frac{|A-B|\{1+|B|\}}{\{1 + (\varphi(\delta,\mu,q,3) - 1)\lambda\}\Psi(\alpha,\beta,q,3)(\varphi(\delta,\mu,q,3))^m}.$$

□

If in Theorem 2.3, we take

$$\mu = 0 = \alpha \quad \text{and} \quad \beta = \delta = 1 = A = -B$$

and let $q \to 1-$, we have the following known result.

Corollary 2.4. *([11]). Let $f \in \mathcal{B}_\Sigma(p,\lambda,\varphi)$. Then*

$$|a_2| \leq \min\left\{\frac{1}{(1+\lambda)2^{m-1}}, \frac{2}{\sqrt{(1+2\lambda)3^m}}\right\},$$

$$|a_3| \leq \min\left\{\frac{1}{(1+\lambda)^2 2^{2m-2}} + \frac{2}{(1+2\lambda)3^m}, \frac{2}{(1+2\lambda)3^{m-1}}\right\},$$

$$\left|a_3 - a_2^2\right| \leq \frac{2}{(1+2\lambda)3^m}$$

and

$$\left|a_3 - 2a_2^2\right| \leq \frac{4}{(1+2\lambda)3^m}.$$

3. Conclusions

Basic (or q-) Calculus is particularly applicable in many deserve areas of mathematics and physics. In our present investigations, we have first introduced the q-analogus of generalized differential operator involving q-Mittag-Leffler function in open unit disk

$$E = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and then defined certain new subclasses of analytic and bi-univalent functions. Furthermore, By applying the Faber polynomial expansion method, we have determined general coefficient bounds $|a_n|$, for $n \geq 3$. We have also highlight some known consequences of our main results.

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