Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2142–2159.
DOI: 10.3934/math.2022122
Received: 21 August 2021
Accepted: 19 October 2021
Published: 08 November 2021

Research article

Global attractivity for uncertain differential systems

Nana Tao[1,*] **and Chunxiao Ding**[2]

[1] School of Information Engineering, Kaifeng University, Kaifeng, 475004, Henan, China
[2] School of Mathematical Sciences, Liaocheng University, Liaocheng, 252059, Shandong, China

* **Correspondence:** Email: nanatao1981@126.com.

Abstract: This paper studies global attractivity for uncertain differential systems, which are effective tools to solve the problems with uncertainty. And They have been applied in many areas. This article presents several global attractivity concepts. Based on the knowledge of uncertainty theory, some sufficient conditions of global attractivity for linear uncertain differential systems are given. In particular, the attractivity on the solutions and α-path of uncertain differential systems is studied. Last, as an application of attractivity, an interest rate model with uncertainty is shown.

Keywords: global attractivity; uncertain differential systems; sufficient condition; α-path; interest rate model with uncertainty
Mathematics Subject Classification: 34D45, 37C75

1. Introduction

In 1940s, for dealing with dynamic stochastic systems, stochastic differential equations involving Wiener process were proposed by Itô for the first time. And they have played a very important role in many branches of science and industry. Ruelle [1] initiated the study of global random attractors. In 2004, for a kind of higer order nonlinear difference equation, The literature [2] considered global attractivity. In particular, for neutral SPDEs, Liu and Li [3] analyzed global attracting set, exponential decay and stability in distribution. Recently, much research has been done for various differential systems [4–6].

An uncertain differential system (U-D-S) [7] involving the canonical Liu process [8] was proposed for the first time. Then Chen and Liu made an in-depth analysis of the U-D-S and obtained the existence and uniqueness theorem [9]. After that, The U-D-Ss have been applied in more and more fields. For instance, they have been applied to uncertain optimal control [10–12], and uncertain financial market [7, 13, 14].

Since Liu [15] presented the definition of stability, many scholars have done a lot of research. For instance, existence and uniqueness theorem for uncertain differential equations [16–18], Stability in mean for uncertain differential equation [19], and stability for multi-dimensional uncertain differential equation [20, 21]. In 2021, Some new stability theorems of uncertain differential equations with time-dependent delay were studied by Jia and Liu [22].

From the perspective of application, attractive domain estimation has been applied to many fields, such as in power system. If the voltage disturbance exceeds a certain level, it may cause a large area of power failure of the entire grid, or even make the whole grid collapse. Therefore, it is particularly important to determine the allowable value of deviation from the stable state, that is, the size of the attractive domain of the stable fixed point. For example, in an ecosystem, the initial population density range, namely the size of the viable attractive region, is determined to ensure that the system will not become extinct or explode under the given parameters. Therefore, the estimation of the attractive region has strong practical value. Tao and Zhu [23, 24] studied attractivity and got some the judgement conditions for U-D-Ss.

In this paper, our aim is to study global attractivity for U-D-Ss. Furthermore, we will deduce some judging conditions for linear U-D-Ss. In Section 2, Several basic uncertainty definitions and theorems will be reviewed. Section 3 will present several global attractivity concepts for U-D-Ss. In Section 4, for linear U-D-Ss, we will deduce some sufficient conditions. Furthermore, we will find the relationship of of global attractivity between the solution of the U-D-S and its α-path. In Section 5, we will present some examples having locally but not globally attractivity. Last, Section 6 will show an interest rate model with uncertainty, which is a global attractive in mean.

2. Preliminary

Definition 2.1. *([25]) ζ defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertain variable. If at least one of the two integrals is finite, then*

$$E[\zeta] = \int_0^{+\infty} \mathcal{M}\{\zeta \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\zeta \leq r\}\mathrm{d}r$$

is called the expected value $E[\zeta]$ of ζ.

Definition 2.2. *([7]) Let f_1 and f_2 be two given binary functions, and C_t be a canonical process. Then*

$$\mathrm{d}Y_t = f_1(t, Y_t)\mathrm{d}t + f_2(t, Y_t)\mathrm{d}C_t \tag{2.1}$$

is called a U-D-S.

Definition 2.3. *([26]) Suppose a number α satisfies $0 < \alpha < 1$. If Y_t^α solves*

$$\mathrm{d}Y_t^\alpha = f_1(t, Y_t^\alpha)\mathrm{d}t + |f_2(t, Y_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t, \tag{2.2}$$

In the above equation, $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, and

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}\ln\frac{\alpha}{1-\alpha}}{\pi}.$$

Then U-D-S (2.1) is said to have an α-path Y_t^α.

Definition 2.4. *([23]) Let Y_t and Z_t satisfy U-D-S (2.1), and their initial values be Y_0 and Z_0, respectively. Then the U-D-S (2.1) is called*

(i) attractive in measure (i.e., locally attractive in measure) if for any given $\varepsilon > 0$, here exists $\sigma > 0$ satisfying when $|Y_0 - Z_0| < \sigma$, we can get

$$\lim_{t\to+\infty} \mathcal{M}\{|Y_t - Z_t| > \varepsilon\} = 0;$$

(ii) attractive almost surely (i.e., locally attractive almost surely) if here exists $\sigma > 0$ satisfying when $|Y_0 - Z_0| < \sigma$, one can obtain

$$\lim_{t\to+\infty} |Y_t - Z_t| = 0 \text{ almost surely};$$

(iii) attractive in mean (i.e., locally attractive in mean) if there exists $\sigma > 0$ satisfying when $|Y_0 - Z_0| < \sigma$, we can get

$$\lim_{t\to+\infty} E[|Y_t - Z_t|] = 0.$$

(iv) attractive in distribution (i.e., locally attractive in distribution) if Y_t and Z_t have uncertainty distributions $\Upsilon_t(x)$ and $\Psi_t(x)$, respectively. And here exists $\sigma > 0$ satisfying when $|Y_0 - Z_0| < \sigma$, we can get

$$\lim_{t\to+\infty} |\Upsilon_t(x) - \Psi_t(x)| = 0, \ x \in \Re,$$

Theorem 2.1. *([25]) Let ζ be an uncertain variable. Then , the following inequality holds.*

$$\mathcal{M}\{|\zeta| \geq t\} \leq \frac{E[f(\zeta)]}{f(t)} \quad \forall t > 0.$$

provided that f is a even function with $f \geq 0$ and increasing on $[0, \infty)$.

Theorem 2.2. *([9]) Suppose that Y_t defined on $[a_1, a_2]$ is an integrable uncertain process, C_t is a canonical process, and the sample path $C_t(\gamma)$ has the Lipschitz constant $K(\gamma)$. Then, the following inequality holds.*

$$\left| \int_{a_1}^{a_2} Y_t(\gamma) \mathrm{d}C_t(\gamma) \right| \leq K(\gamma) \int_{a_1}^{a_2} |Y_t(\gamma)| \, \mathrm{d}t$$

Theorem 2.3. *([26]) Let Y_t and Y_t^α satisfy (2.1) and (2.2), respectively. Then at each time t,*

$$\Psi_t^{-1}(\alpha) = Y_t^\alpha, \ 0 < \alpha < 1.$$

is said to be an inverse uncertainty distribution of Y_t.

