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Research article

Stability of the mixed Caputo fractional integro-differential equation by means of weighted space method

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Abstract: In this research work, we consider a class of nonlinear fractional integro-differential equations containing Caputo fractional derivative and integral derivative. We discuss the stabilities of Ulam-Hyers, Ulam-Hyers-Rassias, semi-Ulam-Hyers-Rassias for the nonlinear fractional integro-differential equations in terms of weighted space method and Banach fixed-point theorem. After the demonstration of our results, an example is given to illustrate the results we obtained.

Keywords: mixed Caputo fractional derivative; weighted space method; Ulam-Hyers stability; Ulam-Hyers-Rassias stability; semi-Ulam-Hyers-Rassias stability
Mathematics Subject Classification: 34A08, 34D20, 34K37, 47H10

1. Introduction

Since the end of the 20th century, fractional derivatives are widely used in various fields, such as biomedical engineering, biomechanics, viscoelastic mechanics, soft matter physics and mechanics, porous media mechanics, and so forth. A amount of research papers in the literature show interesting slights of fractional calculus both on theories results and applications [3–6, 10]. In view of the broad application background of fractional calculus, it is difficult for researchers to directly obtain solutions to most fractional differential equations. Therefore, it is necessary to discuss the existence and uniqueness of the solutions to fractional differential equations. Researchers have obtained a lot of results about the existence and uniqueness of solutions for boundary value problems of fractional differential equations [11, 27–31]. At the same time, a large number of numerical solutions of fractional differential equations have been obtained [7–9, 12–14].

The Ulam stability of differential equations is an important subject in nonlinear analysis. When a given problem has Ulam-type stability, this shows that there must be the exact solution of the differential equation near an approximate solution. Ulam stability theory is closely related to many branches of mathematics and is one of the most active research topics in the field of fractional calculus. So far, the mathematicians have obtained a number of important conclusions [15, 16, 18, 21, 24].

In 2016, S. Sevgin and H. Sevli [17] investigated the Ulam-Hyers-Rassias stability and the Ulam-Hyers stability for a class of nonlinear Volterra integro-differential equations by means of fixed point theorems

$$u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))ds.$$

Vu and Hoa [25] studied the initial value problem of a class of nonlinear Volterra integro-differential equations, and they discussed the Ulam-Hyers-Rassias stability and the Ulam-Hyers stability of the following equations in terms of successive approximation method

$$\begin{aligned} u'(t) &= f(t, u(t)) + \int_a^t g(t, s, u(s))ds, \quad t \in [a, b], \\ u(a) &= u_0. \end{aligned}$$

J.Vanterler da C. Sousa et al. [19,20] presented the $\psi-Hilfer$ fractional derivative and discussed the Ulam-Hyers stability for the fractional Volterra integro-differential equation by the Banach fixed-point theorem [24]

$${}^H D_{0+}^{\alpha,\beta;\psi} x(t) = g(t, x(t)) + \int_0^t K(t, s, x(s))ds,$$

where ${}^H D_{0+}^{\alpha,\beta;\psi}$ is the $\psi - Hilfer$ fractional derivative.

J.Vanterler da C. Sousa et al. [26] discussed the Ulam-type stability of solutions for initial value problem of the following class of fractional differential equations

$$\begin{aligned} {}^H D_{a+}^{\alpha,\beta;\psi} y(x) &= f\left(x, y(x), \int_a^x K(t, s, y(s), y(\delta(s)))ds\right), \\ I_{a+}^{1-\gamma;\psi} y(a) &= c, \end{aligned}$$

where $I_{a+}^{1-\gamma;\psi}$ is $\psi - Riemann - Liouville$ fractional integral, $\gamma = \alpha + \beta(1 - \alpha)$.

Motivated by the above references, we consider the following class of the mixed Caputo fractional integro-differential equations in this paper

$$u'(t) +^c D_{0+}^{\alpha} u(t) = g(t, u(t)) + \int_0^t K(t, s, u(s))ds, \tag{1.1}$$

where ${}^c D_{0+}^{\alpha}$ is the Caputo derivative, with $0 < \alpha < 1$, $t \in [0, 1]$, $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $u \in C^1[0, 1]$.

