



Research article

Jensen-Mercer variant of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional operator

Jia-Bao Liu¹, Saad Ihsan Butt^{2,*}, Jamshed Nasir³, Adnan Aslam⁴, Asfand Fahad⁵ and Jarunee Soontharanon^{6,*}

¹ School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China

² COMSATS University Islamabad, Lahore Campus, Pakistan

³ Virtual University Lahore Campus, Pakistan

⁴ University of Engineering and Technology, Lahore (RCET), Pakistan

⁵ COMSATS University Islamabad, Vehari Campus Campus, Pakistan

⁶ Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand

* **Correspondence:** Email: saadihsanbutt@gmail.com, jarunee.s@sci.kmutnb.ac.th.

Abstract: We present new Mercer variants of Hermite-Hadamard (HH) type inequalities via Atangana-Baleanu (AB) fractional integral operators pertaining non-local and non-singular kernels. We establish trapezoidal type identities for fractional operator involving non-singular kernel and give Jensen-Mercer (JM) variants of Hermite-Hadamard type inequalities for differentiable mapping Υ possessing convex absolute derivatives. We establish connections of our results with several renowned results in the literature and also give applications to special functions.

Keywords: Jensen-Mercer inequality; Atangana-Baleanu fractional operators; q -digamma function; convex function

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1. Introduction

In recent years, the concept of convex functions has been generalized extensively. In several zones of Mathematics, applications of convex functions are widely seen in many areas of modern analysis. Especially in optimization theory they are magical due to number of expedient properties. Convex functions have significant relation with inequality theory. Many important and useful inequalities can be obtained by using convex functions. As a consequence of its growing variety of applications, it

is one of the more intensively developed fields of mathematical analysis. Due to their applicability in theory of inequality and convex programming, convex functions are the subject of researchers in several fields.

Definition 1.1. A function $\Upsilon : [\mu, \nu] \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$ is called convex function if

$$\Upsilon(\zeta x + (1 - \zeta)y) \leq \zeta \Upsilon(x) + (1 - \zeta)\Upsilon(y), \quad (1.1)$$

holds for all $x, y \in [\mu, \nu]$ and $\zeta \in [0, 1]$. We say that Υ is concave if $(-\Upsilon)$ is convex.

For more material on convexity of different types and their role in inequalities, see [1–3]. The literature and results presented in these books are the major motivation behind the progress and investigation of the integral inequalities pertaining convex functions. The most important and notable inequalities for convex functions are the Jensen and related inequalities due their immense variety of applications in real world problems like optimization, computational problems, information theory, probability theory see [4–6] and references there in.

The following variant of Jensen's inequality has been given by Mercer [7]:

Theorem 1.1. For a convex function Υ on $[\mu, \nu]$, we have

$$\Upsilon\left(\mu + \nu - \sum_{i=1}^n \xi_i \omega_i\right) \leq \Upsilon(\mu) + \Upsilon(\nu) - \sum_{i=1}^n \xi_i \Upsilon(\omega_i), \quad (1.2)$$

$\forall \omega_i \in [\mu, \nu]$ and all $\xi_i \in [0, 1]$, $(i = 1, 2, \dots, n)$.

For recent results concerning Jensen-Mercer inequality and its applications, we refer to [8–11]. Fractional calculus is the generalization of the classical calculus and is effectively utilized in sciences, particularly in engineering. In most sections of the applied areas of the sciences, the classical calculus offers a powerful method for modeling and describing many significant dynamic processes. The main motivation behind the theory of Fractional calculus is to unify and generalize the integer-order differentiation and n-fold integration. Fractional analysis is an important part of applied sciences. Many results on the fractional models appeared in different areas of science. Fractional calculus is more equipped to deal with the time-dependent changes analyzed in real-world processes including fractional-order relaxation-oscillation model (and memristive chaotic circuit), mathematical biology and economics see [12, 13].

Due to the fact that the solutions obtained from the fractional integral and derivative operators are more closer to real world problems, these operators contribute to improve the relationships between mathematics and other disciplines. One can see the evolution of fractional integral and derivative operators time to time by looking at the few selected papers [14–17] and references there in. The most latest compact review about fractional calculus is by two eminent Professors D. Balenu and R. P. Agrwal in there review article “Fractional calculus in the sky” [18].