3. Some global attractivity concepts

In this section, several global attractivity definitions are given for the U-D-S

$$\mathrm{d}Y_t = f_1(t, Y_t)\mathrm{d}t + f_2(t, Y_t)\mathrm{d}C_t. \tag{3.1}$$

We suppose that Y_t and Z_t satisfy the above system (3.1), and the initial values are Y_0 and Z_0, respectively.

Definition 3.1. *If for any $0 < \sigma < +\infty$ and for any $\varepsilon > 0$, when $|Y_0 - Z_0| < \sigma$, the following equation*

$$\lim_{t\to+\infty} \mathcal{M}\{|Y_t - Z_t| > \varepsilon\} = 0.$$

holds. Then the U-D-S (3.1) is called globally attractive in measure.

Example 3.1. Assume that U-D-S has the below form

$$\mathrm{d}Y_t = -\exp(2t)Y_t\mathrm{d}t + \exp(t)Y_t\mathrm{d}C_t. \tag{3.2}$$

It follows that

$$\frac{\mathrm{d}(Y_t - Z_t)}{(Y_t - Z_t)} = -\exp(2t)\mathrm{d}t + \exp(t)\mathrm{d}C_t.$$

Thus

$$\begin{aligned} Y_t - Z_t &= (Y_0 - Z_0)\exp\left(\int_0^t [-\exp(2s)]\,\mathrm{d}s + \int_0^t \exp(s)\mathrm{d}C_s\right) \\ &= (Y_0 - Z_0)\exp\left(\frac{1}{2} - \frac{1}{2}\exp(2t) + \int_0^t \exp(s)\mathrm{d}C_s\right). \end{aligned}$$

For any given $\sigma > 0$ and $\varepsilon > 0$. We prove it in two cases. Case 1: Assume $0 < \sigma \le \varepsilon$. When $|Y_0 - Z_0| < \sigma$, it is easy to see that $\ln\frac{\varepsilon}{|Y_0-Z_0|} > \ln\frac{\varepsilon}{\sigma} \ge 0$. According to $\int_0^t -\exp(2s)\mathrm{d}s = \frac{1}{2} - \frac{1}{2}\exp(2t) < 0$, where $t > 0$. By the Theorem 2.1, we can obtain

$$\begin{aligned} \mathcal{M}\{|Y_t - Z_t| > \varepsilon\} &= \mathcal{M}\left\{|Y_0 - Z_0|\exp\left(\frac{1}{2} - \frac{1}{2}\exp(2t) + \int_0^t \exp(s)\mathrm{d}C_s\right) > \varepsilon\right\} \\ &= \mathcal{M}\left\{\int_0^t \exp(s)\mathrm{d}C_s > \ln\frac{\varepsilon}{|Y_0 - Z_0|} + \frac{1}{2}\exp(2t) - \frac{1}{2}\right\} \\ &\le \frac{E\left[\left(\int_0^t \exp(s)\mathrm{d}C_s\right)^2\right]}{\left(\ln\frac{\varepsilon}{|Y_0-Z_0|} + \frac{1}{2}\exp(2t) - \frac{1}{2}\right)^2} \\ &= \frac{\left(\int_0^t |\exp(s)|\mathrm{d}s\right)^2}{\left(\ln\frac{\varepsilon}{|Y_0-Z_0|} + \frac{1}{2}\exp(2t) - \frac{1}{2}\right)^2} \\ &\le \frac{(\exp(t) - 1)^2}{\left(\ln\frac{\varepsilon}{\sigma} + \frac{1}{2}\exp(2t) - \frac{1}{2}\right)^2} \\ &\to 0 \end{aligned}$$

as $t \to +\infty$. Case 2: Assume $\sigma > \varepsilon$. When $|Y_0 - Z_0| < \sigma$, it is easy to see that $\ln\frac{\varepsilon}{\sigma} < \ln\frac{\varepsilon}{|Y_0-Z_0|} < 0$. Then there exists $T_0 = \frac{1}{2}\ln\left(2 + 2|\ln\frac{\varepsilon}{\sigma}|\right) > 0$, such that when $t > T_0$, we can obtain $\ln\frac{\varepsilon}{|Y_0-Z_0|} + \frac{1}{2}\exp(2t) - \frac{1}{2} > 0$. It follows from Theorem 2.1 that

$$\mathcal{M}\{|Y_t - Z_t| > \varepsilon\} = \mathcal{M}\left\{\int_0^t \exp(s)\mathrm{d}C_s > \ln\frac{\varepsilon}{|Y_0 - Z_0|} + \frac{1}{2}\exp(2t) - \frac{1}{2}\right\}$$

$$
\begin{aligned}
&\leq \frac{E\left[\left(\int_0^t \exp(s)\mathrm{d}C_s\right)^2\right]}{\left(\ln\frac{\varepsilon}{|Y_0-Z_0|}+\frac{1}{2}\exp(2t)-\frac{1}{2}\right)^2}\\
&= \frac{\left(\int_0^t |\exp(s)|\mathrm{d}s\right)^2}{\left(\ln\frac{\varepsilon}{|Y_0-Z_0|}+\frac{1}{2}\exp(2t)-\frac{1}{2}\right)^2}\\
&\leq \frac{(\exp(t)-1)^2}{\left(\ln\frac{\varepsilon}{\sigma}+\frac{1}{2}\exp(2t)-\frac{1}{2}\right)^2}\\
&\to 0
\end{aligned}
$$

as $t \to +\infty$. Thus the U-D-S (3.2) is globally attractive in measure.

Definition 3.2. *The U-D-S (3.1) is called globally attractive almost surely if for any* σ *with* $0 < \sigma < +\infty$*, when* $|Y_0 - Z_0| < \sigma$*, we get*

$$\lim_{t\to+\infty} |Y_t - Z_t| = 0 \textit{ almost surely.}$$

Definition 3.3. *The U-D-S (3.1) is called globally attractive in mean if for any* $0 < \sigma < +\infty$*, when* $|Y_0 - Z_0| < \sigma$*, we get*

$$\lim_{t\to+\infty} E[|Y_t - Z_t|] = 0.$$

Definition 3.4. *Suppose* Y_t *and* Z_t *have uncertainty distributions* $\Upsilon_t(x)$ *and* $\Psi_t(x)$*, respectively. Then the U-D-S (3.1) is called globally attractive in distribution if for any* $0 < \sigma < +\infty$*, when* $|Y_0 - Z_0| < \sigma$*, one can obtain*

$$\lim_{t\to+\infty} |\Upsilon_t(x) - \Psi_t(x)| = 0,\ x \in \Re,$$

Remark 3.1. Global attractivity in measure implies local attractivity in measure of the uncertainty solutions. However the reverse implication may not hold (Several other definitions have similar results). For the global and local attractivity of zero solution, Figure 1 shows us an intuitional comprehension. (a) Global attractivity means for any $0 < \sigma < +\infty$, If Y_0 is in the σ-neighborhood of zero, the state Y_t with initial value Y_0 converges to 0 ($t \to +\infty$). (b) Local attractivity describes here exists $0 < \sigma < +\infty$, if Y_0 is in the σ-neighborhood of zero, then Y_t with initial value Y_0 converges to 0 ($t \to +\infty$).