There are a couple of physical motivations for Eq (1.1). First of all, the Eq (1.1) is used to represent constitutive relation for viscoelastic model of fractional differential equation, and secondly the Eq (1.1) is used to describe macroscopic models II for electric diffusion of ions in nerve cells when molecular diffusion is anomalous secondary diffusion due to binding, crowding or trapping [23].

The paper is arranged in the following manner: In section 2, we introduce some fundamental definitions and important results of fractional calculus. Besides that, we present the concepts of Ulam-type stability for problem (1.1). We discuss the stabilities of Ulam-Hyers, Ulam-Hyers-Rassias and semi-Ulam-Hyers-Rassias for problem (1.1) in section 3. We provide an example to illustrate the main results in section 4. Finally, the concluding remarks are presented in section 5.

2. Preliminaries

In this section, we will briefly introduce some necessary definitions, notations, the fundamental results related to fractional calculus that will be used throughout the paper, for details, see [1,2]. Also, we will present the concepts of stabilities of Ulam-Hyers, Ulam-Hyers-Rassias and semi-Ulam-Hyers-Rassias for Eq (1.1).

Let $C^1[0,1] = \{u|u$ is a differentiable function on $[0,1]$ and its derivative is continuous$\}$. We consider $C^1[0,1]$ endowed with the weighted metric

$$\rho(u,v) = \sup_{t\in[0,1]} \frac{|u(t)-v(t)|}{\sigma(t)}, u,v \in C^1[0,1],$$

where $\sigma : [0,1] \to (0,+\infty)$ is a non-decreasing continuous function, and suppose that there exists a constant $\xi \in [0,1)$, such that

$$\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\sigma(s)ds \leq \xi\sigma(t).$$

Obviously, $(C^1[0,1],\rho)$ is a complete metric space.

Definition 1. *[1, 2] For a real valued integrable function* $f : (0,+\infty) \to \mathbb{R}$*, the Riemann-Liouville fractional integral with order* $0 < \alpha < 1$ *is defined as*

$$I^{\alpha}_{0+} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, x > 0,$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2. *[1, 2] The left Caputo fractional derivative* ${}^cD^{\alpha}_{0+}$ *of an absolutely continuous(or differentiable) function* $f(t)$ *with order* $0 < \alpha < 1$ *is defined as*

$${}^cD^{\alpha}_{0+} f(x) = I^{1-\alpha}_{0+} f'(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\,dt.$$

Definition 3. *[1, 2] The two-parametric Mittag-Leffler function is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \alpha,\beta,z \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0.$$

In addition, We need the following Laplace transform formulas of fractional derivative operators and Mittag-Leffler function to develop our results [1,2].

The Laplace transform of the Caputo derivative ${}^cD^{\alpha}_{0+} f(t)$ is

$$\mathcal{L}\{{}^cD^{\alpha}_{0+} f(t)\}(s) = s^{\alpha}\widetilde{f}(s) - s^{\alpha-1} f(0), 0 < \alpha < 1. \tag{2.1}$$

The Laplace transforms of the two-parametric Mittag-Leffler function are

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})\}(s) = \frac{s^{\alpha-\beta}}{(s^{\alpha} \mp a)}, Re(s) >| a |^{\frac{1}{\alpha}}, Re(\alpha) > 0, Re(\beta) > 0, \alpha,\beta \in \mathbb{C}, \tag{2.2}$$

$$\mathcal{L}\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\}(s)=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}}, Re(s)>\mid a\mid^{\frac{1}{\alpha}}, Re(\alpha)>0, Re(\beta)>0, \alpha,\beta\in\mathbb{C},$$

where $E_{\alpha,\beta}^{(k)}(y)=\frac{d^k}{dy^k}E_{\alpha,\beta}(y)=\sum_{j=0}^{\infty}\frac{(j+k)!y^j}{j!\Gamma(\alpha j+\alpha k+\beta)}, k=0,1,2,....$

Next, we will present the concepts of the stabilities of Ulam-Hyers, Ulam-Hyers-Rassias and semi-Ulam-Hyers-Rassias for Eq (1.1). The following Definition 4, Definition 5 and Definition 6 are adapted from paper [22].