In recent years, fractional derivative and the corresponding integral operators have drawn attention of several researchers to solve many complex real world models in an intrinsic manner. The operators with nonsingular kernels are highly effective in solving non-locality of real world problems in an appropriate desired manner. Therefore, we recall the notion of the Caputo-Fabrizio (CF) derivative operator:

Definition 1.2. [19] Let $\Upsilon \in H^1(0, \nu)$, $\eta \in [0, 1]$ and $\nu > \mu$. Then the new CF derivative is defined as:

$${}^{CF}D^\eta \Upsilon(t) = \frac{M(\eta)}{1-\eta} \int_\mu^t \Upsilon'(s) \exp\left[-\frac{\eta}{(1-\eta)}(t-s)\right] ds, \quad (1.3)$$

where $M(\eta)$ denotes the normalization function.

Moreover, the corresponding CF fractional integral operator is given as:

Definition 1.3. [20] Let $\Upsilon \in H^1(0, \nu)$, $\nu > \mu$, $\eta \in [0, 1]$.

$$\left({}^{CF}I_\mu^\eta\right)(t) = \frac{1-\eta}{B(\eta)} \Upsilon(t) + \frac{\eta}{B(\eta)} \int_\mu^t \Upsilon(y) dy,$$

and

$$\left({}^{CF}I_\nu^\eta\right)(t) = \frac{1-\eta}{B(\eta)} \Upsilon(t) + \frac{\eta}{B(\eta)} \int_t^\nu \Upsilon(y) dy,$$

with normalization function $B(\eta)$.

The CF operator is a highly functional operator by definition, which is of significance importance in order to study discrete and continuous dynamical systems, physical phenomena, COVID-19 and highly epidemic models. Besides its usefulness in many ways, it has one drawback of not retrieving original function for $\eta = 1$. Recently, Atangana & Baleanu successfully proposed a new fractional derivative operator which also over comes this deficiency. They formulate a new fractional operator pertaining the Mittag-Leffler (ML) function in the kernel, that solve the problem of retrieving original function (a clear advantage on CF operator). In modeling physical and natural phenomena, ML function is found to be more suitable than power law. This made this operator more effective and helpful. As a result many researchers have shown keen interest in utilizing this operator [21–23]. Atangana & Baleanu introduced the derivative operator both in Caputo and Reimann-Liouville sense:

Definition 1.4. [24] Let $\nu > \mu$, $\eta \in [0, 1]$ and $\Upsilon \in H^1(\mu, \nu)$. The new fractional derivative is given as:

$${}^{ABC}D_t^\eta [\Upsilon(t)] = \frac{B(\eta)}{1-\eta} \int_\mu^t \Upsilon'(x) E_\eta \left[-\eta \frac{(t-x)^\eta}{(1-\eta)} \right] dx. \quad (1.4)$$

Definition 1.5. [24] Let $\Upsilon \in H^1(\mu, \nu)$, $\mu > \nu$, $\eta \in [0, 1]$. The new fractional derivative is given as:

$${}^{ABR}D_t^\eta [\Upsilon(t)] = \frac{B(\eta)}{1-\eta} \frac{d}{dt} \int_\mu^t \Upsilon(x) E_\eta \left[-\eta \frac{(t-x)^\eta}{(1-\eta)} \right] dx. \quad (1.5)$$

However in the same paper they give corresponding Atangana-Baleanu fractional integral operator as:

Definition 1.6. [24] Let $\Upsilon \in H^1(\mu, \nu)$, then its fractional integral operator with non-local kernel is defined as:

$${}^{AB}I_t^\eta \{\Upsilon(t)\} = \frac{1-\eta}{B(\eta)} \Upsilon(t) + \frac{\eta}{B(\eta)\Gamma(\eta)} \int_\mu^t \Upsilon(y)(t-y)^{\eta-1} dy,$$

where $\nu > \mu$, $\eta \in [0, 1]$.