Figure 1. (a) Global attractivity and (b) Local attractivity.

4. Some results

Firstly, this section studies the global attractivity of linear U-D-Ss, and deduces several sufficient and necessary conditions for global attractivity. Secondly, global attractivity relationships between the solution and α-path are discussed.

Assuming that $A_1(t)$, $A_2(t)$, $B_1(t)$, $B_2(t)$ are continuous functions on $[0, +\infty)$. For the linear U-D-S

$$\mathrm{d}Y_t = (A_1(t)Y_t + A_2(t))\,\mathrm{d}t + (B_1(t)Y_t + B_2(t))\,\mathrm{d}C_t \tag{4.1}$$

we suppose that Y_t and Z_t satisfy the above system (4.1), and the initial values are Y_0 and Z_0, respectively.

Theorem 4.1. *The linear U-D-S (4.1) is globally attractive in measure if*

$$\int_0^{+\infty} |B_1(s)|\,\mathrm{d}s = +\infty \tag{4.2}$$

and here exists one number $p > 1$ that satisfies

$$\limsup_{t\to+\infty} \frac{\int_0^t A_1(s)\mathrm{d}s}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^p} < 0. \tag{4.3}$$

Proof: According to the system (4.1), It is easy to see that

$$\frac{\mathrm{d}\,(Y_t - Z_t)}{(Y_t - Z_t)} = A_1(t)\mathrm{d}t + B_1(t)\mathrm{d}C_t.$$

Thus

$$Y_t - Z_t = (Y_0 - Z_0)\exp\left(\int_0^t A_1(s)\mathrm{d}s + \int_0^t B_1(s)\mathrm{d}C_s\right).$$

For any given $\sigma > 0$ and $\varepsilon > 0$. We prove it in two cases. Case 1: Assume $\sigma \le \varepsilon$. When $|Y_0 - Z_0| < \sigma$, it is easy to see that $\ln\frac{\varepsilon}{|Y_0-Z_0|} > \ln\frac{\varepsilon}{\sigma} \ge 0$. According to the (4.3), here exists $\tau > 0$ satisfying when $t > \tau$, it can be seen $\int_0^t A_1(s)\mathrm{d}s < 0$. From Theorem 2.1, (4.2) and (4.3), if $t \to +\infty$, we can get

$$\begin{aligned}
\mathcal{M}\{|Y_t - Z_t| > \varepsilon\} &= \mathcal{M}\left\{|Y_0 - Z_0|\exp\left(\int_0^t A_1(s)\mathrm{d}s + \int_0^t B_1(s)\mathrm{d}C_s\right) > \varepsilon\right\}\\
&= \mathcal{M}\left\{\int_0^t B_1(s)\mathrm{d}C_s > \ln\frac{\varepsilon}{|Y_0 - Z_0|} - \int_0^t A_1(s)\mathrm{d}s\right\}\\
&\le \frac{E\left[\left(\int_0^t B_1(s)\mathrm{d}C_s\right)^2\right]}{\left(\ln\frac{\varepsilon}{|Y_0-Z_0|} - \int_0^t A_1(s)\mathrm{d}s\right)^2} = \frac{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^2}{\left(\ln\frac{\varepsilon}{|Y_0-Z_0|} - \int_0^t A_1(s)\mathrm{d}s\right)^2}\\
&\le \frac{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^2}{\left(\ln\frac{\varepsilon}{\sigma} - \int_0^t A_1(s)\mathrm{d}s\right)^2}
\end{aligned}$$

$$= \frac{1}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^{2(p-1)} \left[\frac{\ln\frac{\varepsilon}{\sigma}}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^p} - \frac{\int_0^t A_1(s)\mathrm{d}s}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^p}\right]^2} \to 0$$

Case 2: Assume $\sigma > \varepsilon$. When $|Y_0 - Z_0| < \sigma$, it is easy to see that $\ln\frac{\varepsilon}{\sigma} < \ln\frac{\varepsilon}{|Y_0 - Z_0|} < 0$. According to the (4.3), here exists $\tau > 0$ such that when $t > \tau$, the inequality $\ln\frac{\varepsilon}{\sigma} - \int_0^t A_1(s)\mathrm{d}s > 0$ holds. By the Theorem 2.1, (4.2) and (4.3), we can get

$$\begin{aligned}
\mathcal{M}\{|Y_t - Z_t| > \varepsilon\} &= \mathcal{M}\left\{|Y_0 - Z_0|\exp\left(\int_0^t A_1(s)\mathrm{d}s + \int_0^t B_1(s)\mathrm{d}C_s\right) > \varepsilon\right\} \\
&= \mathcal{M}\left\{\int_0^t B_1(s)\mathrm{d}C_s > \ln\frac{\varepsilon}{|Y_0 - Z_0|} - \int_0^t A_1(s)\mathrm{d}s\right\} \\
&\leq \frac{E\left[\left(\int_0^t B_1(s)\mathrm{d}C_s\right)^2\right]}{\left(\ln\frac{\varepsilon}{|Y_0 - Z_0|} - \int_0^t A_1(s)\mathrm{d}s\right)^2} \\
&\leq \frac{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^2}{\left(\ln\frac{\varepsilon}{\sigma} - \int_0^t A_1(s)\mathrm{d}s\right)^2} \\
&= \frac{1}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^{2(p-1)} \left[\frac{\ln\frac{\varepsilon}{\sigma}}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^p} - \frac{\int_0^t A_1(s)\mathrm{d}s}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^p}\right]^2} \\
&\to 0
\end{aligned}$$

as $t \to +\infty$. Thus the linear U-D-S (4.1) is globally attractive in measure.

Example 4.1. Analyze a following linear U-D-S

$$\mathrm{d}Y_t = -4t^3 Y_t \mathrm{d}t + tY_t \mathrm{d}C_t.$$

Obviously $A_1(t) = -4t^3$ and $B_1(t) = t$, then it is easy to get

$$\int_0^{+\infty} |B_1(s)|\mathrm{d}s = +\infty \text{ and } \lim_{t\to+\infty} \frac{\int_0^t A_1(s)\mathrm{d}s}{\left(\int_0^t |B_1(s)|\mathrm{d}s\right)^2} = -\lim_{t\to+\infty} \frac{t^4}{\left(\frac{t^2}{2}\right)^2} = -4 < 0.$$

Thus $dY_t = -4t^3 Y_t \mathrm{d}t + tY_t \mathrm{d}C_t$ is globally attractive in measure by Theorem 4.1.

