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Research article

$\theta\beta$-ideal approximation spaces and their applications

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Abstract: The essential aim of the current work is to enhance the application aspects of Pawlak rough sets. Using the notion of a j-neighborhood space and the related concept of $\theta\beta$-open sets, different methods for generalizing Pawlak rough sets are proposed and their characteristics will be examined. Moreover, in the context of ideal notion, novel generalizations of Pawlak's models and some of their generalizations are presented. Comparisons between the suggested methods and the previous approximations are calculated. Finally, an application from real-life problems is proposed to explain the importance of our decision-making methods.

Keywords: rough sets; topology; j-neighborhood spaces; j-near open sets; $\boldsymbol{\theta\beta_j}$-open sets and $\boldsymbol{\theta\beta_j}$-ideal approximations
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1. Introduction

Rough set philosophy [1] deals with ambiguity. In fact, this methodology was based on an equivalence relation which limited the application fields. Accordingly, to extend the application scope of these approaches, the equivalence relation is generalized to a rough set model constructed by any binary relation. Numerous proposals established in this line [2–31]. Nowadays, there are many applications of topology and its extensions, for instance [5,11,13,17,32,33].

Abd El-Monsef et al. [19] presented a structure to generalize the standard rough set concept. In fact, they presented the notion of a j-neighborhood space (briefly, j-*NS*) constructed by a binary relation. Moreover, they used a general topology generated from the binary relation to set up generalized rough sets. This methodology paved the way for extra topological presentations in the rough set contexts and helped to formalize various applications from daily-life problems. After then, Amer et al. [20] applied some near open sets in a j-$\boldsymbol{NS}$, and thus they managed to in generating new generalized rough set approximations, namely j-near approximations. In 2018, Hosny [21] extended these approximations to different approximations using the concepts of $\delta\beta$-open and $\wedge_{\beta}$-open sets. Using the interesting notion "Ideal", Hosny [22] proposed another idea for generalizing Pawlak's theory. She has introduced the ideas of I_j-lower and I_j-upper approximations as extensions for Pawlak's approximations and several of their generalizations.

Ideal represents an important notion in topological spaces and has a vital part in the study of topological problems. This concept is very interesting in rough set theory since it is considered as a bridge between rough sets and topological structures. Moreover, the ideal can be considered as a class of some objects in the information system that has some conditions and the expert wants to study and make a new granulation for the data collected from real-life problem according to this class. Few authors have used the term "ideal" to describe and produce non-granular rough approximations over general approximation spaces in recent years. These relation-based rough set studies have aimed to obtain fine properties analogous to those of classical rough approximations. The authors severely generalize these ideas in this research paper, and they investigate associated semantic features. Granules are used implicitly in the building of approximations, and thus the idea of $\boldsymbol{\theta\beta_j}$-ideal approximations are introduced. In other words, we suggested two different methods depending on topological concepts, and hence the significance of these approaches was that they were based on ideals, which were topological tools, and the two proposed methods represent two opinions rather than one. Therefore, these techniques open the way for more topological applications in the rough set context.

The fundamental contributions of the present work are to introduce different extensions of Pawlak's rough context and its generalizations. In fact, we suggest two different types to extend and strengthen the already methods in the literature (namely, Abd El-Monsef et al. [19], Amer et al. [20], and Hosny [21,22] techniques). Moreover, we illustrate that the proposed methods have extended the application fields in real-life problems and help in extracting the data in the information system.

This paper is organized into five different sections besides the introduction and the conclusion. Section 2 is devoted to state the main concepts, results, and methods that are used throughout the paper. Additionally, we give a summary of previous methods which are compared with our approaches. By using the new notion of $\boldsymbol{\theta\beta_j}$-open sets, we proposed a new technique to generalize Pawlak's rough models and some of its generalizations in section 3. The central properties of this method are examined and compared to Abd El-Monsef et al. [19], Amer et al. [20], and Hosny [21,22] methods. Additionally, many results with counterexamples are investigated to reason that the suggested method is stronger than the other methods. In setion 4, we present and examine the models of $\boldsymbol{\theta\beta_j}$-ideal lower and $\boldsymbol{\theta\beta_j}$-ideal upper approximations for any subset. Some properties of these operators will be examined and demonstrated that $\boldsymbol{\theta\beta_j}$-ideal approximations are more accurate than the other methods. Finally, in section 5, a practical example is presented to clarify the importance of the proposed techniques in decision-making. Besides, the proposed methods are compared with the previous methodologies. From real-life problems, we use an information system (with decision attributes). This system depends on a binary relation which means that Pawlak's rough sets can't be applied here. So, we apply our methods and will show that the recommended methods are stronger than the other methods. Therefore, we can say that our methods extend the application fields for rough sets from a topological point of view.

2. Preliminaries

In the next, we give the basic ideas and consequences used in current research.

Definition 2.1. [34] A non-empty class I of subsets of a set X is called an ideal on X, if it satisfies the subsequent conditions

1) If $A \in I$ and $B \subseteq A$, then $B \in I$ (hereditary),
2) If $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity).

Definition 2.2. [19] Suppose that R is an arbitrary binary relation on a non-empty finite set X. Therefore, the j-neighborhoods of $x \in X$ (Symbolically, $N_j(x), \forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, i, u, \langle i\rangle, \langle u\rangle\}$), are given by:

(i) Right neighborhood (briefly, r-neighborhood):

$$N_r(x) = \{y \in X: xRy\}$$

(ii) Left neighborhood (briefly, ℓ-neighborhood):

$$N_\ell(x) = \{y \in X: yRx\}$$

(iii) Minimal right neighborhood (briefly, $\langle r\rangle$-neighborhood):

$$\mathrm{N}_{\langle r\rangle}(\mathrm{x}) = \cap_{\mathrm{x}\in \mathrm{N}_{\mathrm{r}}(\mathrm{y})} \mathrm{N}_{\mathrm{r}}(\mathrm{y})$$

(iv) Minimal left neighborhood (briefly, $\langle \ell\rangle$-neighborhood):

$$N_{\langle \ell\rangle}(x) = \cap_{x\in N_\ell(y)} N_\ell(y).$$

(v) Intersection of right and left neighborhoods (briefly, i-neighborhood):

$$N_i(x) = N_r(x) \cap N_\ell(x).$$

(vi) Union of right and left neighborhoods (briefly, u-neighborhood):

$$N_u(x) = N_r(x) \cup N_\ell(x).$$

(vii) Intersection of minimal right and minimal left neighborhoods (briefly, $\langle i\rangle$-neighborhood):

$$N_{\langle i\rangle}(x) = N_{\langle r\rangle}(x) \cap N_{\langle \ell\rangle}(x).$$

(viii) Union of minimal right and minimal left neighborhoods (briefly, $\langle u\rangle$-neighborhood):

$$N_{\langle u\rangle}(x) = N_{\langle r\rangle}(x) \cup N_{\langle \ell\rangle}(x).$$

Definition 2.3. [19] Suppose that R is an arbitrary binary relation on a non-empty finite set X and $\psi_j: X \longrightarrow P(X)$ is a mapping which assigns an j-neighborhood for each $x \in X$ in $P(X)$. The triple (X, R, ψ_j) is named a j-neighborhood space (in briefly, j-***NS***).

Theorem 2.1. [19] If (X, R, ψ_j) is a j-***NS***, then $\forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ the family $\tau_j = \{A \subseteq X: \forall p \in A,\ N_j(p) \subseteq A\}$ forms a topology on X.

Definition 2.4. [19] Let (X, R, ψ_j) be a j-***NS***. A subset $A \subseteq X$ is said to be a j-open set if $A \in \tau_j$, the complement of a j-open set is called a j-closed set. A class Γ_j of all j-closed sets is given by $\Gamma_j = \{F \subseteq X: F^c \in \tau_j\}$, such that F^c represents a complement of F.

Definition 2.5. [19] Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. Then, $\forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, we define the j-lower and the j-upper approximations, the j-boundary regions and the j-accuracy of approximations of A, respectively, by:

$\underline{R}_j(A) = \bigcup\{G \in \tau_j : G \subseteq A\} = int_j(A)$, where $int_j(A)$ represents j-interior of A.

$\overline{R}_j(A) = \bigcap\{F \in \Gamma_j : F \supseteq A\} = cl_j(A)$, where $cl_j(A)$ represents j-closure of A.

$B_j(A) = \overline{R}_j(A) - \underline{R}_j(A)$.

$\sigma_j(A) = \frac{|\underline{R}_j(A)|}{|\overline{R}_j(A)|}$, where $|\overline{R}_j(A)| \neq 0$.

Definition 2.6. [19] Suppose that (X, R, ψ_j) is a j-***NS***, and $A \subseteq X$, then for each $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, the subset A is named a j-exact set if $\underline{R}_j(A) = \overline{R}_j(A) = A$. Else, A is called a j-rough set.