Let $\Gamma(\eta)$ is the Gamma function, then the right hand side of AB-fractional integral operator [25] is

$${}_{\nu}^{AB}I_t^{\eta}\{\Upsilon(t)\} = \frac{1-\eta}{B(\eta)}\Upsilon(t) + \frac{\eta}{B(\eta)\Gamma(\eta)} \int_t^{\nu} \Upsilon(y)(y-t)^{\eta-1} dy.$$

The positivity of the normalization function $B(\eta)$ implies that for any positive function, its fractional AB-integral is positive. We obtain the classical integral and the initial function when $\eta \rightarrow 1$ and $\eta \rightarrow 0$ respectively. Some recent development in theory of integral inequalities involving AB operators can be seen in [26–28].

In this study, we present new versions of Hermite-Mercer type inequalities for convex functions through AB integral operators. The ultimate goal to employ AB operators is to obtain results which yield general inequalities of Hermite-Mercer type. The case $\alpha = 1$ in obtained results produce certain variants of the classical Hermite-Mercer type inequalities. At the end, the obtained results have been supported by reduced results and applications.

2. Hermite-Jensen-Mercer type inequalities for Atangana-Baleanu fractional integrals

By using JM inequalities, we can express the HH type inequalities in AB-fractional integrals form as follows:

Theorem 2.1. *For a positive convex function $\Upsilon : [\mu, \nu] \rightarrow \mathfrak{R}$ with $0 \leq \mu < \nu$ and $\Upsilon \in X_c(\mu, \nu)$, the inequalities for Atangana-Baleanu fractional integrals hold:*

$$\begin{aligned} & \frac{2}{B(\eta)\Gamma(\eta)}\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) \\ & \leq \frac{1}{(\kappa_2 - \kappa_1)^{\eta}} \left[\left\{ {}_{(\mu+\nu-\kappa_2)}^{AB}I_{(\mu+\nu-\kappa_1)}^{\eta} \Upsilon(\mu + \nu - \kappa_1) + {}_{(\mu+\nu-\kappa_1)}^{AB}I_{(\mu+\nu-\kappa_2)}^{\eta} \Upsilon(\mu + \nu - \kappa_2) \right\} \right. \\ & \quad \left. - \frac{1-\eta}{B(\eta)} \{ \Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \} \right] \\ & \leq \frac{1}{B(\eta)\Gamma(\eta)} (2[\Upsilon(\mu) + \Upsilon(\nu)] - [\Upsilon(\kappa_1) + \Upsilon(\kappa_2)]), \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} & \frac{1}{B(\eta)\Gamma(\eta)}\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) \\ & \leq \frac{1}{B(\eta)\Gamma(\eta)} (\Upsilon(\mu) + \Upsilon(\nu)) - \frac{1}{2(\kappa_2 - \kappa_1)^{\eta}} \left[\left\{ {}_{\kappa_1}^{AB}I_{\kappa_2}^{\eta} \Upsilon(\kappa_2) + {}_{\kappa_2}^{AB}I_{\kappa_1}^{\eta} \Upsilon(\kappa_1) \right\} \right. \\ & \quad \left. - \frac{1-\eta}{B(\eta)} \{ \Upsilon(\kappa_1) + \Upsilon(\kappa_2) \} \right] \\ & \leq \frac{1}{B(\eta)\Gamma(\eta)} \left[\Upsilon(\mu) + \Upsilon(\nu) - \Upsilon\left(\frac{\kappa_1 + \kappa_2}{2}\right) \right], \end{aligned} \quad (2.2)$$

for all $\kappa_1, \kappa_2 \in [\mu, \nu]$, $\eta > 0$.

Proof. As we know that Υ is convex on $[\mu, \nu]$, we obtain

$$2\Upsilon\left(\mu + \nu - \frac{u + v}{2}\right) \leq \Upsilon(\mu + \nu - u) + \Upsilon(\mu + \nu - v) \quad (2.3)$$

$$\forall u, v \in [\mu, \nu].$$

Replacing $u = \zeta\kappa_1 + (1 - \zeta)\kappa_2$ and $v = (1 - \zeta)\kappa_1 + \zeta\kappa_2$, $\forall \kappa_1, \kappa_2 \in [\mu, \nu]$ and $\zeta \in [0, 1]$ in (2.3), we have

$$2\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) \leq \Upsilon(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) + \Upsilon(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)).$$