Theorem 4.2. *The linear U-D-S (4.1) is globally attractive almost surely if and only if*

$$\int_0^{+\infty} A_1(s)\mathrm{d}s = -\infty.$$

Proof: Since Y_t and Z_t both satisfy the linear U-D-S (4.1), the following equation holds

$$Y_t - Z_t = (Y_0 - Z_0)\exp\left(\int_0^t A_1(s)\mathrm{d}s + \int_0^t B_1(s)\mathrm{d}C_s\right).$$

According to $\int_0^t B_1(s)\mathrm{d}C_s \sim \mathcal{N}(0, \int_0^t |B_1(s)|\mathrm{d}s)$, $\int_0^t |B_1(s)|\mathrm{d}s \le \int_0^{+\infty} |B_1(s)|\mathrm{d}s < +\infty$. Let $K(\gamma)$ be the Lipschitz constant of $C_t(\gamma)$. From the Theorem 2.2, we can get

$$\int_0^t B_1(s)\mathrm{d}C_s < K(\gamma)\int_0^t |B_1(s)|\mathrm{d}s,$$

It is easy to see that

$$\int_0^{+\infty} B_1(s)\mathrm{d}C_s < +\infty \text{ almost surely.}$$

For any given $\sigma > 0$, when $|Y_0 - Z_0| < \sigma$, we know that

$$\begin{aligned}\lim_{t\to+\infty} |Y_t - Z_t| &= \lim_{t\to+\infty}\left\{|Y_0 - Z_0|\exp\left(\int_0^t A_1(s)\mathrm{d}s\right)\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right)\right\}\\ &< \lim_{t\to+\infty}\left\{\sigma\exp\left(\int_0^t A_1(s)\mathrm{d}s\right)\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right)\right\}.\end{aligned}$$

The linear U-D-S (4.1) is globally attractive almost surely if and only if

$$\int_0^{+\infty} A_1(s)\mathrm{d}s = -\infty.$$

Example 4.2. Analyze a following linear U-D-S

$$\mathrm{d}Y_t = -(t^2 + t)Y_t\mathrm{d}t + \exp(-3t)Y_t\mathrm{d}C_t.$$

Since $A_1(t) = -(t^2 + t)$ and $B_1(t) = \exp(-3t)$, the following equations

$$\int_0^{+\infty} A_1(s)\mathrm{d}s = -\infty,$$

and

$$\int_0^{+\infty} |B_1(s)|\mathrm{d}s = \frac{1}{3} < +\infty$$

hold. Thus $\mathrm{d}Y_t = -(t^2 + t)Y_t\mathrm{d}t + \exp(-3t)Y_t\mathrm{d}C_t$ is globally attractive almost surely.

Theorem 4.3. *Suppose that* $\int_0^{+\infty} |B_1(s)|\mathrm{d}s < \frac{\pi}{\sqrt{3}}$. *Then the linear U-D-S (4.1) is globally attractive in mean if and only if*

$$\int_0^{+\infty} A_1(s)\mathrm{d}s = -\infty.$$

Proof: We can get easily the following equation from (4.1)

$$Y_t - Z_t = (Y_0 - Z_0)\exp\left(\int_0^t A_1(s)\mathrm{d}s + \int_0^t B_1(s)\mathrm{d}C_s\right).$$

For the above equation, by taking the expected value, we can obtain

$$E\left[|Y_t - Z_t|\right] = |Y_0 - Z_0|\exp\left(\int_0^t A_1(s)\mathrm{d}s\right)E\left[\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right)\right].$$

Now that $\int_0^t B_1(s)\mathrm{d}C_s \sim \mathcal{N}(0, \int_0^t |B_1(s)|\mathrm{d}s)$.

$$\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right) \sim \mathcal{LOGN}\left(0, \int_0^t |B_1(s)|\mathrm{d}s\right).$$

Since

$$\int_0^t |B_1(s)|\mathrm{d}s \leq \int_0^{+\infty} |B_1(s)|\mathrm{d}s < \frac{\pi}{\sqrt{3}},$$

According to the [27], we can obtain

$$E\left[\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right)\right] = \sqrt{3}\int_0^t |B_1(s)|\mathrm{d}s \csc\left(\sqrt{3}\int_0^t |B_1(s)|\mathrm{d}s\right).$$

For any given $\sigma > 0$, when $|Y_0 - Z_0| < \sigma$, it is easy to see that

$$\begin{aligned} E\left[|Y_t - Z_t|\right] &= |Y_0 - Z_0| \exp\left(\int_0^t A_1(s)\mathrm{d}s\right) E\left[\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right)\right] \\ &< \sigma \exp\left(\int_0^t A_1(s)\mathrm{d}s\right) E\left[\exp\left(\int_0^t B_1(s)\mathrm{d}C_s\right)\right]. \end{aligned}$$

Obviously,

$$\lim_{t\to+\infty} E\left[|Y_t - Z_t|\right] = 0$$

is equivalent to

$$\int_0^{+\infty} A_1(s)ds = -\infty.$$

Hence (4.1) is globally attractive in mean if and only if

$$\int_0^{+\infty} A_1(s)\mathrm{d}s = -\infty.$$

Example 4.3. For the linear U-D-S with following form

$$\mathrm{d}Y_t = -\frac{1}{t+1}Y_t\mathrm{d}t + \frac{1}{t^2+1}Y_t\mathrm{d}C_t.$$

Note that $A_1(t) = -\frac{1}{t+1}$ and $B_1(t) = \frac{1}{t^2+1}$, we immediately obtain

$$\int_0^{+\infty} A_1(s)\mathrm{d}s = -\infty \text{ and } \int_0^{+\infty} |B_1(s)|\mathrm{d}s = \frac{\pi}{2} < \frac{\pi}{\sqrt{3}}.$$

Thus $\mathrm{d}Y_t = -\frac{1}{t+1}Y_t\mathrm{d}t + \frac{1}{t^2+1}Y_t\mathrm{d}C_t$ is globally attractive in mean.

Next, global attractivity on the solutions and α-paths for U-D-Ss will be studied.

Theorem 4.4. *The U-D-S (3.1) is globally attractive in distribution if and only if the differential system (2.2) is globally attractive.*

Proof: Suppose $\Upsilon_t(x)$ and $\Psi_t(x)$ are corresponding regular distributions of Y_t and Z_t, respectively. Let Y_t^α and Z_t^α satisfy (2.2), and their initial values be Y_0 and Z_0, respectively.

If (3.1) is globally attractive in distribution, In other words, for any given $0 < \sigma < +\infty$, when $|Y_0 - Z_0| < \sigma$, we can get

$$\lim_{t\to+\infty} |\Upsilon_t(x) - \Psi_t(x)| = 0 \ \forall x \in \Re.$$

Then

$$\lim_{t\to+\infty} |\Upsilon_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)| = 0 \ \ for\ all\ \alpha \in (0,1)$$

By Yao-Chen Formula (Theorem 2.3), the following equation holds.

$$\lim_{t\to+\infty} |Y_t^\alpha - Z_t^\alpha| = 0, \ \forall x \in \Re.$$

with $|Y_0 - Z_0| < \sigma$. Hence, (2.2) is globally attractive.