Definition 2.7. [20] Suppose that (X, R, ψ_j) is a j-***NS***. A subset $A \subseteq X$ is named

(1) j-regular open (R_j^*-open), if $A = int_j(cl_j(A))$;
(2) j-preopen (P_j-open), if $A \subseteq int_j(cl_j(A))$;
(3) j-semi open (S_j-open), if $A \subseteq cl_j(int_j(A))$;
(4) γ_j-open (b_j-open), if $A \subseteq int_j(cl_j(A)) \cup cl_j(int_j(A))$;
(5) α_j-open, if $A \subseteq int_j[cl_j(int_j(A))]$;
(6) β_j-open (semi preopen), if $A \subseteq cl_j[int_j(cl_j(A))]$;
(7) $\delta\beta_j$-open, if $A \subseteq cl_j[int_j(cl_j^{\delta}(A))]$.

Remark 2.1. [20]
(i) The previous types of sets are called j-near open sets and the families of j-near open sets of X symbolized by $K_jO(X)$, $\forall K \in \{R^*, P, S, \gamma, \alpha, \beta, \delta\beta\}$.
(ii) The complements of the j-near open sets are called j-near closed sets and the families of j-near closed sets of X symbolized by $K_jC(X)$, $\forall K \in \{R^*, P, S, \gamma, \alpha, \beta, \delta\beta\}$.

Definition 2.8. [21] Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. A subset $\wedge_{\beta_j}(A)$ is assumed as follows: $\wedge_{\beta_j}(A) = \bigcap\{G : A \subseteq G, G \in \beta_jO(X)\}$. The complement of $\wedge_{\beta_j}(A)$-set is called $\vee_{\beta_j}(A)$-set.

Definition 2.9. [21] Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. A subset A is said to be a $\wedge_{\beta_j}$-set if $A = \wedge_{\beta_j}(A)$. The family of all $\wedge_{\beta_j}$-sets and $\vee_{\beta_j}$-sets are symbolized by $\wedge_{\beta_j}O(X)$ and $\vee_{\beta_j}C(X)$, respectively.

Definition 2.10. [20,21] Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. Then, for each $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ and $K \in \{R^*, P, S, \gamma, \alpha, \beta, \delta\beta, \wedge_\beta\}$, the j-near lower, j-near upper approximations, j-near boundary regions and j-near accuracy of the approximations of A are assumed respectively by:

$$\underline{R}_j^K(A) = \bigcup\{G \in K_jO(X) : G \subseteq A\}.$$

$$\overline{R}_j^K(A) = \bigcap\{F \in K_jC(X) : F \supseteq A\}.$$

$$B_j^K(A) = \overline{R}_j^K(A) - \underline{R}_j^K(A).$$

$$\sigma_j^K(A) = \frac{|\underline{R}_j^K(A)|}{|\overline{R}_j^K(A)|}, \text{ where } |\overline{R}_j^K(A)| \neq 0.$$

Definition 2.11. [20,21] Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. Then, for each $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ and $K \in \{R^*, P, S, \gamma, \alpha, \beta, \delta\beta, \wedge_\beta\}$, A is called a j-near definable (j-near exact) set if $\underline{R}_j^K(A) = \overline{R}_j^K(A)$. Else, A is called a j-near rough set.

Proposition 2.1. [20,21] If (X, R, ψ_j) is a j-***NS***, and $A \subseteq X$, then for each $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ and $K \in \{P, S, \gamma, \alpha, \beta, \delta\beta, \wedge_\beta\}$, $K \neq R^*$:

$$\underline{R}_j(A) \subseteq \underline{R}_j^K(A) \subseteq A \subseteq \overline{R}_j^K(A) \subseteq \overline{R}_j(A).$$

Proposition 2.2. [20,21] Consider (X, R, ψ_j) is a j-***NS***, and $A \subseteq X$. Then, the following properties are satisfied:

1) $\underline{R}_j^\alpha(A) \subseteq \underline{R}_j^P(A) \subseteq \underline{R}_j^\gamma(A) \subseteq \underline{R}_j^\beta(A) \subseteq \underline{R}_j^{\delta\beta}(A)$.

2) $\underline{R}_j^\alpha(A) \subseteq \underline{R}_j^P(A) \subseteq \underline{R}_j^\gamma(A) \subseteq \underline{R}_j^\beta(A) \subseteq \underline{R}_j^{\wedge_\beta}(A)$.

3) $\underline{R}_j^\alpha(A) \subseteq \underline{R}_j^S(A) \subseteq \underline{R}_j^\gamma(A) \subseteq \underline{R}_j^\beta(A) \subseteq \underline{R}_j^{\delta\beta}(A)$.

4) $\underline{R}_j^\alpha(A) \subseteq \underline{R}_j^S(A) \subseteq \underline{R}_j^\gamma(A) \subseteq \underline{R}_j^\beta(A) \subseteq \underline{R}_j^{\wedge_\beta}(A)$.

5) $\overline{R}_j^{\delta\beta}(A) \subseteq \overline{R}_j^\beta(A) \subseteq \overline{R}_j^\gamma(A) \subseteq \overline{R}_j^P(A) \subseteq \overline{R}_j^\alpha(A)$.

6) $\overline{R}_j^{\wedge_\beta}(A) \subseteq \overline{R}_j^\beta(A) \subseteq \overline{R}_j^\gamma(A) \subseteq \overline{R}_j^P(A) \subseteq \overline{R}_j^\alpha(A)$.

7) $\overline{R}_j^{\delta\beta}(A) \subseteq \overline{R}_j^\beta(A) \subseteq \overline{R}_j^\gamma(A) \subseteq \overline{R}_j^S(A) \subseteq \overline{R}_j^\alpha(A)$.

8) $\overline{R}_j^{\wedge_\beta}(A) \subseteq \overline{R}_j^\beta(A) \subseteq \overline{R}_j^\gamma(A) \subseteq \overline{R}_j^S(A) \subseteq \overline{R}_j^\alpha(A)$.

Proposition 2.3. [20,21] Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. Then, the following statements are verified:

1) $B_j^{\delta\beta}(A) \subseteq B_j^\beta(A) \subseteq B_j^\gamma(A) \subseteq B_j^P(A) \subseteq B_j^\alpha(A)$.

2) $B_j^{\wedge_\beta}(A) \subseteq B_j^\beta(A) \subseteq B_j^\gamma(A) \subseteq B_j^P(A) \subseteq B_j^\alpha(A)$.

3) $B_j^{\delta\beta}(A) \subseteq B_j^\beta(A) \subseteq B_j^\gamma(A) \subseteq B_j^S(A) \subseteq B_j^\alpha(A)$.

4) $B_j^{\wedge_\beta}(A) \subseteq B_j^\beta(A) \subseteq B_j^\gamma(A) \subseteq B_j^S(A) \subseteq B_j^\alpha(A)$.

5) $\sigma_j^\alpha(A) \leq \sigma_j^P(A) \leq \sigma_j^\gamma(A) \leq \sigma_j^\beta(A) \leq \sigma_j^{\delta\beta}(A)$.

6) $\sigma_j^\alpha(A) \leq \sigma_j^P(A) \leq \sigma_j^\gamma(A) \leq \sigma_j^\beta(A) \leq \sigma_j^{\wedge_\beta}(A)$.

7) $\sigma_j^\alpha(A) \leq \sigma_j^S(A) \leq \sigma_j^\gamma(A) \leq \sigma_j^\beta(A) \leq \sigma_j^{\delta\beta}(A)$.

8) $\sigma_j^\alpha(A) \leq \sigma_j^S(A) \leq \sigma_j^\gamma(A) \leq \sigma_j^\beta(A) \leq \sigma_j^{\wedge_\beta}(A)$.

Theorem 2.2. [22] Assume that (X, R, ψ_j) is a j-***NS***, $A \subseteq X$ and I is an ideal on X, then for each $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ the collection $\tau_j^I = \{A \subseteq X: \forall\, p \in A,\ N_j(p) \cap A^c \in I\}$ is a topology on X.

Definition 2.12. [22] Let (X, R, ψ_j) be a j-***NS***, and I be an ideal on X. A subset $A \subseteq X$ is named an I_j-open set if $A \in \tau_j^I$, the complement of an I_j-open set is named an I_j-closed set. A family $\mathcal{T}_j^I$ of all I_j-closed sets is given by $\mathcal{T}_j^I = \{F \subseteq X: F^c \in \tau_j^I\}$, such that F^c represents a complement of F.