Multiplying $\frac{\eta}{B(\eta)\Gamma(\eta)}\zeta^{\eta-1}$ on both sides and integrating the inequality w.r.t $\zeta \in [0, 1]$, we have

$$\begin{aligned} \frac{2}{B(\eta)\Gamma(\eta)}\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{\eta}{B(\eta)\Gamma(\eta)} \int_0^1 \zeta^{\eta-1} \Upsilon(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta \\ &\quad + \frac{\eta}{B(\eta)\Gamma(\eta)} \int_0^1 \zeta^{\eta-1} \Upsilon(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) d\zeta, \end{aligned}$$

$$\begin{aligned} \frac{2}{B(\eta)\Gamma(\eta)}\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{1}{(\kappa_2 - \kappa_1)^\eta} \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \Upsilon(\mu + \nu - \kappa_1) + {}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \Upsilon(\mu + \nu - \kappa_2) \right. \\ &\quad \left. - \frac{1 - \eta}{B(\eta)} \{ \Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \} \right], \end{aligned}$$

and we obtain the first inequality in (2.1).

Now, to prove inequality of (2.1), since Υ is convex on $[\mu, \nu]$, so for $\zeta \in [0, 1]$, it gives

$$\Upsilon(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \leq \Upsilon(\mu) + \Upsilon(\nu) - \zeta\Upsilon(\kappa_1) - (1 - \zeta)\Upsilon(\kappa_2), \quad (2.4)$$

and

$$\Upsilon(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) \leq \Upsilon(\mu) + \Upsilon(\nu) - (1 - \zeta)\Upsilon(\kappa_1) - \zeta\Upsilon(\kappa_2). \quad (2.5)$$

By adding (2.4) and (2.5) we get

$$\begin{aligned} &\Upsilon(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) + \Upsilon(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) \\ &\leq 2\Upsilon(\mu) + 2\Upsilon(\nu) - \zeta\Upsilon(\kappa_1) - (1 - \zeta)\Upsilon(\kappa_2) - (1 - \zeta)\Upsilon(\kappa_1) - \zeta\Upsilon(\kappa_2). \end{aligned}$$

Multiplying $\frac{\eta}{B(\eta)\Gamma(\eta)}\zeta^{\eta-1}$ on both sides and integrating w.r.t $\zeta \in [0, 1]$, we get the required inequality of (2.1). Next we prove inequality (2.2). Since Υ is convex, we have

$$\Upsilon\left(\mu + \nu - \frac{u + v}{2}\right) \leq \Upsilon(\mu) + \Upsilon(\nu) - \frac{\Upsilon(u) + \Upsilon(v)}{2}. \quad (2.6)$$

Let $\zeta \in [0, 1]$, by replacing $u = \zeta\kappa_1 + (1 - \zeta)\kappa_2$ and $v = (1 - \zeta)\kappa_1 + \zeta\kappa_2$ in (2.6) we obtain

$$\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) \leq \Upsilon(\mu) + \Upsilon(\nu) - \frac{\Upsilon(\zeta\kappa_1 + (1 - \zeta)\kappa_2) + \Upsilon(\zeta\kappa_2 + (1 - \zeta)\kappa_1)}{2}. \quad (2.7)$$

Multiplying $\frac{\eta}{B(\eta)\Gamma(\eta)}\zeta^{\eta-1}$ on both sides and integrating w.r.t ζ we get

$$\begin{aligned} & \frac{1}{B(\eta)\Gamma(\eta)}\Upsilon\left(\mu + \nu - \frac{\kappa_1 + \kappa_2}{2}\right) \\ & \leq \frac{1}{B(\eta)\Gamma(\eta)}(\Upsilon(\mu) + \Upsilon(\nu)) - \frac{1}{2(\kappa_2 - \kappa_1)^\eta} \left[\left\{ {}^{AB}I_{\kappa_2}^\eta \Upsilon(\kappa_2) + {}^{AB}I_{\kappa_1}^\eta \Upsilon(\kappa_1) \right\} \right. \\ & \quad \left. - \frac{1-\eta}{B(\eta)} \left\{ \Upsilon(\kappa_1) + \Upsilon(\kappa_2) \right\} \right], \end{aligned} \quad (2.8)$$

and we obtain the first inequality in (2.2). To prove the other inequality in (2.2), we observe that if Υ is convexity, then for $\zeta \in [0, 1]$,

$$\begin{aligned} \Upsilon\left(\frac{\kappa_1 + \kappa_2}{2}\right) &= \Upsilon\left(\frac{\zeta\kappa_1 + (1-\zeta)\kappa_2 + \zeta\kappa_2 + (1-\zeta)\kappa_1}{2}\right), \\ \Upsilon\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{1}{2}[\Upsilon(\zeta\kappa_1 + (1-\zeta)\kappa_2) + \Upsilon(\zeta\kappa_2 + (1-\zeta)\kappa_1)]. \end{aligned} \quad (2.9)$$