Each of these steps is reversible. Then the proof has been completed.

Example 4.4. Let us analyze the following linear system

$$\mathrm{d}Y_t = -Y_t^3\mathrm{d}t + \sigma\mathrm{d}C_t.$$

Suppose Y_t^α is an α-path and the initial value is Y_0, i.e., it satisfies the ordinary differential system

$$\mathrm{d}Y_t^\alpha = -(Y_t^\alpha)^3\mathrm{d}t + \sigma\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\mathrm{d}t. \tag{4.4}$$

Since the differential system (4.4) and $\mathrm{d}Y_t^\alpha = -(Y_t^\alpha)^3\mathrm{d}t$ have the same as attractivity, we only study $\mathrm{d}Y_t^\alpha = -(Y_t^\alpha)^3\mathrm{d}t$. By solving the equation, we get $Y_t^\alpha = Y_0[1 + 2(Y_0)^2t]^{-\frac{1}{2}}$ with the initial value Y_0. For any $\sigma > 0$ and $\varepsilon > 0$, here exists $T_1 = \frac{1}{\varepsilon^2} > 0$ such that when $|Y_0| < \sigma$ and $t > T_1$, one can obtain

$$|Y_t^\alpha| = |Y_0[1 + 2(X_0)^2t]^{-\frac{1}{2}}| \leq |Y_0|\frac{1}{\sqrt{Y_0^2t}} \leq \frac{1}{\sqrt{T_1}} < \varepsilon.$$

Thus $\mathrm{d}Y_t^\alpha = -(Y_t^\alpha)^3\mathrm{d}t$ is globally attractive. It follows from the globally attractivity of $\mathrm{d}Y_t^\alpha = -(Y_t^\alpha)^3\mathrm{d}t$ that (4.4) is globally attractive. According to the Theorem 4.4, the uncertain differential system $\mathrm{d}Y_t = -Y_t^3\mathrm{d}t + \sigma\mathrm{d}C_t$ is globally attractive in distribution.

Corollary 4.1. *The U-D-S (3.1) is not globally attractive in distribution if here exists* $0 < \alpha < 1$ *such that the differential system (2.2) is not globally attractive.*

Proof: We suppose $\alpha_0 \in (0,1)$ and the differential system

$$\mathrm{d}Y_t^{\alpha_0} = f_1(t, Y_t^{\alpha_0})\mathrm{d}t + |f_2(t, Y_t^{\alpha_0})|\Phi^{-1}(\alpha_0)\mathrm{d}t$$

is not globally attractive. By the Theorem 4.4, the U-D-S (3.1) is not globally attractive in distribution.

5. Examples having local but not global attractivity

In this section, we give some examples to show that local attractivity does not implies global attractivity.

Example 5.1. *(Locally but not globally attractive in measure)* Let us consider the following form of system

$$\mathrm{d}Y_t = \left(-Y_t + Y_t^2\right)\mathrm{d}t - \exp(-t)\left(-Y_t + Y_t^2\right)\mathrm{d}C_t. \tag{5.1}$$

We have

$$\frac{\mathrm{d}Y_t}{(-Y_t + Y_t^2)} = \left(\frac{1}{Y_t - 1} - \frac{1}{Y_t}\right)\mathrm{d}Y_t = \mathrm{d}t - \exp(-t)\mathrm{d}C_t.$$

If we integrate both sides, we can obtain

$$\frac{Y_t - 1}{Y_t} = \frac{Y_0 - 1}{Y_0}\exp\left(t - \int_0^t \exp(-s)\mathrm{d}C_s\right).$$

That is

$$Y_t = \frac{Y_0\exp\left(-t + \int_0^t \exp(-s)\mathrm{d}C_s\right)}{Y_0\exp\left(-t + \int_0^t \exp(-s)\mathrm{d}C_s\right) - Y_0 + 1}.$$

We know that 0 satisfies (5.1) with the initial $Y_0 = 0$. For any given $0 < \varepsilon < 1$, let $|Y_0| < 1$. We can prove it in two cases. Case 1: If $0 \le Y_0 < 1$, it follows that

$$\begin{aligned}
\mathcal{M}\{|Y_t| > \varepsilon\} &= \mathcal{M}\left\{\exp\left(-t + \int_0^t \exp(-s)\mathrm{d}C_s\right) > \frac{\varepsilon(1 - Y_0)}{Y_0(1 - \varepsilon)}\right\} \\
&= \mathcal{M}\left\{\int_0^t \exp(-s)\mathrm{d}C_s > \ln\left(\frac{\varepsilon(1 - Y_0)}{Y_0(1 - \varepsilon)}\right) + t\right\} \\
&\le \frac{E\left[(\int_0^t \exp(-s)\mathrm{d}C_s)^2\right]}{\left(\ln\left(\frac{\varepsilon(1-Y_0)}{Y_0(1-\varepsilon)}\right) + t\right)^2} \\
&= \frac{\left(\int_0^t \exp(-s)\mathrm{d}s\right)^2}{\left(\ln\left(\frac{\varepsilon(1-Y_0)}{Y_0(1-\varepsilon)}\right) + t\right)^2} \\
&= \left(\frac{-\exp(-t) + 1)}{\ln\left(\frac{\varepsilon(1-Y_0)}{Y_0(1-\varepsilon)}\right) + t}\right)^2 \\
&\to 0
\end{aligned}$$

as $t \to +\infty$.

Case 2: Assume $-1 < Y_0 < 0$, we can easily obtain

$$\mathcal{M}\{|Y_t| > \varepsilon\} = \mathcal{M}\left\{\frac{1}{|1 - \frac{Y_0 - 1}{Y_0\exp\left(-t + \int_0^t \exp(-s)\mathrm{d}C_s\right)}|} > \varepsilon\right\}$$

$$
\begin{aligned}
&\leq \mathcal{M}\left\{\exp\left(-t+\int_0^t \exp(-s)\mathrm{d}C_s\right) > \frac{(Y_0-1)\varepsilon}{Y_0(1+\varepsilon)}\right\} \\
&= \mathcal{M}\left\{\int_0^t \exp(-s)\mathrm{d}C_s > t+\ln\left(\frac{(Y_0-1)\varepsilon}{Y_0(1+\varepsilon)}\right)\right\} \\
&\leq \frac{E\left[\left(\int_0^t \exp(-s)\mathrm{d}C_s\right)^2\right]}{\left(t+\ln\left(\frac{\varepsilon(Y_0-1)}{Y_0(1+\varepsilon)}\right)\right)^2} \\
&= \frac{\left(\int_0^t \exp(-s)\mathrm{d}s\right)^2}{\left(t+\ln\left(\frac{\varepsilon(Y_0-1)}{Y_0(1+\varepsilon)}\right)\right)^2} \\
&= \left(\frac{-\exp(-t)+1}{t+\ln\left(\frac{\varepsilon(Y_0-1)}{Y_0(1+\varepsilon)}\right)}\right)^2 \\
&\to 0
\end{aligned}
$$

as $t \to +\infty$. No matter what case happens, we always have

$$\lim_{t\to+\infty} \mathcal{M}\{|Y_t| > \varepsilon\} = 0.$$

That is to say that (5.1) is locally attractive in measure. However if $Y_0 = 1$, $Y_t = 1$ is the solution of (5.1). Then we have

$$\mathcal{M}\{|Y_t| > \varepsilon\} = 1.$$

Thus (5.1) is not globally attractive in measure.