Definition 2.13. [22] Consider (X, R, ψ_j) is a j-***NS***, I is an ideal on X and $A \subseteq X$. $\forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, the I_j-lower, I_j-upper approximations, I_j-boundary regions and I_j-accuracy of the approximations of A are assumed, respectively, by:

$$\underline{R}_j^I(A) = \cup\{O \in \tau_j^I: O \subseteq A\} = int_j^I(A), \text{ where } int_j^I(A) \text{ represents } I\text{-}j\text{-interior of } A.$$

$$\overline{R}_j^I(A) = \cap\{F \in \mathcal{T}_j^I: F \supseteq A\} = cl_j^I(A), \text{ where } cl_j^I(A) \text{ represents } I\text{-}j\text{-closure of A.}$$

$$B_j^I(A) = \overline{R}_j^I(A) - \underline{R}_j^I(A).$$

$$\sigma_j^I(A) = \frac{\left|\underline{R}_j^I(A)\right|}{\left|\overline{R}_j^I(A)\right|}, \text{ where } \left|\overline{R}_j^I(A)\right| \neq 0.$$

3. $\theta\beta_j$-rough sets by using $\theta\beta_j$-open sets

A new method for defining generalized rough sets using the idea of $\theta\beta_j$-open sets is proposed. The current method's properties are investigated and compared to those of Abd El-Monsef et al. [19], Amer et al. [20], and Hosny [21].

Definition 3.1. Consider (X, R, ψ_j) is a j-***NS***, $A \subseteq X$ and for each $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$. The θ_j-closure of A is defined by $cl_j^\theta(A) = \{x \in X: A \cap cl_j(G) \neq \emptyset, \forall G \in \tau_j \text{ and } x \in G\}$. Moreover, A is called a θ_j-closed if $A = cl_j^\theta(A)$. The complement of a θ_j-closed set is θ_j-open. Note that: $int_j^\theta(A) = X - cl_j^\theta(X - A)$.

Definition 3.2. Consider (X, R, ψ_j) is a j-***NS*** and $A \subseteq X$. A subset A is called a $\theta\beta_j$-open, if $A \subseteq cl_j[int_j(cl_j^\theta(A))]$. A is a $\theta\beta_j$-closed set if its complement is a $\theta\beta_j$-open set and the family of all $\theta\beta_j$-open and $\theta\beta_j$-closed sets are symbolized by $\theta\beta_jO(X)$ and $\theta\beta_jC(X)$, respectively.

Example 3.1. Let $X = \{a, b, c, d, e\}$ and $= \{(a, a), (a, e), (b, c), (b, d), (b, e), (c, c), (c, d), (d, c), (d, d), (e. e)\}$ be a binary relation given on X. Then, we get

$$N_r(a) = \{a, e\},\ N_r(b) = \{c, d, e\},\ N_r(c) = N_r(d) = \{c, d\},\ N_r(e) = \{e\}.$$

Therefore, the topology associated with this relation is $\tau_r = \{X, \emptyset, \{e\}, \{a, e\}, \{c, d\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}$, and hence we obtain $\theta\beta_rO(X) = P(X)$.

The next results demonstrate that $\theta\beta_j$-open sets are more accurate than $\delta\beta_j$-open sets and $\wedge_{\beta_j}$-open sets. Therefore, $\theta\beta_j$-open sets are stronger than other j-near open sets such as R_j^* -open, P_j-open, S_j-open, γ_j-open, α_j-open and β_j-open sets. Moreover, by appling the characteristics of the definitions of j-interior, j-closure, δ_j-closure and θ_j-closure operators, it is easy to prove these results, so we omit the proof.

Proposition 3.1. Each $\delta\beta_j$-open set is $\theta\beta_j$-open.

Proposition 3.2. Each $\wedge_{\beta_j}$-open set is $\theta\beta_j$-open.

Remark 3.1. The reverse of Propositions 3.1 and 3.2 is not essentially correct as showing in Example 3.1.

Definition 3.3. Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. Then, the $\theta\beta_j$-lower, $\theta\beta_j$-upper approximations, $\theta\beta_j$-boundary regions and $\theta\beta_j$-accuracy of the approximations of A are defined, respectively, by:

$$\underline{R}_j^{\theta\beta}(A) = \cup\{G \in \theta\beta_j O(X): G \subseteq A\} = \theta\beta_j\text{- interior of } A.$$

$$\overline{R}_j^{\theta\beta}(A) = \cap\{F \in \theta\beta_j C(X): F \supseteq A\} = \theta\beta_j\text{- closure of } A.$$

$$B_j^{\theta\beta}(A) = \overline{R}_j^{\theta\beta}(A) - \underline{R}_j^{\theta\beta}(A).$$

$$\sigma_j^{\theta\beta}(A) = \frac{\left|\underline{R}_j^{\theta\beta}(A)\right|}{\left|\overline{R}_j^{\theta\beta}(A)\right|}, \text{ where } \left|\overline{R}_j^{\theta\beta}(A)\right| \neq 0.$$

To illustrate the connections among the existing approaches in Definition 3.3 and the preceding one in Definitions 2.5 [19] and 2.10 [20,21], we present the following results.

Theorem 3.1. Let (X, R, ψ_j) be a j-***NS***, and $A \subseteq X$. Then, the next statements are verified:

$$1)\ \underline{R}_j^{\alpha}(A) \subseteq \underline{R}_j^{P}(A) \subseteq \underline{R}_j^{\gamma}(A) \subseteq \underline{R}_j^{\beta}(A) \subseteq \underline{R}_j^{\delta\beta}(A) \subseteq \underline{R}_j^{\theta\beta}(A).$$

$$2)\ \underline{R}_j^{\alpha}(A) \subseteq \underline{R}_j^{P}(A) \subseteq \underline{R}_j^{\gamma}(A) \subseteq \underline{R}_j^{\beta}(A) \subseteq \underline{R}_j^{\wedge_\beta}(A) \subseteq \underline{R}_j^{\theta\beta}(A).$$

$$3)\ \underline{R}_j^{\alpha}(A) \subseteq \underline{R}_j^{S}(A) \subseteq \underline{R}_j^{\gamma}(A) \subseteq \underline{R}_j^{\beta}(A) \subseteq \underline{R}_j^{\delta\beta}(A) \subseteq \underline{R}_j^{\theta\beta}(A).$$

$$4)\ \underline{R}_j^{\alpha}(A) \subseteq \underline{R}_j^{S}(A) \subseteq \underline{R}_j^{\gamma}(A) \subseteq \underline{R}_j^{\beta}(A) \subseteq \underline{R}_j^{\wedge_\beta}(A) \subseteq \underline{R}_j^{\theta\beta}(A).$$

$$5)\ \underline{R}_j(A) \subseteq \underline{R}_j^{\theta\beta}(A).$$

$$6)\ \overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j^{\delta\beta}(A) \subseteq \overline{R}_j^{\beta}(A) \subseteq \overline{R}_j^{\gamma}(A) \subseteq \overline{R}_j^{P}(A) \subseteq \overline{R}_j^{\alpha}(A).$$

$$7)\ \overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j^{\wedge_\beta}(A) \subseteq \overline{R}_j^{\beta}(A) \subseteq \overline{R}_j^{\gamma}(A) \subseteq \overline{R}_j^{P}(A) \subseteq \overline{R}_j^{\alpha}(A).$$

$$8)\ \overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j^{\delta\beta}(A) \subseteq \overline{R}_j^{\beta}(A) \subseteq \overline{R}_j^{\gamma}(A) \subseteq \overline{R}_j^{S}(A) \subseteq \overline{R}_j^{\alpha}(A).$$

$$9)\ \overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j^{\wedge_\beta}(A) \subseteq \overline{R}_j^{\beta}(A) \subseteq \overline{R}_j^{\gamma}(A) \subseteq \overline{R}_j^{S}(A) \subseteq \overline{R}_j^{\alpha}(A).$$

$$10)\ \overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j(A).$$

Proof.

1) $\underline{R}_j^{\alpha}(A) \subseteq \underline{R}_j^{P}(A) \subseteq \underline{R}_j^{\gamma}(A) \subseteq \underline{R}_j^{\beta}(A) \subseteq \underline{R}_j^{\delta\beta}(A)$, by Proposition 2.2 (1) and $\underline{R}_j^{\delta\beta}(A) = \cup\{G \in \delta\beta_j O(X): G \subseteq A\} \subseteq \cup\{G \in \theta\beta_j O(X): G \subseteq A\} = \underline{R}_j^{\theta\beta}(A)$ (Proposition 3.1).

2) $\underline{R}_j^{\alpha}(A) \subseteq \underline{R}_j^{P}(A) \subseteq \underline{R}_j^{\gamma}(A) \subseteq \underline{R}_j^{\beta}(A) \subseteq \underline{R}_j^{\wedge_\beta}(A)$, by Proposition 2.2 (2) and

$\underline{R}_j^{\wedge_\beta}(A) = \cup\left\{G \in \wedge_{\beta_j} O(X): G \subseteq A\right\} \subseteq \cup\{G \in \theta\beta_j O(X): G \subseteq A\} = \underline{R}_j^{\theta\beta}(A)$ (Proposition 3.2).