Multiplying both sides by $\frac{\eta}{B(\eta)\Gamma(\eta)}\zeta^{\eta-1}$ and integrating w.r.t $\zeta \in [0, 1]$, we get

$$\begin{aligned} & \frac{1}{B(\eta)\Gamma(\eta)}\Upsilon\left(\frac{\kappa_1 + \kappa_2}{2}\right) \\ & \leq \frac{1}{2(\kappa_2 - \kappa_1)^\eta} \left[\left\{ {}^{AB}I_{\kappa_2}^\eta \Upsilon(\kappa_2) + {}^{AB}I_{\kappa_1}^\eta \Upsilon(\kappa_1) \right\} - \frac{1-\eta}{B(\eta)} \left\{ \Upsilon(\kappa_1) + \Upsilon(\kappa_2) \right\} \right]. \end{aligned} \quad (2.10)$$

Multiplying (-1) by (2.10)

$$\begin{aligned} & -\frac{1}{B(\eta)\Gamma(\eta)}\Upsilon\left(\frac{\kappa_1 + \kappa_2}{2}\right) \\ & \geq -\frac{1}{2(\kappa_2 - \kappa_1)^\eta} \left[\left\{ {}^{AB}I_{\kappa_2}^\eta \Upsilon(\kappa_2) + {}^{AB}I_{\kappa_1}^\eta \Upsilon(\kappa_1) \right\} - \frac{1-\eta}{B(\eta)} \left\{ \Upsilon(\kappa_1) + \Upsilon(\kappa_2) \right\} \right]. \end{aligned}$$

By adding $\frac{1}{B(\eta)\Gamma(\eta)}(\Upsilon(\mu) + \Upsilon(\nu))$ in above, we get

$$\begin{aligned} & \frac{1}{B(\eta)\Gamma(\eta)}(\Upsilon(\mu) + \Upsilon(\nu)) - \frac{1}{2(\kappa_2 - \kappa_1)^\eta} \left[\left\{ {}^{AB}I_{\kappa_2}^\eta \Upsilon(\kappa_2) + {}^{AB}I_{\kappa_1}^\eta \Upsilon(\kappa_1) \right\} \right. \\ & \quad \left. - \frac{1-\eta}{B(\eta)} \left\{ \Upsilon(\kappa_1) + \Upsilon(\kappa_2) \right\} \right] \\ & \leq \frac{1}{B(\eta)\Gamma(\eta)} \left[\Upsilon(\mu) + \Upsilon(\nu) - \Upsilon\left(\frac{\kappa_1 + \kappa_2}{2}\right) \right]. \end{aligned}$$

Thus, the second inequality is proved. $\square$

Remark 2.1. If we choose $\eta = 1$ in Theorem 2.1 is proved by Kian and Moslehian in ([8], Theorem 2.1).

3. Main results

Lemma 3.1. For a differentiable Υ from $[\mu, \nu]$ to $\mathfrak{R}$ with $0 \leq \mu < \nu$, the equality for Atangana-Baleanu fractional integral holds:

$$\begin{aligned} & \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \\ & - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \\ & = \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 (\zeta^\eta - (1 - \zeta)^\eta) \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta. \end{aligned} \quad (3.1)$$

for all $\kappa_1, \kappa_2 \in [\mu, \nu]$, $\eta > 0$, $\zeta \in [0, 1]$.