Example 5.2. *(Locally but not globally attractive almost surely)* Analyze the U-D-S

$$\mathrm{d}Y_t = \left(-Y_t^2 - Y_t\right)\mathrm{d}t - \left(-Y_t^2 - Y_t\right)\exp(-t)\mathrm{d}C_t. \tag{5.2}$$

It is easy to see that 0 is the solution of (5.2) and the initial is $Y_0 = 0$. It can be obtained that

$$Y_t = \frac{Y_0 \exp\left(-t+\int_0^t \exp(-s)\mathrm{d}C_s\right)}{-Y_0\exp\left(-t+\int_0^t \exp(-s)\mathrm{d}C_s\right)+Y_0+1}.$$

From Theorem 2.2, we can deduce

$$\left|\int_0^t \exp(-s)\mathrm{d}C_s\right| \leq K(\gamma)\int_0^t \exp(-s)\mathrm{d}s \leq K(\gamma),$$

where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$. It is easy to see that

$$\int_0^{+\infty} \exp(-s)\mathrm{d}C_s < +\infty \text{ almost surely.}$$

Then we have

$$\exp\left(\int_0^{+\infty} \exp(-s)\mathrm{d}C_s\right) < +\infty \text{ almost surely.}$$

It follows that

$$\begin{aligned}\lim_{t\to+\infty}|Y_t| &= \lim_{t\to+\infty}\left|\frac{Y_0\exp\left(-t+\int_0^t\exp(-s)\mathrm{d}C_s\right)}{-Y_0\exp\left(-t+\int_0^t\exp(-s)\mathrm{d}C_s\right)+Y_0+1}\right|\\ &= \lim_{t\to+\infty}\left|\frac{Y_0\exp\left(\int_0^t\exp(-s)\mathrm{d}C_s\right)}{-Y_0\exp\left(\int_0^t\exp(-s)\mathrm{d}C_s\right)+\exp(t)(Y_0+1)}\right|\\ &= 0 \ \text{ almost surely}\end{aligned}$$

for $|Y_0| < 1$. Thus (5.2) is locally attractive almost surely. But if $Y_0 = -1$, we have $Y_t = -1$ and then

$$|Y_t| \equiv 1.$$

Then (5.2) is not globally attractive almost surely.

Example 5.3. *(Locally but not globally attractive in mean)* Supposed the U-D-S is

$$\mathrm{d}Y_t = (-Y_t + Y_t^3)\mathrm{d}t - (-Y_t + Y_t^3)\frac{1}{t^2+1}\mathrm{d}C_t. \tag{5.3}$$

We know that 0 is the solution of (5.3) if the initial is $Y_0 = 0$. In addition, we can obtain

$$\frac{\mathrm{d}Y_t}{(-Y_t+Y_t^3)} = \frac{1}{2}\left(\frac{1}{Y_t-1}+\frac{1}{Y_t+1}-\frac{2}{Y_t}\right)\mathrm{d}Y_t = \mathrm{d}t - \frac{1}{t^2+1}\mathrm{d}C_t.$$

Obviously,

$$\left(\frac{1}{Y_t-1}+\frac{1}{Y_t+1}-\frac{2}{Y_t}\right)\mathrm{d}Y_t = 2\mathrm{d}t - \frac{2}{t^2+1}\mathrm{d}C_t.$$

Taking integral on both sides, we get

$$\frac{Y_t^2-1}{Y_t^2} = \frac{Y_0^2-1}{Y_0^2}\exp\left(2t - \int_0^t\frac{2}{s^2+1}\mathrm{d}C_s\right).$$

Thus

$$|Y_t| = \sqrt{\frac{Y_0^2\exp(-2t+\int_0^t\frac{2}{s^2+1}\mathrm{d}C_s)}{Y_0^2\exp(-2t+\int_0^t\frac{2}{s^2+1}\mathrm{d}C_s)-Y_0^2+1}}.$$

Note that $\int_0^t\frac{1}{s^2+1}\mathrm{d}C_s \sim \mathcal{N}(0, \int_0^t\frac{1}{s^2+1}\mathrm{d}s)$. Hence

$$\exp\left(\int_0^t\frac{1}{s^2+1}\mathrm{d}C_s\right) \sim \mathcal{LOGN}\left(0, \int_0^t\frac{1}{s^2+1}\mathrm{d}s\right).$$

Since

$$\int_0^t\frac{1}{s^2+1}\mathrm{d}s \le \int_0^{+\infty}\frac{1}{s^2+1}\mathrm{d}s = \frac{\pi}{2} < \frac{\pi}{\sqrt{3}},$$

from [27], we have

$$E\left[\exp\left(\int_0^t\frac{1}{s^2+1}\mathrm{d}C_s\right)\right] = \sqrt{3}\int_0^t\frac{1}{s^2+1}\mathrm{d}s\,\csc\left(\sqrt{3}\int_0^t\frac{1}{s^2+1}\mathrm{d}s\right).$$

When $|Y_0| < 1$, we have

$$\begin{aligned}
E\left[|Y_t|\right] &= E\left[\sqrt{\frac{Y_0^2\exp(-2t+\int_0^t \frac{2}{s^2+1}\mathrm{d}C_s)}{Y_0^2\exp(-2t+\int_0^t \frac{2}{s^2+1}\mathrm{d}C_s)-Y_0^2+1}}\right]\\
&\le E\left[\sqrt{\frac{Y_0^2\exp(-2t+\int_0^t \frac{2}{s^2+1}\mathrm{d}C_s)}{-Y_0^2+1}}\right]\\
&= E\left[\frac{|Y_0|\exp(-t+\int_0^t \frac{1}{s^2+1}\mathrm{d}C_s)}{\sqrt{-Y_0^2+1}}\right]\\
&= \frac{|Y_0|\exp(-t)E\left[\exp(\int_0^t \frac{1}{s^2+1}\mathrm{d}C_s)\right]}{\sqrt{-Y_0^2+1}}\\
&= \frac{|Y_0|\exp(-t)}{\sqrt{-Y_0^2+1}}\sqrt{3}\int_0^t \frac{1}{s^2+1}\mathrm{d}s\csc\left(\sqrt{3}\int_0^t \frac{1}{s^2+1}\mathrm{d}s\right)\\
&\to 0
\end{aligned}$$

as $t \to +\infty$. So (5.3) is locally attractive in mean. But if $Y_0 = 1$, (5.3) has a solution $Y_t = 1$, and then we have

$$E[|Y_t|] \equiv 1 \ (t \to +\infty).$$

Thus (5.3) is not globally attractive in mean.