Definition 4. *If $x(t)$ is a given differential function, satisfying*

$$|x'(t)+{}^{c}D_{0+}^{\alpha}x(t)-g(t,x(t))-\int_0^t K(t,s,x(s))\,ds|\le\theta, t\in[0,1], \theta>0,$$

there is a solution $u(t)$ of the Eq (1.1) and a constant $C>0$ such that

$$|x(t)-u(t)|\le C\theta, t\in[0,1],$$

where C is independent of $x(t)$ and $u(t)$, then we say that the Eq (1.1) has the Ulam-Hyers stability.

Definition 5. *If $x(t)$ is a given differential function, satisfying*

$$|x'(t)+{}^{c}D_{0+}^{\alpha}x(t)-g(t,x(t))-\int_0^t K(t,s,x(s))\,ds|\le\theta, t\in[0,1], \theta>0,$$

there exists a solution $u(t)$ of the Eq (1.1) and a constant $C>0$ independent of $x(t)$ and $u(t)$ such that

$$|x(t)-u(t)|\le C\phi(t), t\in[0,1],$$

where $\phi:[0,1]\to(0,+\infty)$ is a nonnegative continuous function, then we say that the Eq (1.1) has the semi-Ulam-Hyers-Rassias stability.

Definition 6. *If $x(t)$ is a given differential function, satisfying*

$$|x'(t)+{}^{c}D_{0+}^{\alpha}x(t)-g(t,x(t))-\int_0^t K(t,s,x(s))\,ds|\le\varphi(t),$$

where $\varphi:[0,1]\to(0,+\infty)$ is a continuous function, there exists a solution $u(t)$ of the Eq (1.1) and a constant $C>0$ independent of $x(t)$ and $u(t)$ such that

$$|x(t)-u(t)|\le C\varphi(t), t\in[0,1],$$

then we say that the Eq (1.1) has the Ulam-Hyers-Rassias stability.

Theorem 1. *(Banach) Let (X,d) be a generalized complete metric space. Assume that $\Lambda:X\to X$ is a strictly contractive operator with the Lipschitz constant $K<1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x,\Lambda^k x)<\infty$, for some $x\in X$, then the following three propositions hold true:*

(1) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;

(2) x^* is the unique fixed point of Λ in $X^*=\{y\in X: d(\Lambda^k x,y)<\infty\}$;

(3) If $y\in X^*$, then $d(y,x^*)\le\frac{1}{1-K}d(\Lambda y,y)$.

3. Main results

In this section, we will investigate the stabilities of Ulam-Hyers-Rassias, semi-Ulam-Hyers-Rassias and Ulam-Hyers for Eq (1.1) in $C^1[0,1]$ endowed with a weighted metric by means of the Banach's fixed point theorem.

3.1. Ulam-Hyers-Rassias stability

In this subsection, we will obtain the equivalent fractional integral equation of Eq (1.1). Moreover, we will discuss the Ulam-Hyers-Rassias stability for Eq (1.1) in the weighted space $(C^1[0,1],\rho)$.