Proof. Note that

$$\begin{aligned} K_1 &= \int_0^1 \zeta^\eta \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta \\ &= \frac{\Upsilon(\mu + \nu - \kappa_1)}{(\kappa_2 - \kappa_1)} - \frac{\eta}{\kappa_2 - \kappa_1} \int_0^1 \zeta^{\eta-1} \Upsilon(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta. \end{aligned}$$

and

$$\begin{aligned} K_2 &= \int_0^1 (1 - \zeta)^\eta \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta \\ &= - \frac{\Upsilon(\mu + \nu - \kappa_2)}{(\kappa_2 - \kappa_1)} + \frac{\eta}{\kappa_2 - \kappa_1} \int_0^1 (1 - \zeta)^{\eta-1} \Upsilon(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta. \end{aligned}$$

Now $K_1 - K_2$ and multiplying by $\frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)}$, using definition of the AB-fractional Integrals, we get (3.1). $\square$

Remark 3.1. If we choose $\eta = 1$ and $\kappa_1 = \mu$ and $\kappa_2 = \nu$ in Lemma (3.1), we get in ([29], Lemma 2.1).

Theorem 3.1. For a differentiable $\Upsilon : [\mu, \nu] \rightarrow \mathfrak{R}$ with $0 \leq \mu < \nu$ and $\Upsilon' \in L_1[\mu, \nu]$ such that $|\Upsilon'|$ is convex function on $[\mu, \nu]$, the inequality for Atangana-Baleanu fractional integrals holds:

$$\begin{aligned} & \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ & \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ & \leq \frac{2(\kappa_2 - \kappa_1)^{\eta+1}}{(\eta + 1)B(\eta)\Gamma(\eta)} \left(1 - \frac{1}{2^\eta} \right) \left\{ |\Upsilon'(\mu)| + |\Upsilon'(\nu)| - \left(\frac{|\Upsilon'(\kappa_1)| + |\Upsilon'(\kappa_2)|}{2} \right) \right\}. \end{aligned}$$

Proof. By Lemma 3.1 and using JM inequality, we have

$$\left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right|$$

$$\begin{aligned}
& - \left[\frac{{}^{AB}_{(\mu+\nu-\kappa_2)^+} I_{(\mu+\nu-\kappa_1)}^\eta \{f(\mu+\nu-\kappa_1)\} + \frac{{}^{AB}_{(\mu+\nu-\kappa_1)^-} I_{(\mu+\nu-\kappa_2)}^\eta \{f(\mu+\nu-\kappa_2)\}}{B(\eta)\Gamma(\eta)} \right] \\
& \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 |(\zeta^\eta - (1-\zeta)^\eta)| |\Upsilon'(\mu+\nu - (\zeta\kappa_1 + (1-\zeta)\kappa_2))| d\zeta \\
& \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 |\zeta^\eta - (1-\zeta)^\eta| \\
& \times \left\{ |\Upsilon'(\mu)| + |\Upsilon'(\nu)| - (\zeta |\Upsilon'(\kappa_1)| + (1-\zeta) |\Upsilon'(\kappa_2)|) \right\} d\zeta \\
& = \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} [I_1 + I_2].
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} ((1-\zeta)^\eta - \zeta^\eta) \\
&\times \left\{ |\Upsilon'(\mu)| + |\Upsilon'(\nu)| - (\zeta |\Upsilon'(\kappa_1)| + (1-\zeta) |\Upsilon'(\kappa_2)|) \right\} d\zeta \\
&= (|\Upsilon'(\mu)| + |\Upsilon'(\nu)|) \left(\frac{1}{\eta+1} - \frac{2^{-\eta}}{\eta+1} \right) \\
&- \left\{ |\Upsilon'(\kappa_1)| \left(\frac{1}{(\eta+1)(\eta+2)} - \frac{2^{-\eta-1}}{\eta+1} \right) + |\Upsilon'(\kappa_2)| \left(\frac{1}{(\eta+2)} - \frac{2^{-\eta-1}}{\eta+1} \right) \right\},
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^1 (\zeta^\eta - (1-\zeta)^\eta) \\
&\times \left\{ |\Upsilon'(\mu)| + |\Upsilon'(\nu)| - (\zeta |\Upsilon'(\kappa_1)| + (1-\zeta) |\Upsilon'(\kappa_2)|) \right\} d\zeta \\
&= (|\Upsilon'(\mu)| + |\Upsilon'(\nu)|) \left(\frac{1}{(\eta+1)} - \frac{2^{-\eta}}{\eta+1} \right) \\
&- \left\{ |\Upsilon'(\kappa_1)| \left(\frac{1}{(\eta+2)} - \frac{2^{-\eta-1}}{(\eta+1)} \right) + |\Upsilon'(\kappa_2)| \left(\frac{1}{(\eta+1)(\eta+2)} - \frac{2^{-\eta-1}}{(\eta+1)} \right) \right\}.
\end{aligned}$$