Example 5.4. *(Locally but not globally attractive in distribution)* Analyze the U-D-S with the following form

$$\mathrm{d}Y_t = \left(-Y_t + Y_t^2\right)\mathrm{d}t + \sigma \mathrm{d}C_t.$$

Let Y_t^α be an α-path of the above system and the initial value is Y_0, i.e., it satisfies the ordinary differential system

$$\mathrm{d}Y_t^\alpha = \left(-Y_t^\alpha + (Y_t^\alpha)^2\right)\mathrm{d}t + \frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\mathrm{d}t. \tag{5.4}$$

Since the differential system (5.4) and $\mathrm{d}Y_t^\alpha = (-Y_t^\alpha + (Y_t^\alpha)^2)\mathrm{d}t$ have the same attractivity, we only study $\mathrm{d}Y_t^\alpha = (-Y_t^\alpha + (Y_t^\alpha)^2)\mathrm{d}t$, which has zero solution and a solution $Y_t^\alpha = \frac{Y_0\exp(-t)}{Y_0\exp(-t)-Y_0+1}$ with initial value Y_0. Let $|Y_0| < 1$, we can prove it in two cases. Case 1: Let $0 \le Y_0 < 1$.

$$\begin{aligned}
\left|Y_t^\alpha\right| &= \left|\frac{Y_0\exp(-t)}{Y_0\exp(-t)-X_0+1}\right|\\
&\le \frac{Y_0\exp(-t)}{1-Y_0}\\
&\to 0
\end{aligned}$$

as $t \to +\infty$. Case 2: Assume $-1 < Y_0 < 0$. Then

$$\left|Y_t^\alpha\right| = \frac{|Y_0|\exp(-t)}{1-Y_0\left[1-\exp(-t)\right]}$$

$$\leq |Y_0| \exp(-t)$$
$$\to 0$$

as $t \to +\infty$. No matter what case happens, the ordinary differential system $\mathrm{d}Y_t^\alpha = (-Y_t^\alpha + (Y_t^\alpha)^2)\mathrm{d}t$ is locally attractive. But if $Y_0 = 1$, then $Y_t^\alpha = 1$ and we have

$$\left|Y_t^\alpha\right| \equiv 1 \ (t \to +\infty).$$

Thus the ordinary differential system $\mathrm{d}Y_t^\alpha = (-Y_t^\alpha + (Y_t^\alpha)^2)\mathrm{d}t$ is not globally attractive. we obtain that the ordinary differential system (5.4) is locally attractive but not globally attractive. By [23], the uncertain differential system

$$\mathrm{d}Y_t = (-Y_t + Y_t^2)\mathrm{d}t + \sigma \mathrm{d}C_t$$

is locally attractive in distribution but not globally attractive in distribution by Corollary 4.1.

6. Interest rate model with uncertainty

The real interest rate has not kept unchanged. Assumed that the interest rate follows an U-D-S, Chen and Gao [14] introduced a following model

$$\mathrm{d}Z_t = (-aZ_t + b)\mathrm{d}t + \sigma \mathrm{d}C_t \tag{6.1}$$

where a, b, σ are all positive numbers. Z_t and Y_t are assumed to satisfy (6.1) with initial values Z_0 and Y_0, respectively. It is easy to get the following equation

$$\mathrm{d}\left(Z_t - Y_t\right) = -aZ_t\mathrm{d}t + aY_t\mathrm{d}t = -a(Z_t - Y_t)\mathrm{d}t.$$

Thus

$$Z_t - Y_t = \left(Z_0 - Y_0\right)\exp\left(-at\right).$$

For any $0 < \sigma < +\infty$, when $|Z_0 - Y_0| < \sigma$, we can easily obtain the following result

$$E[|Z_t - Y_t|] < \sigma \exp\left(-at\right) \to 0$$

as $t \to +\infty$. Thus

$$E[|Z_t - Y_t|] \to 0$$

as $t \to +\infty$. Therefore, this model (6.1) is globally attractive in mean.

The above result shows that if Z_0 is higher than $\frac{b}{a}$, the drift of (6.1) is negative, and the interest rate will drop down in the $\frac{b}{a}$ direction. Similarly, if Z_0 is less than $\frac{b}{a}$, the drift of (6.1) is positive, so the rate will rise in the direction of $\frac{b}{a}$.

Next, let $a = 2, b = 2, \sigma = 1$, using MATLAB, Figure 2 shows the simulation diagram of $E(X_t)$ with several different initial values.

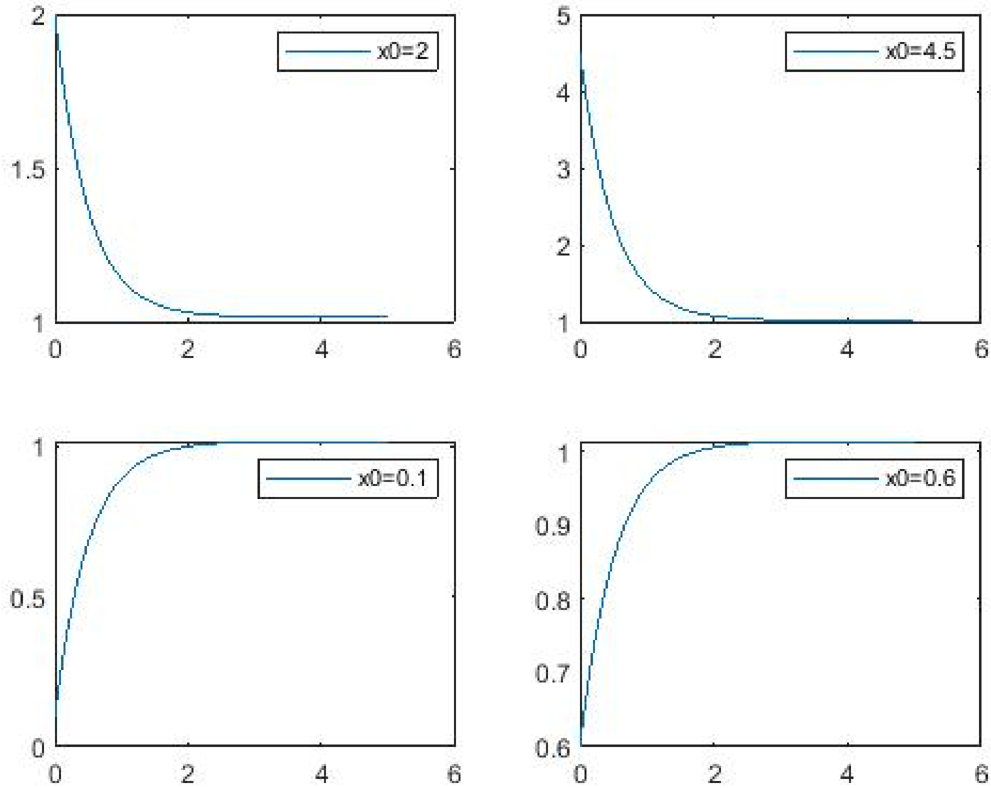

Figure 2. $E(X_t)$ for $\mathrm{d}X_t = (-2X_t + 2)\mathrm{d}t + X_t\mathrm{d}C_t$.