Lemma 1. *Assume that $u(t)\in C^1[0,1]$, $0<\alpha<1$ and $f:[0,1]\to\mathbb{R}$ is a continuous function, the unique solution of the following nonlinear fractional differential equation*

$$u'(t)+{}^c D^{\alpha}_{0+}u(t)=f(t) \tag{3.1}$$

is given by

$$u(t)=u(0)+\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})f(s)ds.$$

proof: The Laplace transforms of both $u'(t)$ and ${}^cD^{\alpha}_{0+}u(t)$ exist for $u(t)\in C^1[0,1]$ (refer to [16]), So we take the Laplace transform on both sides of Eq (3.1). By Eq (2.1), we get

$$s\widetilde{u}(s)-u(0)+s^{\alpha}\widetilde{u}(s)-s^{\alpha-1}u(0)=\widetilde{f}(s).$$

$$\widetilde{u}(s)=\frac{1}{s}u(0)+\frac{1}{s^{\alpha}+s}\widetilde{f}(s). \tag{3.2}$$

By Eq (2.2) and applying the inverse Laplace transform on the two sides of Eq (3.2), we obtain

$$u(t)=u(0)+\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})f(s)ds. \tag{3.3}$$

We conclude that $u(t)$ satisfies Eq (3.1) if and only if $u(t)$ satisfies Eq (3.3). Consequently, the equivalent fractional integral equation of Eq (3.1) is Eq (3.3).

Theorem 2. *Let $\sigma:[0,1]\to(0,\infty)$ is a non-decreasing continuous function, and suppose that there exists a constant $\xi\in[0,1)$, such that*

$$\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\sigma(s)ds\le\xi\sigma(t). \tag{3.4}$$

Let $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the Lipschitz condition

$$|g(t,h_1)-g(t,h_2)|\le M_1|h_1-h_2|, t\in[0,1], h_1,h_2\in\mathbb{R}, \tag{3.5}$$

with $M_1>0$. Moreover, suppose that the kernel $K:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$|K(t,s,h_1)-K(t,s,h_2)|\le M_2|h_1-h_2|, t,s\in[0,1], h_1,h_2\in\mathbb{R}, \tag{3.6}$$

with $M_2 > 0$. *If* $x \in C^1[0,1]$ *satisfies*

$$|x'(t) + ^c D^{\alpha}_{0+}x(t) - g(t, x(t)) - \int_0^t K(t, s, x(s))ds| \leq \sigma(t), t \in [0, 1]. \tag{3.7}$$

and $(M_1 + M_2)\xi < 1$, *then there exists a solution* $u(t)$ *of Eq (1.1) such that*

$$|x(t) - u(t)| \leq \frac{\xi\sigma(t)}{1 - (M_1 + M_2)\xi}, t \in [0, 1]. \tag{3.8}$$

proof: Applying Lemma 1, the equivalent fractional integral equation of Eq (1.1) is given by

$$u(t) = u(0) + \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\left[g(s, u(s)) + \int_0^s K(s, \tau, u(\tau))d\tau\right]ds. \tag{3.9}$$

The proof of Eq (3.9) follows from the proof of Eq (3.3). We set that $u(0) = \eta$.

Define the operator $\Omega : C^1[0,1] \to C^1[0,1]$ by

$$\begin{aligned}(\Omega v)(t) &= \eta + \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})g(s, v(s))ds \\ &+ \int_0^t \int_0^s E_{1-\alpha,1}(-(t-s)^{1-\alpha})K(s, \tau, v(\tau))d\tau ds, t \in [0, 1], v \in C^1[0, 1].\end{aligned}$$

Because of the continuous hypotheses of g, K and Mittag-Leffler function, then the operator Ω is continuous.

Firstly, we will prove that the operator Ω is strictly contractive in $(C^1[0, 1], \rho)$. By Eqs (3.4)–(3.6) and using the definition of the weighted metric ρ, for any $v, w \in C^1[0, 1]$, we obtain

$$\begin{aligned}\rho(\Omega v, \Omega w) &\leq \sup_{t\in[0,1]} \frac{|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})[g(s, v(s)) - g(s, w(s))]ds|}{\sigma(t)} \\ &+ \sup_{t\in[0,1]} \frac{|\int_0^t \int_0^s E_{1-\alpha,1}(-(t-s)^{1-\alpha})[K(s, \tau, v(\tau)) - K(s, \tau, w(\tau))]d\tau ds|}{\sigma(t)} \\ &\leq M_1 \sup_{t\in[0,1]} \frac{|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})|v(s) - w(s)|ds|}{\sigma(t)} \\ &+ M_2 \sup_{t\in[0,1]} \frac{|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) \int_0^s |v(\tau) - w(\tau)|d\tau ds|}{\sigma(t)} \\ &\leq M_1\xi\rho(v, w) + M_2\xi\rho(v, w) = (M_1 + M_2)\xi\rho(v, w).\end{aligned}$$

From the hypothesis $(M_1 + M_2)\xi < 1$, so the operator Ω is strictly contractive.