Placing the values of the evaluated integrals I_1 and I_2 gives require inequality. $\square$

Theorem 3.2. For a differentiable $\Upsilon : [\mu, \nu] \rightarrow \mathfrak{R}$ with $0 \leq \mu < \nu$ and $\Upsilon' \in L_1[\mu, \nu]$ such that $|\Upsilon'|^r$ is convex on $[\mu, \nu]$, the inequality for Atangana-Baleanu fractional integrals holds:

$$\begin{aligned}
& \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1-\eta}{B(\eta)} \right) [\Upsilon(\mu+\nu-\kappa_1) + \Upsilon(\mu+\nu-\kappa_2)] \right. \\
& \left. - \left[\frac{{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \{\Upsilon(\mu+\nu-\kappa_1)\} + \frac{{}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \{\Upsilon(\mu+\nu-\kappa_2)\}}{B(\eta)\Gamma(\eta)} \right] \right| \\
& \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \left(\frac{2-2^{1-\eta}}{(s\eta+1)} \right)^{\frac{1}{s}} \left\{ |\Upsilon'(\mu)|^r + |\Upsilon'(\nu)|^r - \left(\frac{|\Upsilon'(\kappa_1)|^r + |\Upsilon'(\kappa_2)|^r}{2} \right) \right\}^{\frac{1}{r}}, \quad (3.2)
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $\zeta \in [0, 1]$ and $\kappa_1, \kappa_2 \in [\mu, \nu]$.

Proof. By Lemma 3.1, we have

$$\begin{aligned} & \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ & \quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I^\eta_{(\mu+\nu-\kappa_1)} \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I^\eta_{(\mu+\nu-\kappa_2)} \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 |(\zeta^\eta - (1 - \zeta)^\eta)| \left| \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right| d\zeta. \end{aligned}$$

By applying Hölder's inequality with Jensen Mercer's inequality and convexity of $|\Upsilon'|^r$, we get

$$\begin{aligned} & \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ & \quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I^\eta_{(\mu+\nu-\kappa_1)} \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I^\eta_{(\mu+\nu-\kappa_2)} \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \left[\left(\int_0^1 |\zeta^\eta - (1 - \zeta)^\eta|^s d\zeta \right)^{\frac{1}{s}} \right. \\ & \quad \left. \left(\int_0^1 \left| \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right|^r d\zeta \right)^{\frac{1}{r}} \right]. \end{aligned}$$

Where

$$\begin{aligned} & \int_0^1 \left| \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right|^r d\zeta \\ & \leq \int_0^1 \left[\left| \Upsilon'(\mu) \right|^r + \left| \Upsilon'(\nu) \right|^r - \zeta \left| \Upsilon'(\kappa_1) \right|^r - (1 - \zeta) \left| \Upsilon'(\kappa_2) \right|^r \right] d\zeta. \end{aligned}$$

It can be easily notice that

$$\int_0^1 \zeta d\zeta = \frac{1}{2} = \int_0^1 (1 - \zeta) d\zeta.$$

After simplification, we get (3.2). $\square$

Theorem 3.3. For a differentiable $\Upsilon : [\mu, \nu] \rightarrow \mathfrak{R}$ with $0 \leq \mu < \nu$ and $\Upsilon' \in L_1[\mu, \nu]$ such that $|\Upsilon'|^r$ is convex on $[\mu, \nu]$, the inequality for Atangana-Baleanu fractional integrals holds:

$$\begin{aligned} & \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ & \quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I^\eta_{(\mu+\nu-\kappa_1)} \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I^\eta_{(\mu+\nu-\kappa_2)} \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \left\{ \left(\frac{2 - 2^{1-\eta}}{s(s\eta + 1)} \right) + \frac{\left| \Upsilon'(\mu) \right|^r + \left| \Upsilon'(\nu) \right|^r}{r} - \left(\frac{\left| \Upsilon'(\kappa_1) \right|^r + \left| \Upsilon'(\kappa_2) \right|^r}{2r} \right) \right\}. \end{aligned}$$

Where $\frac{1}{s} + \frac{1}{r} = 1$.