7. Conclusions

This article gave several concepts of global attractivity. Global attractivity (in measure, in mean, almost surely, in distribution) implies local attractivity (in measure, in mean, almost surely, in distribution). However the reverse implication may not hold. We gave some locally but not globally attractive examples. For linear U-D-Ss, some sufficient conditions of global attractivitywere presented. Furthermore, this paper found the relationship of attractivity and stability between the solution of the U-D-S and its α-path. The attractivity and stability of the general differential system solution can be determined by constructing Lyapunov function, thus the difficulty of determining the attractivity and stability of the U-D-Ss is greatly reduced. Last, an uncertain interest rate model which is global attractive in mean was considered. It is deduced that the solution of the model is attractive in mean. Future work will focus on the application of stability and attractivity.

Acknowledgments

This work is supported by the Key Scientific Research Projects of Henan Higher Education Institutions (No.18B11012).

Conflict of interest

No potential conflict of interest was reported by the author(s).

References

1. D. Ruelle, Characteristic exponents for a viscous fluid subjected to time dependent forces, *Commun. Math. Phys.,* **93** (1984). doi:285–300.10.1007/BF01258529.
2. X. Yan, W. Li, Global attractivity for a class of higer order nonlinear difference equations, *Appl. Math. Comput.,* **149** (2004), 533–546. doi: 10.1016/S0096-3003(03)00159-0.
3. K. Liu, Z. Li, Global attracting set, exponential decay and stability in distribution of neutral SPDEs driven by additive α-stable processes, *Discrete Contin. Dyn. Syst.,* **21** (2016), 3551–3573. doi:10.3934/dcdsb.2016110.
4. D. Xu, H. Zhao, Invariant and attracting sets of Hopfield neural networks with delay, *Int. J. Syst. Sci.,* **32** (2002), 863–866. doi: 10.1080/002077201300306207.
5. H. Zhao, Invariant set and attractor of nonautonomous functional differential systems, *J. Math. Anal. Appl.,* **282** (2003), 437–443. doi: 10.1016/S0022-247X(02)00370-0.
6. D. Xu, Z. Yang, Attracting and invariant sets for a class of impulsive functional differential equations, *J. Math. Anal. Appl.,* **329** (2007), 1036–1044. doi: 10.1016/j.jmaa.2006.05.072.
7. B. Liu, Fuzzy process, hybrid process and uncertain process, *J. Uncertain Syst.,* **2** (2008), 3–16.
8. B. Liu, *Uncertainty Theory: A branch of mathematics for modeling human uncertainty*, Springer-Verlag, Berlin, 2010.
9. X. Chen, B. Liu, Existence and uniqueness theorem for uncertain differential equations, *Fuzzy Optimiz. Decis. Mak.,* **9** (2010), 69–81. doi: 10.1007/s10700-010-9073-2.
10. Y. Zhu, Uncertain optimal control with application to a portfolio selection model, *Cybern. Syst.,* **41** (2010), 535-547. doi: 10.1080/01969722.2010.511552.
11. C. Ding, N. Tao, Y. Sun, Y. Zhu, The effect of time delays on transmission dynamics of schistosomiasis, *Chaos, Soliton. Fract.,* **91** (2016), 360–371. doi: 10.1016/j.chaos.2016.06.017.
12. C. Ding, Y. Sun, Y. Zhu, A NN-based hybrid intelligent algorithm for a discrete nonlinear uncertain optimal control problem, *Neural. Process. Lett.,* (2016), doi: 10.1007/s11063-016-9536-8.
13. J. Peng, Y. Gao, A new option pricing model for stocks in uncertainty markets, *Int. J. Oper. Res.,* **8** (2011), 18–26. doi: http://dx.doi.org/.
14. X. Chen, J, Gao, Uncertain term structure model of interest rate, *Soft Comput.,* **17** (2013), 597–604. doi: 10.1007/s00500-012-0927-0.
15. B. Liu, Some research problems in uncertainty theory, *J. Uncertain Syst.,* **3** (2009), 3–10. doi: 10.1007/978-3-642-01156-6.
16. L. Deng, Y. Zhu, Existence and uniqueness theorem of solution for uncertain differential equations with jump, *ICIC Expr. Lett.,* **6** (2012), 2693–2698.
17. Y. Gao, Existence and uniqueness theorems of uncertain differential equations with local Lipschitz condition, *J. Uncertain Syst.,* **6** (2012), 223–232.

18. X. Ge, Y. Zhu, Existence and uniqueness theorem for uncertain delay differential equations, *J. Comput. Inform. Syst.,* **8** (2012), 8341–8347. doi:http://dx.doi.org/.

19. K. Yao, H. Ke, Y. Sheng, Stability in mean for uncertain differential equation, *Fuzzy Optimiz. Decis. Mak.,* **14** (2015), 365–379. doi: 10.1007/s10700-014-9204-2.

20. T. Su, H. Wu, J. Zhou, Stability of multi-dimensional uncertain differential equation, *Soft Comput.,* **20** (2016), 4991–4998. doi: 10.1007/s00500-015-1788-0.

21. Y. Feng, X. Yang, G. Cheng, Stability in mean for multi-dimensional uncertain differential equation, *Soft Comput.,* **22** (2018), 5783–5789. doi: 10.1007/s00500-017-2659-7.

22. Z. Jia, X. Liu, New stability theorems of uncertain differential equations with time-dependent delay, *AIMS Math.,* **6** (2021), 623–642. doi: 10.3934/math.2021038.

23. N. Tao, Y. Zhu, Attractivity and stability analysis of uncertain differential systems, *Int. J. Bifurcat. Chaos.,* **25** (2015), 1550022-1-1550022-10. doi: 10.1142/S0218127415500224.

24. N. Tao, Y. Zhu, Stability and attractivity in optimistic value for dynamical systems with uncertainty, *Int. J. Gen. Syst.,* **45** (2016), 418–433. doi: 10.1080/03081079.2015.1072522.

25. B. Liu, *Uncertainty Theory*, 2Eds., Berlin: Springer-Verlag, 2007.

26. K. Yao, X. Chen, A numerical method for solving uncertain differential equations, *J. Intell. Fuzzy Syst.,* **25** (2013), 825–832. doi: 10.3233/IFS-120688.

27. K. Yao, Expected value of lognormal uncertain variable, Proceedings of the First International Conference on Uncertainty Theory, Urumchi, China, August 11–19, (2010), 241–243.