Next, suppose that $x(t) \in C^1[0, 1]$ satisfies Eq (3.7). Applying Eq (3.7), Eq (3.4) and Lemma 1, we get

$$\begin{aligned}&\left|x(t) - \eta - \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})[g(s, x(s)) + \int_0^s K(s, \tau, x(\tau))d\tau]ds\right| \\ &\leq \left|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\sigma(s)ds\right| \leq \xi\sigma(t),\end{aligned} \tag{3.10}$$

the proof of Eq (3.10) follows the same steps as in the proof of Lemma 1.

From the definition of the operator Ω and Eq (3.10), we have that

$$|(\Omega x)(t) - x(t)| \leq \xi\sigma(t).$$

Therefore, by the definition of the weighted metric ρ, we obtain

$$\rho(\Omega x, x) \leq \xi < 1 < \infty. \tag{3.11}$$

Let $C^*[0,1] = \{y \in C^1[0,1] : \rho(\Omega x, y) < \infty\}$. Applying theorem 1, there is a unique element $u \in C^*[0,1]$ such that $\Omega u = u$, that means u is a solution of Eq (3.9).

Since Eq (3.9) is the equivalent integral equation of Eq (1.1), we conclude that u is the solution of Eq (1.1). By means of item 3 of theorem 1 and Eq (3.11), then

$$\rho(x, u) \leq \frac{1}{1 - (M_1 + M_2)\xi}\rho(\Omega x, x) \leq \frac{\xi}{1 - (M_1 + M_2)\xi}.$$

From the definition of the weighted metric ρ, so Eq (3.9) holds. This completes the proof.

Theorem 2 shows that under the conditions of theorem 2, the Eq (1.1) has the Ulam-Hyers- Rassias stability.

3.2. Semi-Ulam-Hyers-Rassias and Ulam-Hyers stabilities

In this subsection, by means of the weighted metric and Banach fixed-point theorem, we will investigate the stabilities of semi-Ulam-Hyers-Rassias and Ulam-Hyers in $C^1[0,1]$ for Eq (1.1).

Theorem 3. *Let $\sigma : [0,1] \to (0,\infty)$ is a non-decreasing continuous function, and suppose that there exists a constant $\xi \in [0,1)$, such that*

$$\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\sigma(s)ds \leq \xi\sigma(t).$$

Suppose M_1, M_2 are positive constants for which $(M_1 + M_2)\xi < 1$. Let $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying

$$|g(t, h_1) - g(t, h_2)| \leq M_1|h_1 - h_2|, t \in [0,1], h_1, h_2 \in \mathbb{R}.$$
$$|K(t, s, h_1) - K(t, s, h_2)| \leq M_2|h_1 - h_2|, t, s \in [0,1], h_1, h_2 \in \mathbb{R}.$$

If $x \in C^1[0,1]$ satisfies

$$|x'(t) +^c D_{0+}^{\alpha}x(t) - g(t, x(t)) - \int_0^t K(t, s, x(s))ds| \leq \theta, t \in [0,1], \tag{3.12}$$

with $\theta > 0$, then there exists a solution $u(t)$ of Eq (1.1) such that

$$|x(t) - u(t)| \leq \frac{\theta\sigma(t)}{[1 - (M_1 + M_2)\xi]\sigma(0)}\left|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})ds\right|, t \in [0,1]. \tag{3.13}$$

This means that under above conditions, the fractional differential equation Eq (1.1) has the semi-Ulam-Hyers-Rassias stability.