3) Similar to (1).

4) Similar to (2).

5) By Proposition 2.1, $\underline{R}_j \subseteq \underline{R}_j^{K}(A)$, and $K \in \{P, S, \gamma, \alpha, \beta, \delta\beta, \wedge_\beta\}$, such that $K \neq R^*$, and from (1) $\underline{R}_j^{K}(A) \subseteq \underline{R}_j^{\theta\beta}(A)$. Therefore, $\underline{R}_j \subseteq \underline{R}_j^{\theta\beta}(A)$.

(6)–(9) Similar to (1) and (2).

10) By Proposition 2.1, $\overline{R}_j^K(A) \subseteq \overline{R}_j(A)$, and $K \in \{P, S, \gamma, \alpha, \beta, \delta\beta, \Lambda_\beta\}$, $K \neq R^*$, and by (6) $\overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j^K(A)$. Therefore, $\overline{R}_j^{\theta\beta}(A) \subseteq \overline{R}_j(A)$.

Corollary 3.1. If (X, R, ψ_j) is a j-***NS***, and $A \subseteq X$, then the following statements are satisfied:

1) $B_j^{\theta\beta}(A) \subseteq B_j^{\delta\beta}(A) \subseteq B_j^{\beta}(A) \subseteq B_j^{\gamma}(A) \subseteq B_j^{P}(A) \subseteq B_j^{\alpha}(A)$.

2) $B_j^{\theta\beta}(A) \subseteq B_j^{\Lambda_\beta}(A) \subseteq B_j^{\beta}(A) \subseteq B_j^{\gamma}(A) \subseteq B_j^{P}(A) \subseteq B_j^{\alpha}(A)$.

3) $B_j^{\theta\beta}(A) \subseteq B_j^{\delta\beta}(A) \subseteq B_j^{\beta}(A) \subseteq B_j^{\gamma}(A) \subseteq B_j^{S}(A) \subseteq B_j^{\alpha}(A)$.

4) $B_j^{\theta\beta}(A) \subseteq B_j^{\Lambda_\beta}(A) \subseteq B_j^{\beta}(A) \subseteq B_j^{\gamma}(A) \subseteq B_j^{S}(A) \subseteq B_j^{\alpha}(A)$.

5) $B_j^{\theta\beta}(A) \subseteq B_j(A)$.

6) $\sigma_j^{\alpha}(A) \leq \sigma_j^{P}(A) \leq \sigma_j^{\gamma}(A) \leq \sigma_j^{\beta}(A) \leq \sigma_j^{\delta\beta}(A) \leq \sigma_j^{\theta\beta}(A)$.

7) $\sigma_j^{\alpha}(A) \leq \sigma_j^{P}(A) \leq \sigma_j^{\gamma}(A) \leq \sigma_j^{\beta}(A) \leq \sigma_j^{\Lambda_\beta}(A) \leq \sigma_j^{\theta\beta}(A)$.

8) $\sigma_j^{\alpha}(A) \leq \sigma_j^{S}(A) \leq \sigma_j^{\gamma}(A) \leq \sigma_j^{\beta}(A) \leq \sigma_j^{\delta\beta}(A) \leq \sigma_j^{\theta\beta}(A)$.

9) $\sigma_j^{\alpha}(A) \leq \sigma_j^{S}(A) \leq \sigma_j^{\gamma}(A) \leq \sigma_j^{\beta}(A) \leq \sigma_j^{\Lambda_\beta}(A) \leq \sigma_j^{\theta\beta}(A)$.

10) $\sigma_j(A) \leq \sigma_j^{\theta\beta}(A)$.

Remark 3.2. According to Theorem 3.1, we noted that the proposed technique decreases the boundary region by enhancing the $\theta\beta_j$ -lower approximation and reducing the $\theta\beta_j$ -upper approximation comparing them with the methods of Abd El-Monsef et al. (in Definition 2.5), Amer et al. and Hosny (in Definition 2.10). Furthermore, the accuracy that given by Definition 3.3 is higher than the other accuracies in Definitions 2.5 and 2.10. In Example 3.1 as demonstrated by Corollary 3.1. To this end, we compute the approximations, boundary regions, and the accuracy measure using the proposed method in Definition 3.3 and compare them with the previous techniques "Abd El-Monsef et al., Amer et al., Hosny" as exposed in Table 1.

Table 1. Comparison of the boundary and accuracy methods of Abd El-Monsef et al. [19], Amer et al. [20], and Hosny [21], as well as the existing technique (Definition 3.3).

$A \subseteq X$	Abd El Monsef et al. method		Amer et al. method		Hosny methods				The current method	
	$B_r(A)$	$\sigma_r(A)$	$B_r^{\beta}(A)$	$\sigma_r^{\beta}(A)$	$B_r^{\delta\beta}(A)$	$\sigma_r^{\delta\beta}(A)$	$B_r^{\wedge\beta}(A)$	$\sigma_r^{\wedge\beta}(A)$	$B_r^{\theta\beta}(A)$	$\sigma_r^{\theta\beta}(A)$
$\{a\}$	$\{a\}$	0	$\{a\}$	0	$\emptyset$	1	$\{a\}$	0	$\emptyset$	1
$\{b\}$	$\{b\}$	0	$\{b\}$	0	$\{b\}$	0	$\emptyset$	1	$\emptyset$	1
$\{c\}$	$\{b,c,d\}$	0	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{d\}$	$\{b,c,d\}$	0	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{e\}$	$\{a,b\}$	1/3	$\{a\}$	1/2	$\emptyset$	1	$\{a\}$	1/2	$\emptyset$	1
$\{a,b\}$	$\{a,b\}$	0	$\{a,b\}$	0	$\emptyset$	1	$\{a\}$	1/2	$\emptyset$	1
$\{a,c\}$	$\{a,b,c,d\}$	0	$\{a\}$	1/2	$\emptyset$	1	$\{a\}$	1/2	$\emptyset$	1
$\{a,d\}$	$\{a,b,c,d\}$	0	$\{a\}$	1/2	$\emptyset$	1	$\{a\}$	1/2	$\emptyset$	1
$\{a,e\}$	$\{b\}$	2/3	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{b,c\}$	$\{b,c,d\}$	0	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{b,d\}$	$\{b,c,d\}$	0	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{b,e\}$	$\{a,b\}$	1/3	$\{a\}$	2/3	$\emptyset$	1	$\{a\}$	2/3	$\emptyset$	1
$\{c,d\}$	$\{b\}$	2/3	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{c,e\}$	$\{a,b,c,d\}$	1/5	$\{a\}$	2/3	$\emptyset$	1	$\{a\}$	2/3	$\emptyset$	1
$\{d,e\}$	$\{a,b,c,d\}$	1/5	$\{a\}$	2/3	$\emptyset$	1	$\{a\}$	2/3	$\emptyset$	1
$\{a,b,c\}$	$\{a,b,c,d\}$	0	$\{a\}$	2/3	$\emptyset$	1	$\{a\}$	2/3	$\emptyset$	1
$\{a,b,d\}$	$\{a,b,c,d\}$	0	$\{a\}$	2/3	$\emptyset$	1	$\{a\}$	2/3	$\emptyset$	1
$\{a,b,e\}$	$\{b\}$	2/3	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{a,c,d\}$	$\{a,b\}$	1/2	$\{a\}$	2/3	$\emptyset$	1	$\{a\}$	2/3	$\emptyset$	1
$\{a,c,e\}$	$\{b,c,d\}$	2/5	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{a,d,e\}$	$\{b,c,d\}$	2/5	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{b,c,d\}$	$\{b\}$	2/3	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{b,c,e\}$	$\{a,b,c,d\}$	1/5	$\{a\}$	3/4	$\emptyset$	1	$\{a\}$	3/4	$\emptyset$	1
$\{b,d,e\}$	$\{a,b,c,d\}$	1/5	$\{a\}$	3/4	$\emptyset$	1	$\{a\}$	3/4	$\emptyset$	1
$\{c,d,e\}$	$\{a,b\}$	3/5	$\{a,b\}$	3/5	$\emptyset$	1	$\{a\}$	3/4	$\emptyset$	1
$\{a,b,c,d\}$	$\{a,b\}$	1/2	$\{a\}$	3/4	$\emptyset$	1	$\{a\}$	3/4	$\emptyset$	1
$\{a,b,c,e\}$	$\{b,c,d\}$	2/5	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{a,b,d,e\}$	$\{b,c,d\}$	2/5	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{a,c,d,e\}$	$\{b\}$	4/5	$\{b\}$	4/5	$\{b\}$	4/5	$\emptyset$	1	$\emptyset$	1
$\{b,c,d,e\}$	$\{a\}$	4/5	$\{a\}$	4/5	$\emptyset$	1	$\{a\}$	4/5	$\emptyset$	1
U	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1

4. $\boldsymbol{\theta\beta_j}$- ideal approximation spaces

We present the ideas of $\boldsymbol{\theta\beta_j}$-ideal lower and $\boldsymbol{\theta\beta_j}$-ideal upper approximations for any subset in this section. Some of the related characteristics of them will be examined; demonstrating that the $\boldsymbol{\theta\beta_j}$-ideal approximations represent the finest ones and precise than the other approaches.

Definition 4.1. If (X, R, ψ_j) is a j-***NS***, I be an ideal on X. Then a subset A in X is called an I-$\theta\beta_j$-open, if $A \subseteq cl_j[int_j(cl_j^{*\theta}(A))]$. A complement of an I-$\theta\beta_j$-open set is an I-$\theta\beta_j$-closed. A family of all I-$\theta\beta_j$-open and I-$\theta\beta_j$ closed sets are denoted by $I-\theta\beta_j O(X)$ and $I-\theta\beta_j C(X)$, respectively.
Note that: $cl_j^{*\theta}(A) = A \cup A_j^{*\theta}$, where $A_j^{*\theta} = \{x \in X: A \cap cl_j(G) \notin I, \forall G \in \tau_j$ and $x \in G\}$.
Definition 4.2. Consider (X, R, ψ_j) is a j-***NS***, I is an ideal on X and $A \subseteq X$. For all $j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, I-$\theta\beta_j$-lower, I-$\theta\beta_j$-upper approximations, I-$\theta\beta_j$-boundary regions and I-$\theta\beta_j$-accuracy of the approximations of A are defined respectively, by:

$\underline{R}_j^{I-\theta\beta}(A) = \cup\{G \in I-\theta\beta_j O(X): G \subseteq A\} = int_j^{I-\theta\beta}(A)$, where $int_j^{I-\boldsymbol{\theta\beta}}(A)$ represents I-$\boldsymbol{\theta\beta_j}$-interior of A.

$\overline{R}_j^{I-\theta\beta}(A) = \cap\{F \in I-\theta\beta_j C(X): F \supseteq A\} = cl_j^{I-\theta\beta}(A)$, where $cl_j^{I-\boldsymbol{\theta\beta}}(A)$ represents I-$\boldsymbol{\theta\beta_j}$-closure of A.

$B_j^{I-\theta\beta}(A) = \overline{R}_j^{I-\theta\beta}(A) - \underline{R}_j^{I-\theta\beta}(A)$.

$\sigma_j^{I-\theta\beta}(A) = \frac{\left|\underline{R}_j^{I-\theta\beta}(A)\right|}{\left|\overline{R}_j^{I-\theta\beta}(A)\right|}$, where $\left|\overline{R}_j^{I-\theta\beta}(A)\right| \neq 0$.

The following proposition proposes the essential properties of the existing I-$\theta\beta_j$-lower and I-$\theta\beta_j$-upper approximations.

Proposition 4.1. Let (X, R, ψ_j) be a j-***NS***, I be an ideal on X and $A, B \subseteq X$. Then:

(i) $\underline{R}_j^{I-\theta\beta}(A) \subseteq A \subseteq \overline{R}_j^{I-\theta\beta}(A)$.

(ii) $\underline{R}_j^{I-\theta\beta}(\emptyset) = \overline{R}_j^{I-\theta\beta}(\emptyset) = \emptyset$ and $\underline{R}_j^{I-\theta\beta}(X) = \overline{R}_j^{I-\theta\beta}(X) = X$

(iii) If $A \subseteq B$, then $\underline{R}_j^{I-\theta\beta}(A) \subseteq \underline{R}_j^{I-\theta\beta}(B)$ and $\overline{R}_j^{I-\theta\beta}(A) \subseteq \overline{R}_j^{I-\theta\beta}(B)$

(iv) $\underline{R}_j^{I-\theta\beta}(A) \cup \underline{R}_j^{I-\theta\beta}(B) \subseteq \underline{R}_j^{I-\theta\beta}(A \cup B)$

(v) $\overline{R}_j^{I-\theta\beta}(A) \cup \overline{R}_j^{I-\theta\beta}(B) \subseteq \overline{R}_j^{I-\theta\beta}(A \cup B)$

(vi) $\underline{R}_j^{I-\theta\beta}(A \cap B) \subseteq \underline{R}_j^{I-\theta\beta}(A) \cap \underline{R}_j^{I-\theta\beta}(B)$

(vii) $\overline{R}_j^{I-\theta\beta}(A \cap B) \subseteq \overline{R}_j^{I-\theta\beta}(A) \cap \overline{R}_j^{I-\theta\beta}(B)$

(viii) $\underline{R}_j^{I-\theta\beta}(A^c) = (\overline{R}_j^{I-\theta\beta}(A))^c$

(ix) $\overline{R}_j^{I-\theta\beta}(A^c) = (\underline{R}_j^{I-\theta\beta}(A))^c$

(x) $\underline{R}_j^{I-\theta\beta}\left(\underline{R}_j^{I-\theta\beta}(A)\right) = \underline{R}_j^{I-\theta\beta}(A)$

$$\text{(xi)}\ \overline{R}_j^{I-\theta\beta}\left(\overline{R}_j^{I-\theta\beta}(A)\right) = \overline{R}_j^{I-\theta\beta}(A)$$

$$\text{(xii)}\ \underline{R}_j^{I-\theta\beta}\left(\underline{R}_j^{I-\theta\beta}(A)\right) \subseteq \overline{R}_j^{I-\theta\beta}(\underline{R}_j^{I-\theta\beta}(A))$$

$$\text{(xiii)}\ \underline{R}_j^{I-\theta\beta}\left(\overline{R}_j^{I-\theta\beta}(A)\right) \subseteq \overline{R}_j^{I-\theta\beta}(\overline{R}_j^{I-\theta\beta}(A))$$

The proof of this proposition is simple using the properties of I-$\boldsymbol{\theta\beta_j}$-interior and I-$\boldsymbol{\theta\beta_j}$-closure operators, so we omit it.

Remark 4.1. In the next example, we explain that the relation of implication in parts (i), (iv), (v), (vi), (vii), (xii), and (xiii) of Proposition 4.1 cannot be replaced by equality relation:

Example 4.1. Let $X = \{a, b, c, d, e\}$, $R = \{(a, a), (a, e), (b, a), (b, c), (b, d), (b, e), (c, c), (c, d), (d, c), (d, d), (e. e)\}$ and $I = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$.

Thus $N_r(a) = \{a, e\}$, $N_r(b) = \{a, c, d, e\}$, $N_r(c) = N_r(d) = \{c, d\}$, $N_r(e) = \{e\}$.

Then, $I\text{-}\theta\beta_r O(X) = \{X, \emptyset, \{b\}, \{d\}, \{e\}, \{a, b\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, d, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$.

Thus we have

(1) For part (i), assume that $A = \{a\}$. Therfore, $\underline{R}_r^{I-\theta\beta}(A) = \emptyset$ and thus $A \nsubseteq \underline{R}_r^{I-\theta\beta}(A)$. Also, if $A = \{b, d\}$, then $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(A) = \{b, c, d\}$ and hence $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(A) \nsubseteq A$.

(2) For part (iv), assume that $A = \{a\}$ and $B = \{b\}$. Therfore, $\underline{R}_r^{I-\boldsymbol{\theta\beta}}(A) = \emptyset$, $\underline{R}_r^{I-\boldsymbol{\theta\beta}}(B) = \{b\}$, and $\underline{R}_j^{I-\boldsymbol{\theta\beta}}(A \cup B) = \{a, b\}$.

(3) For part (v), assume that $A = \{b\}$ and $B = \{d\}$. Therfore, $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(A) = \{b\}$, $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(B) = \{d\}$, and $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(A \cup B) = \{b, c, d\}$.

(4) For part (vi), if $A = \{a, b\}$ and $B = \{a, e\}$, then $\underline{R}_r^{I-\theta\beta}(A) = \{a, b\}$, $\underline{R}_r^{I-\theta\beta}(B) = \{a, e\}$, and $\underline{R}_j^{I-\boldsymbol{\theta\beta}}(A \cap B) = \emptyset$.

(5) For part (vii), if $A = \{b, c\}$ and $B = \{b, d\}$, then $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(A) = \{b, c\}$, $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(B) = \{b, c, d\}$, and $\overline{R}_r^{I-\boldsymbol{\theta\beta}}(A \cap B) = \{b\}$.