Proof. The first part of the proof follows the same steps as in the proof of Theorem 2. Consider the operator $\Omega : C^1[0,1] \to C^1[0,1]$, defined by

$$(\Omega v)(t) = \eta + \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) \left[g(s, v(s)) + \int_0^s K(s, \tau, v(\tau)) d\tau \right] ds, \tag{3.14}$$

where $t \in [0,1], v \in C^1[0,1]$.

For any $v, w \in C^1[0,1]$, we have

$$\rho(\Omega v, \Omega w) \leq (M_1 + M_2)\xi\rho(v, w).$$

From $(M_1 + M_2)\xi < 1$, then the operator Ω is strictly contractive in $(C^1[0,1], \rho)$.

Next, suppose that $x(t) \in C^1[0,1]$ satisfies Eq (3.12). By Eq (3.12) and Lemma 1, we have

$$\left| x(t) - \eta - \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})[g(s, x(s)) + \int_0^s K(s, \tau, x(\tau)) d\tau]\, ds \right|$$
$$\leq \theta \left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\, ds \right|, t \in [0,1]. \tag{3.15}$$

By the continuity of two parameter Mittag-Leffler function, then $\left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) ds \right|$ is a nonnegative continuous function. From the Eqs (3.14) and (3.15), we get

$$|(\Omega x)(t) - x(t)| \leq \theta \left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) ds \right|.$$

By the definitions of the weighted metric and the continuous function σ, we obtain

$$\rho(\Omega x, x) = \sup_{t\in[0,1]} \frac{|(\Omega x)(t) - x(t)|}{\sigma(t)} \leq \sup_{t\in[0,1]} \frac{\theta \left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) ds \right|}{\sigma(0)} < \infty. \tag{3.16}$$

Let $C^*[0,1] = \{y \in C^1[0,1] : \rho(\Omega x, y) < \infty\}$.

Applying the item 2 of Theorem 1, then there exists a unique element $u \in C^*[0,1]$ such that $\Omega u = u$. That means $u(t)$ is a solution of Eq (1.1).

Using the item 3 of Theorem 1 and Eq (3.16), we have

$$\rho(x, u) \leq \frac{1}{1 - (M_1 + M_2)\xi} \rho(\Omega x, x) \leq \frac{\theta \left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) ds \right|}{[1 - (M_1 + M_2)\xi]\sigma(0)}.$$

Consequently, by the definition of the weighted metric ρ, we get

$$|x(t) - u(t)| \leq \frac{\theta}{[1 - (M_1 + M_2)\xi]\sigma(0)} \sigma(t) \left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha}) ds \right|,$$

where $\sigma(t) \left| \int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\, ds \right|$ is a nonnegative continuous function. This completes the proof.

Remark 1. *For any* $t \in [0,1]$, $\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})ds$ *is convergent series of real number. Therefore, there exists a constant* $N > 0$ *such that*

$$|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})ds| < N. \tag{3.17}$$

Theorem 4. *Assume that* M_1, M_2, ξ *are constants for which* $M_1 > 0, M_2 > 0, 0 \le \xi < 1, (M_1+M_2)\xi < 1$. *Let* $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ *and* $K : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ *are continuous functions satisfying*

$$|g(t,h_1) - g(t,h_2)| \le M_1|h_1 - h_2|, t \in [0,1], h_1, h_2 \in \mathbb{R}.$$
$$|K(t,s,h_1) - K(t,s,h_2)| \le M_2|h_1 - h_2|, t, s \in [0,1], h_1, h_2 \in \mathbb{R}.$$

Let $\sigma : [0,1] \to (0,\infty)$ *be a nondecreasing continuous function, and* σ *satisfies*

$$\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})\sigma(s)ds \le \xi\sigma(t).$$

If $x \in C^1[0,1]$ *satisfies*

$$|x'(t) +^c D_{0+}^{\alpha} x(t) - g(t,x(t)) - \int_0^t K(t,s,x(s))ds| \le \theta, t \in [0,1],$$

with $\theta > 0$, *then there exists a solution* $u(t)$ *of Eq (1.1) such that*

$$|x(t) - u(t)| \le \frac{N\sigma(1)}{[1-(M_1+M_2)\xi]\sigma(0)}\theta, t \in [0,1].$$