(6) For part (xii), if $A = \{b, d\}$, then $\underline{R}_r^{I-\theta\beta}\left(\underline{R}_r^{I-\theta\beta}(A)\right) = \{b, d\}$ and $\overline{R}_j^{I-\boldsymbol{\theta\beta}}\left(\underline{R}_j^{I-\boldsymbol{\theta\beta}}(A)\right) = \{b, c, d\}$, and therefore $\overline{R}_j^{I-\boldsymbol{\theta\beta}}(\underline{R}_j^{I-\boldsymbol{\theta\beta}}(A)) \nsubseteq \underline{R}_j^{I-\boldsymbol{\theta\beta}}\left(\underline{R}_j^{I-\boldsymbol{\theta\beta}}(A)\right)$.

(7) For part (xiii), if $A = \{c\}$, then $\overline{R}_j^{I-\boldsymbol{\theta\beta}}\left(\overline{R}_j^{I-\boldsymbol{\theta\beta}}(A)\right) = \{c\}$ and $\underline{R}_j^{I-\boldsymbol{\theta\beta}}\left(\overline{R}_j^{I-\boldsymbol{\theta\beta}}(A)\right) = \emptyset$, and therefore $\overline{R}_j^{I-\boldsymbol{\theta\beta}}\left(\overline{R}_j^{I-\boldsymbol{\theta\beta}}(\mathrm{A})\right) \nsubseteq \underline{R}_j^{I-\boldsymbol{\theta\beta}}\left(\overline{R}_j^{I-\boldsymbol{\theta\beta}}(A)\right)$.

Definition 4.3. Let (X, R, ψ_j) be a j-$\boldsymbol{NS}$, I be an ideal on X. The subset A in X is named an I-$\boldsymbol{\theta\beta_j}$-definable ($I$-$\boldsymbol{\theta\beta_j}$-exact) set if $\overline{R}_j^{I-\boldsymbol{\theta\beta}}(A) = \underline{R}_j^{I-\boldsymbol{\theta\beta}}(A)$. Else, A is an I-$\boldsymbol{\theta\beta_j}$-rough set.

Note that: In Example 4.1, $A = \{b\}$ is an I-$\boldsymbol{\theta\beta_r}$-exact set, while $B = \{a\}$ is an I-$\boldsymbol{\theta\beta_r}$-rough set.

Remark 4.2. Consider (X, R, ψ_j) is a j-NS, I is an ideal on X, and $A \subseteq X$. Then a finite intersection of two I-$\boldsymbol{\theta\beta_r}$-rough sets essentially not to be an I-$\boldsymbol{\theta\beta_r}$-rough set as in Example 4.1 $\{a, b\}$ and $\{a, e\}$ are I-$\boldsymbol{\theta\beta_r}$-rough sets, $\{a, b\} \cap \{a, e\} = \{a\}$ is not an I-$\boldsymbol{\theta\beta_r}$-rough set.

The subsequent theorem and its corollaries explain the relations amongst the present approximations in Definition 4.2 and the others in Definitions 2.5 [21] and 2.13 [22].

Theorem 4.1. Consider (X, R, ψ_j) is a j-***NS***, I is an ideal on X, and $A \subseteq X$. Then:

$$\text{(i) } \underline{R}_j(A) \subseteq \underline{R}_j^I(A) \subseteq \underline{R}_j^{I-\boldsymbol{\theta\beta}}(A).$$

$$\text{(ii) } \overline{R}_j^{I-\boldsymbol{\theta\beta}}(A) \subseteq \overline{R}_j^I(A) \subseteq \overline{R}_j(A).$$

Proof:

(i) $\underline{R}_j(A) = \cup\{G \in \tau_j : G \subseteq A\} \subseteq \cup\{G \in \tau_j^I : G \subseteq A\} = \underline{R}_j^I(A) \subseteq \cup\{G \in I - \boldsymbol{\theta\beta_j}O(X) : G \subseteq A\} = \underline{R}_j^{I-\boldsymbol{\theta\beta}}(A)$.

(ii) Similar to (i).

Corollary 4.1. Assume that (X, R, ψ_j) is a j-***NS***, I is an ideal on X, and $A \subseteq X$. Then:

$$\text{(i) } B_j^{I-\boldsymbol{\theta\beta}}(A) \subseteq B_j^I(A) \subseteq B_j(A).$$

$$\text{(ii) } \sigma_j(A) \leq \sigma_j^I(A) \leq \sigma_j^{I-\boldsymbol{\theta\beta}}(A).$$

Corollary 4.2. Suppose that (X, R, ψ_j) is a j-***NS***, I is an ideal on X, and $A \subseteq X$. Then:
(1) Each j-exact subset in X is I-$\theta\beta_j$-exact.
(2) Each I-j-exact subset in X is I-$\theta\beta_j$-exact.
(3) Each I-$\theta\beta_j$-rough subset in X is j-rough.
(4) Each I-$\theta\beta_j$-rough subset in X is I-j-rough.

Remark 4.3. Example 4.1 confirms that the opposite of parts of Corollary 4.2 is not necessarily true.
(1) if $A = \{b\}$, then it is I-$\theta\beta_r$-exact, but it is not r-exact.
(2) if $A = \{d\}$, then it is I-$\theta\beta_r$-exact, but it is not I-r-exact.
(3) if $A = \{e\}$, then it is r- rough, but it is not I-$\theta\beta_r$- rough.
(4) if $A = \{a, b\}$, then it is I-r-rough, but it is not I-$\theta\beta_r$- rough.

For example, take $A = \{d\}$: By using the present technique in Definition 4.2, the boundary and accuracy of A are $\emptyset$ and 1 respectively. Whereas, the boundary and accuracy by using Abd El-Monsef et al.'s method in Definition 2.5 are $\{b, c, d\}$ and 0 respectively, and by using Hosny method in Definition 2.13 are $\{b, c\}$ and 1/3 respectively.

The next propositions and corollaries show some the relationships among the I-$\theta\beta_j$-lower, I-$\theta\beta_j$-upper approximations, I-$\theta\beta_j$-boundary regions and I-$\theta\beta_j$-accuracy.

Proposition 4.2. Assume that (X, R, ψ_j) is a j-***NS***, I is an ideal on X, and $A \subseteq X$. Then, the following properties are true.

$$\text{(i) } \underline{R}_u^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_r^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_i^{I-\boldsymbol{\theta\beta}}(A).$$

$$\text{(ii) } \underline{R}_u^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_l^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_i^{I-\boldsymbol{\theta\beta}}(A).$$

$$\text{(iii) } \underline{R}_{<u>}^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_{<r>}^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_{<i>}^{I-\boldsymbol{\theta\beta}}(A).$$

$$\text{(iv) } \underline{R}_{<u>}^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_{<l>}^{I-\boldsymbol{\theta\beta}}(A) \subseteq \underline{R}_{<i>}^{I-\boldsymbol{\theta\beta}}(A).$$

Now, using Example 4.1, Table 2 represents a comparison between the boundary regions and the accuracy of the approximations using different methods.

Table 2. A comparison between the boundary regions and the accuracy of approximations, in Example 4.1, using different methods.