Proof. Since σ is a nondecreasing continuous function,

$$\sigma(t) \le \sigma(1), t \in [0,1].$$

By theorem 3, Eqs (3.13) and (3.17), we obtain

$$\begin{aligned} |x(t) - u(t)| &\le \frac{\theta}{[1-(M_1+M_2)\xi]\sigma(0)}\sigma(t)\left|\int_0^t E_{1-\alpha,1}(-(t-s)^{1-\alpha})ds\right| \\ &\le \frac{N\sigma(1)}{[1-(M_1+M_2)\xi]\sigma(0)}\theta. \end{aligned}$$

Theorem 4 shows that under the conditions of theorem 4, the Eq (1.1) has the Ulam-Hyers stability.

4. Example

Example 1. *We consider the following fractional integro-differential equation*

$$u'(t) +^c D_{0+}^{\frac{1}{2}} u(t) = \frac{1}{100}[t\cos u(t) + u(t)\sin t] + \frac{1}{50}\int_0^t \sin u(s)\, ds. \tag{4.1}$$

By comparing the Eq (1.1), we get

$$\alpha = \frac{1}{2}, g(t, u(t)) = \frac{1}{100}[t\cos u(t) + u(t)\sin t], K(t, s, u(s)) = \frac{1}{50}\sin u(s).$$

We conclude that

$$\begin{aligned} |g(t, h_1) - g(t, h_2)| &\leq \frac{1}{50}|h_1 - h_2|, h_1, h_2 \in \mathbb{R}, t \in [0, 1], \\ |K(t, s, h_1) - K(t, s, h_2)| &\leq \frac{1}{50}|h_1 - h_2|, h_1, h_2 \in \mathbb{R}, t, s \in [0, 1], \end{aligned}$$

Let $\sigma(t) = e^t$, we obtain

$$\int_0^t E_{\frac{1}{2},1}(-(t-s)^{\frac{1}{2}})e^s ds < e^t - 1 < \frac{3}{4}e^t, t \in [0, 1].$$

Here, we have $M_1 = M_2 = \frac{1}{50}, \xi = \frac{3}{4}$, and $(M_1 + M_2)\xi = 0.03 < 1$. Therefore, Theorem 2–Theorem 4 guarantee that Eq (4.1) have Ulam-Hyers-Rassias stability, semi-Ulam-Hyers-Rassias stability and Ulam-Hyers stability.

The stabilities of Ulam-Hyers, Ulam-Hyers-Rassias and semi-Ulam-Hyers-Rassias for Eq (4.1) is independent of the initial value condition. Using MATLAB, the solution $u(t)$ of Eq (4.1) with initial value condition $u(0) = 0$ is computed and depicted in Figure 1.

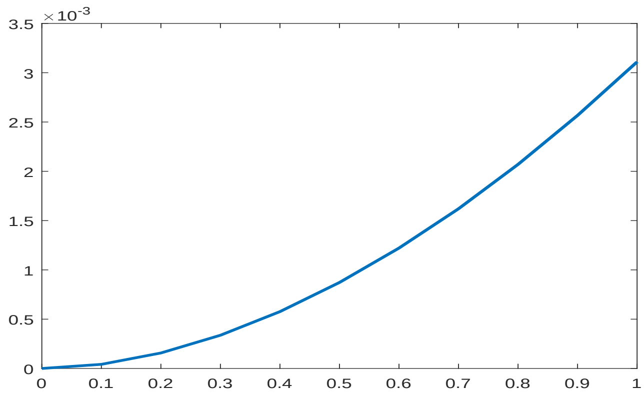

Figure 1. The solution $u(t)$ of Eq (4.1) with initial value condition $u(0) = 0$.

Now, suppose that $x \in C^1[0, 1]$ is the solution of the following fractional integro-differential equation

$$\begin{aligned} x'(t) + {}^c D_{0+}^{\frac{1}{2}} x(t) &= \frac{1}{100}[t\cos x(t) + x(t)\sin t] + \frac{1}{50}\int_0^t \sin x(s)ds + e^t, t \in [0, 1], \\ x(0) &= 0, \end{aligned}$$

Hence x satisfies Eq (3.7), and we have

$$|x(t) - u(t)| \leq \frac{\xi}{1 - (M_1 + M_2)\xi}e^t = \frac{75}{97}e^t, t \in [0, 1]. \qquad \text{(see Figure 2)}$$

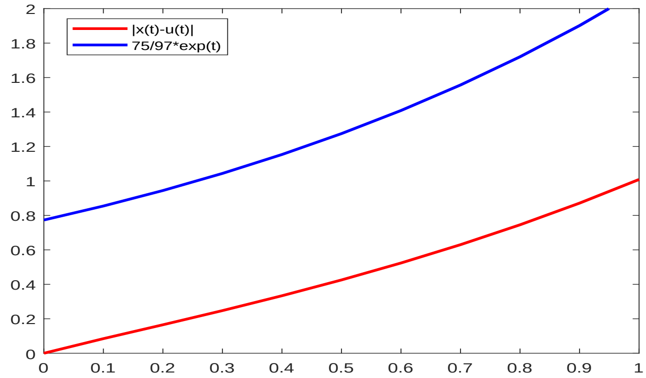

Figure 2. Functions $|x(t) - u(t)|$ and $\dfrac{75}{97}e^t$.

Next, Suppose that $y \in C^1[0, 1]$ is the solution of the following fractional integro-differential equation

$$\begin{aligned} y'(t) +^c D^{\frac{1}{2}}_{0+}y(t) &= \frac{1}{100}[t\cos y(t) + y(t)\sin t] + \frac{1}{50}\int_0^t \sin y(s)ds + t, t \in [0, 1], \\ y(0) &= 0, \end{aligned}$$

Hence x satisfies Eq (3.12) with $\theta = 1$, and we have

$$|y(t) - u(t)| \le \frac{e^t \left|\int_0^t E_{\frac{1}{2},1}(-(t-s)^{\frac{1}{2}})ds\right|}{0.97} = q(t), t \in [0, 1]. \qquad \text{(see Figure 3)}$$

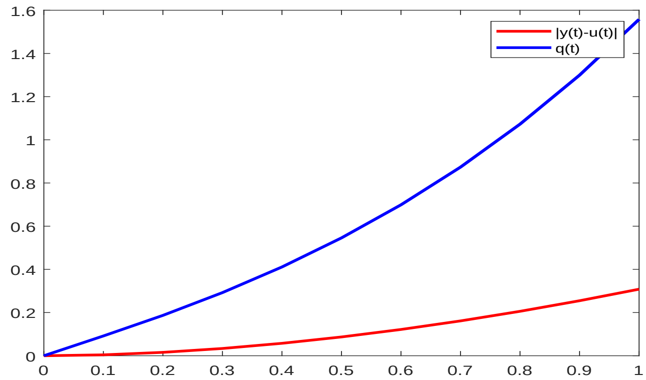

Figure 3. functions $|y(t) - u(t)|$ and $q(t)$.

5. Conclusions

The stability theory of fractional differential equations is an important branch of fractional calculus. If a differential equation has the Ulam-type stability, it means that we can find the exact solution near an approximate solution of the equation. Various forms of Ulam-type stability for the nonlinear fractional differential equations have been studied in the past few decades. Here we use weighted space method and Banach fixed-point theorem to analyze different forms of Ulam-type stability for the mixed Caputo fractional integro-differential equations, including Ulam-Hyers stability, Ulam-Hyers-Rassias stability and semi-Ulam-Hyers-Rassias stability. Besides that, an example is given to illustrate the results.

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Conflict of interest

The authors declare that they have no competing interests.

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