$A \subseteq X$	Abd El-Monsef method		Hosny method		Current method	
	$B_r(A)$	$\sigma_r(A)$	$B_r^I(A)$	$\sigma_r^I(A)$	$B_r^{I-\theta\beta}(A)$	$\sigma_r^{I-\theta\beta}(A)$
$\{a\}$	$\{a,b\}$	0	$\{a\}$	0	$\{a\}$	0
$\{b\}$	$\{b\}$	0	$\{b\}$	0	$\emptyset$	1
$\{c\}$	$\{b,c,d\}$	0	$\{c\}$	0	$\{c\}$	0
$\{d\}$	$\{b,c,d\}$	0	$\{b,c\}$	1/3	$\emptyset$	1
$\{e\}$	$\{a,b\}$	1/3	$\{a,b\}$	1/3	$\emptyset$	1
$\{a,b\}$	$\{a,b\}$	0	$\{a,b\}$	0	$\emptyset$	1
$\{a,c\}$	$\{a,b,c,d\}$	0	$\{a,c\}$	0	$\{a,c\}$	0
$\{a,d\}$	$\{a,b,c,d\}$	0	$\{a,b,c\}$	1/4	$\{a\}$	1/2
$\{a,e\}$	$\{b\}$	2/3	$\{b\}$	2/3	$\emptyset$	1
$\{b,c\}$	$\{b,c,d\}$	0	$\{b,c\}$	0	$\emptyset$	1
$\{b,d\}$	$\{b,c,d\}$	0	$\{b,c\}$	1/3	$\{c\}$	2/3
$\{b,e\}$	$\{a,b\}$	1/3	$\{a,b\}$	1/3	$\{a\}$	2/3
$\{c,d\}$	$\{b\}$	2/3	$\{b\}$	2/3	$\emptyset$	1
$\{c,e\}$	$\{a,b,c,d\}$	1/5	$\{a,b,c\}$	1/4	$\{c\}$	1/2
$\{d,e\}$	$\{a,b,c,d\}$	1/5	$\{a,b,c\}$	2/5	$\emptyset$	1
$\{a,b,c\}$	$\{a,b,c,d\}$	0	$\{a,b,c\}$	0	$\emptyset$	1
$\{a,b,d\}$	$\{a,b,c,d\}$	0	$\{a,b,c\}$	1/4	$\{c\}$	3/4
$\{a,b,e\}$	$\{b\}$	2/3	$\{b\}$	2/3	$\emptyset$	1
$\{a,c,d\}$	$\{a,b\}$	1/2	$\{a,b\}$	1/2	$\{a\}$	2/3
$\{a,c,e\}$	$\{b,c,d\}$	2/5	$\{b,c\}$	1/2	$\{c\}$	2/3
$\{a,d,e\}$	$\{b,c,d\}$	2/5	$\{b,c\}$	3/5	$\emptyset$	1
$\{b,c,d\}$	$\{b\}$	2/3	$\{b\}$	2/3	$\emptyset$	1
$\{b,c,e\}$	$\{a,b,c,d\}$	1/5	$\{a,b,c\}$	1/4	$\{a\}$	3/4
$\{b,d,e\}$	$\{a,b,c,d\}$	1/5	$\{a,c\}$	3/5	$\{a,c\}$	3/5
$\{c,d,e\}$	$\{a,b\}$	3/5	$\{a,b\}$	3/5	$\emptyset$	1
$\{a,b,c,d\}$	$\{a,b\}$	1/2	$\{a,b\}$	1/2	$\emptyset$	1
$\{a,b,c,e\}$	$\{b,c,d\}$	2/5	$\{b,c\}$	1/2	$\emptyset$	1
$\{a,b,d,e\}$	$\{b,c,d\}$	2/5	$\{c\}$	4/5	$\{c\}$	4/5
$\{a,c,d,e\}$	$\{b\}$	4/5	$\{b\}$	4/5	$\emptyset$	1
$\{b,c,d,e\}$	$\{a,b\}$	3/5	$\{a\}$	4/5	$\{a\}$	4/5
X	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1

Proposition 4.3. Suppose that (X, R, ψ_j) is a j-***NS***, I is an ideal on X, and $A \subseteq X$. Then, $\forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, the following statements are valid.

(i) $\overline{R}_i^{I-\theta\beta}(A) \subseteq \overline{R}_r^{I-\theta\beta}(A) \subseteq \overline{R}_u^{I-\theta\beta}(A)$.

(ii) $\overline{R}_i^{I-\theta\beta}(A) \subseteq \overline{R}_l^{I-\theta\beta}(A) \subseteq \overline{R}_u^{I-\theta\beta}(A)$.

(iii) $\overline{R}_{<i>}^{I-\theta\beta}(A) \subseteq \overline{R}_{<r>}^{I-\theta\beta}(A) \subseteq \overline{R}_{<u>}^{I-\theta\beta}(A)$.

(iv) $\overline{R}_{<i>}^{I-\theta\beta}(A) \subseteq \overline{R}_{<l>}^{I-\theta\beta}(A) \subseteq \overline{R}_{<u>}^{I-\theta\beta}(A)$.

The proof of Propositions 4.2 and 4.3 was omitted since it is obvious.

Corollary 4.3. Let (X, R, ψ_j) be a j-**NS**, I be an ideal on X, and $A \subseteq X$. Then, $\forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, we have

(i) $B_i^{I-\theta\beta}(A) \subseteq B_r^{I-\theta\beta}(A) \subseteq B_u^{I-\theta\beta}(A)$.

(ii) $B_i^{I-\theta\beta}(A) \subseteq B_l^{I-\theta\beta}(A) \subseteq B_u^{I-\theta\beta}(A)$.

(iii) $B_{<i>}^{I-\theta\beta}(A) \subseteq B_{<r>}^{I-\theta\beta}(A) \subseteq B_{<u>}^{I-\theta\beta}(A)$.

(iv) $B_{<i>}^{I-\theta\beta}(A) \subseteq B_{<l>}^{I-\theta\beta}(A) \subseteq B_{<u>}^{I-\theta\beta}(A)$.

Corollary 4.4. Consider (X, R, ψ_j) is a j-**NS**, I is an ideal on X, and $A \subseteq X$. Then, $\forall j \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, we have

(i) $\sigma_u^{I-\theta\beta}(A) \leq \sigma_r^{I-\theta\beta}(A) \leq \sigma_i^{I-\theta\beta}(A)$.

(ii) $\sigma_u^{I-\theta\beta}(A) \leq \sigma_l^{I-\theta\beta}(A) \leq \sigma_i^{I-\theta\beta}(A)$.

(iii) $\sigma_{<u>}^{I-\theta\beta}(A) \leq \sigma_{<r>}^{I-\theta\beta}(A) \leq \sigma_{<i>}^{I-\theta\beta}(A)$.

(iv) $\sigma_{<u>}^{I-\theta\beta}(A) \leq \sigma_{<l>}^{I-\theta\beta}(A) \leq \sigma_{<i>}^{I-\theta\beta}(A)$.

5. Some applications of $\theta\beta_j$- ideal approximations in chemistry

Here, a practical example in the arena of chemistry is provided, using the current method in “Definition 4.2” to explain the concepts with an illustrative manner. The main goal is to present the benefits of the proposed methodologies for improving the accuracy measure. This is evident from the comparisons provided between our approaches and previous methods such as those presented via Abd El-Monsef et al. and M. Hosny.

Example 5.1. [35,36] Suppose that $H = \{c_1, c_2, c_3, c_4, c_5\}$ represents five amino acids (AAs). The (AAs) are described in terms of five attributes: a_1 = PIE, a_2 = SAC = surface area, a_3 = MR = molecular refractivity, a_4 = LAM = the side chain polarity and a_5 = Vol = molecular volume as showed in Table 3.

Table 3. Lists all of the quantitative characteristics of the five AAs.

	a_1	a_2	a_3	a_4	a_5
c_1	0.23	254.2	2.126	−0.02	82.2
c_2	−0.48	303.6	2.994	−1.24	112.3
c_3	−0.61	287.9	2.994	−1.08	103.7
c_4	0.45	282.9	2.993	−0.11	99.1
c_5	−0.11	335.0	3.458	−0.19	127.5

Assume the following five reflexive relations on H are defined:

$$R_s = \left\{(c_u, c_v) \in H \times H:\ c_u(a_s) - c_v(a_s) < \frac{\sigma_s}{2},\ u, v, s = 1,2, \ldots, 5\right\}.$$

Where σ_s is the standard deviation of the quantitative attributes a_s, and $s = 1,2, \ldots, 5$. The right neighbourhoods for all elements c_u of H where $u = 1,2, \ldots, 5$, with respect to the relations R_s shown in Table 4.

Table 4. Right Neighborhood of five relations.

	c_uR_1	c_uR_2	c_uR_3	c_uR_4	c_uR_5
c_1	$\{c_1, c_4\}$	H	H	$\{c_1, c_4, c_5\}$	H
c_2	H	$\{c_2, c_5\}$	$\{c_2, c_3, c_4, c_5\}$	H	$\{c_2, c_5\}$
c_3	H	$\{c_2, c_3, c_4, c_5\}$	$\{c_2, c_3, c_4, c_5\}$	H	$\{c_2, c_3, c_4, c_5\}$
c_4	$\{c_4\}$	$\{c_2, c_3, c_4, c_5\}$	$\{c_2, c_3, c_4, c_5\}$	$\{c_1, c_4, c_5\}$	$\{c_2, c_3, c_4, c_5\}$
c_5	$\{c_1, c_4, c_5\}$	$\{c_5\}$	$\{c_5\}$	$\{c_1, c_4, c_5\}$	$\{c_3, c_5\}$

As a result, in order to demonstrate the set of all condition attributes, we compute the intersections of all right neighborhoods for each element of H as follows:
$c_1R = \bigcap_{s=1}^{5} c_1R_s = \{c_1, c_4\}$, $c_2R = \bigcap_{s=1}^{5} c_2R_s = \{c_2, c_5\}$, $c_3R = \bigcap_{s=1}^{5} c_3R_s = \{c_2, c_3, c_4, c_5\}$, $c_4R = \bigcap_{s=1}^{5} c_4R_s = \{c_4\}$ and $c_4R = \bigcap_{s=1}^{5} c_4R_s = \{c_5\}$.

Hence, the topology generated by these right neighborhoods is given by:

$\tau_r = \{H, \emptyset, \{c_4\}, \{c_5\}, \{c_1, c_4\}, \{c_2, c_5\}, \{c_4, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}, \{c_1, c_2, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}\}$.

By Chemistry's expert, if $I = \{\emptyset, \{c_1\}, \{c_4\}, \{c_1, c_4\}\}$ is the selected ideal, then the topology generated by this ideal is:

$\tau_r^I = \{H, \emptyset, \{c_1\}, \{c_4\}, \{c_5\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_5\}, \{c_4, c_5\}, \{c_1, c_2, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_3, c_5\},$

$\{c_2, c_4, c_5\}, \{c_1, c_2, c_3, c_5\}, \{c_1, c_2, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}\}$.

Therefore, $I\text{-}\theta\beta_r O(X) = \{H, \emptyset, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}, \{c_1, c_3\}\{c_1, c_4\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\},$ $\{c_3, c_4\}, \{c_3, c_5\}, \{c_4, c_5\}, \{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \{c_1, c_3, c_4\}, \{c_1, c_3, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_3, c_4\}, \{c_2, c_3, c_5\},$ $\{c_2, c_4, c_5\}, \{c_3, c_4, c_5\}, \{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_5\}, \{c_1, c_2, c_4, c_5\}, \{c_1, c_3, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}\}$.

Now, Table 5 presents a comparison between the boundary (resp. accuracy) using the current technique in Definition 4.2 and the other approaches.

Table 5. A comparison between the boundary and accuracy using the techniques of Abd El-Monsef et al. [19], Hosny [22] and the proposed technique in Definition 4.2.

$A \subseteq H$	Abd El-Monsef method		Hosny method		Current method	
	$B_r(A)$	$\sigma_r(A)$	$B_r^I(A)$	$\sigma_r^I(A)$	$B_r^{I-\theta\beta}(A)$	$\sigma_r^{I-\theta\beta}(A)$
$\{c_1\}$	$\{c_1\}$	0	$\emptyset$	1	$\{c_1\}$	0
$\{c_2\}$	$\{c_2, c_3\}$	0	$\{c_2, c_3\}$	0	$\emptyset$	1
$\{c_3\}$	$\{c_3\}$	0	$\{c_3\}$	0	$\emptyset$	1
$\{c_4\}$	$\{c_1, c_3\}$	1/3	$\emptyset$	1	$\emptyset$	1
$\{c_5\}$	$\{c_2, c_3\}$	1/3	$\{c_2, c_3\}$	1/3	$\emptyset$	1
$\{c_1, c_2\}$	$\{c_1, c_2, c_3\}$	0	$\{c_2, c_3\}$	1/3	$\{c_1\}$	1/2

Continued on next page

$A \subseteq H$	Abd El-Monsef method		Hosny method		Current method	
	$B_r(A)$	$\sigma_r(A)$	$B_r^I(A)$	$\sigma_r^I(A)$	$B_r^{I-\theta\beta}(A)$	$\sigma_r^{I-\theta\beta}(A)$
$\{c_1, c_3\}$	$\{c_1, c_3\}$	0	$\{c_3\}$	1/2	$\emptyset$	1
$\{c_1, c_4\}$	$\{c_3\}$	2/3	$\emptyset$	1	$\emptyset$	1
$\{c_1, c_5\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/2	$\{c_1\}$	1/2
$\{c_2, c_3\}$	$\{c_2, c_3\}$	0	$\{c_2, c_3\}$	0	$\emptyset$	1
$\{c_2, c_4\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/3	$\emptyset$	1
$\{c_2, c_5\}$	$\{c_3\}$	2/3	$\{c_3\}$	2/3	$\emptyset$	1
$\{c_3, c_4\}$	$\{c_1, c_3\}$	1/3	$\{c_3\}$	1/2	$\{c_1\}$	2/3
$\{c_3, c_5\}$	$\{c_2, c_3\}$	1/3	$\{c_2, c_3\}$	1/3	$\emptyset$	1
$\{c_4, c_5\}$	$\{c_1, c_2, c_3\}$	2/5	$\{c_2, c_3\}$	1/2	$\emptyset$	1
$\{c_1, c_2, c_3\}$	$\{c_1, c_2, c_3\}$	0	$\{c_2, c_3\}$	1/3	$\emptyset$	1
$\{c_1, c_2, c_4\}$	$\{c_2, c_3\}$	1/2	$\{c_2, c_3\}$	1/2	$\emptyset$	1
$\{c_1, c_2, c_5\}$	$\{c_1, c_3\}$	1/2	$\{c_3\}$	3/4	$\{c_1\}$	2/3
$\{c_1, c_3, c_4\}$	$\{c_3\}$	2/3	$\{c_3\}$	2/3	$\emptyset$	1
$\{c_1, c_3, c_5\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/2	$\emptyset$	1
$\{c_1, c_4, c_5\}$	$\{c_2, c_3\}$	3/5	$\{c_2, c_3\}$	3/5	$\emptyset$	1
$\{c_2, c_3, c_4\}$	$\{c_1, c_2, c_3\}$	1/4	$\{c_2, c_3\}$	1/3	$\{c_1\}$	3/4
$\{c_2, c_3, c_5\}$	$\{c_3\}$	2/3	$\emptyset$	1	$\emptyset$	1
$\{c_2, c_4, c_5\}$	$\{c_1, c_3\}$	3/5	$\{c_3\}$	3/4	$\emptyset$	1
$\{c_3, c_4, c_5\}$	$\{c_1, c_2, c_3\}$	2/5	$\{c_2, c_3\}$	1/2	$\{c_1\}$	3/4
$\{c_1, c_2, c_3, c_4\}$	$\{c_2, c_3\}$	1/2	$\{c_2, c_3\}$	1/2	$\emptyset$	1
$\{c_1, c_2, c_3, c_5\}$	$\{c_1, c_3\}$	1/2	$\emptyset$	1	$\emptyset$	1
$\{c_1, c_2, c_4, c_5\}$	$\{c_3\}$	4/5	$\{c_3\}$	4/5	$\emptyset$	1
$\{c_1, c_3, c_4, c_5\}$	$\{c_2, c_3\}$	3/5	$\{c_2, c_3\}$	3/5	$\emptyset$	1
$\{c_2, c_3, c_4, c_5\}$	$\{c_1\}$	4/5	$\emptyset$	1	$\{c_1\}$	4/5

Observation: From the previous comparisons in Table 5, we note the following:

1) There are several approaches to approximate the sets. The finest of these approaches is that there are assumed by using $\boldsymbol{I-\theta\beta_j}$ **-**approximations of the current methods in Definition 4.2 for creating the rough approximations because the boundary regions in these cases are minimized (or removed) by maximizing the lower approximation and minimizing the upper approximation. Moreover, the accuracy degree, in these cases, is more accurate than the other types. For example, all proper subsets are rough in Abd El-Monsef's approaches. But there are many $I-\theta\beta_j$- exact sets in the current methods.
2) The suggested method in Definition 4.2 is more accurate and stronger than M. Hosny's methods. Therefore, the suggested methodologies will be useful in decision-making for extracting the information and help in eliminating the ambiguity of the data in real-life problems.
3) The significance of the suggested approximations is not only that it is decreasing or deleting the boundary regions, but also, it's satisfying all characteristics of Pawlak's model without any restrictions as shown in Proposition 4.1.

6. Conclusions and future work

In our daily life, we often face some problems that necessitate complete decision-making. However, in the majority of these cases, we get perplexed as to the best solution. To find the most feasible solution to these problems, we must consider some solution-related parameters. One of the most important topics in rough sets is minimizing the boundary region, which aims to maximize the degree of decision-making accuracy. Topological structures formed by relations are one technique used to accomplish this goal. In this article, using the notion of a j-neighborhood space and the related concept of $\boldsymbol{\theta\beta}$-open sets, different methods for generalizing Pawlak rough sets and some of their enhancements have been proposed and their properties have been studied. Additionally, in the context of ideal notion, other generalizations to Pawlak's models and some of their enhancements such as (Abd El-Monsef et al. [19], Amer et al. [20], and Hosny [21,22] techniques) have been presented. Comparisons were made between the proposed methods and previous approaches published in the literature. Furthermore, numerous results have been proposed to explain why our approximations are more accurate and powerful than other methods.

Finally, an application from Chemistry was proposed to demonstrate the significance of our decision-making methods. Moreover, it provides a comparison between the proposed methods with already existing in the literature. Also, this application proved that the suggested methods improve the accuracy measure which is useful in establishing an accurate decision. So, we can say that the proposed techniques may be useful in applications. In the future, we will apply the proposed techniques in more real-life applications.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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