Proof. By Lemma 3.1, we have

$$\begin{aligned} & \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ & \quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I^\eta_{(\mu+\nu-\kappa_1)} \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I^\eta_{(\mu+\nu-\kappa_2)} \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 |(\zeta^\eta - (1 - \zeta)^\eta)| \left| \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right| d\zeta. \end{aligned}$$

By using Young's inequality $xy \leq \frac{1}{s}x^s + \frac{1}{r}y^r$.

$$\leq \frac{(\kappa_2 - \kappa_1)^{\eta+1}}{B(\eta)\Gamma(\eta)} \left[\frac{1}{s} \int_0^1 |(\zeta^\eta - (1 - \zeta)^\eta)|^s d\zeta + \frac{1}{r} \int_0^1 \left| \Upsilon'(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right|^r d\zeta \right].$$

After further simplifications, we get required result. $\square$

Lemma 3.2. Suppose that $\Upsilon : [\mu, \nu] \rightarrow \mathfrak{R}$ is a differentiable mapping on (μ, ν) with $0 \leq \mu < \nu$. If $\Upsilon'' \in L[\mu, \nu]$, then

$$\begin{aligned} & \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \\ & \quad - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I^\eta_{(\mu+\nu-\kappa_1)} \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I^\eta_{(\mu+\nu-\kappa_2)} \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \\ & = \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\int_0^1 [1 - \zeta^{\eta+1}] \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta \right. \\ & \quad \left. - \int_0^1 \zeta^{\eta+1} \Upsilon''(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) d\zeta \right], \end{aligned} \quad (3.3)$$

for all $\kappa_1, \kappa_2 \in [\mu, \nu]$, $\eta > 0$, $\zeta \in [0, 1]$.

Proof. It is easy to write that

$$\begin{aligned} & \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\int_0^1 [1 - \zeta^{\eta+1}] \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta \right. \\ & \quad \left. - \int_0^1 \zeta^{\eta+1} \Upsilon''(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) d\zeta \right] \end{aligned}$$

$$\frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} [K_1 - K_2]. \quad (3.4)$$

Where

$$K_1 = \int_0^1 [1 - \zeta^{\eta+1}] \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) d\zeta$$

$$\begin{aligned}
&= -\frac{\Upsilon'(\mu + \nu - \kappa_2)}{\kappa_2 - \kappa_1} + \frac{\eta + 1}{\kappa_2 - \kappa_1} \int_0^1 \zeta^\eta \Upsilon'(\mu + \nu - (\zeta \kappa_1 + (1 - \zeta) \kappa_2)) d\zeta \\
&= -\frac{\Upsilon'(\mu + \nu - \kappa_2)}{\kappa_2 - \kappa_1} + \frac{\eta + 1}{(\kappa_2 - \kappa_1)^2} \Upsilon(\mu + \nu - \kappa_1) \\
&\quad - \frac{\eta(\eta + 1)}{(\kappa_2 - \kappa_1)^{\eta+2}} \int_{\mu+\nu-\kappa_2}^{\mu+\nu-\kappa_1} [u - (\mu + \nu - \kappa_2)]^{\eta-1} \Upsilon(u) du,
\end{aligned}$$

and

$$\begin{aligned}
K_2 &= \int_0^1 \zeta^{\eta+1} \Upsilon''(\mu + \nu - ((1 - \zeta) \kappa_1 + \zeta \kappa_2)) d\zeta \\
&= -\frac{\Upsilon(\mu + \nu - \kappa_2)}{\kappa_2 - \kappa_1} + \frac{\eta + 1}{\kappa_2 - \kappa_1} \int_0^1 \zeta^\eta \Upsilon'(\mu + \nu - (\zeta \kappa_2 + (1 - \zeta) \kappa_1)) d\zeta \\
&= -\frac{\Upsilon'(\mu + \nu - \kappa_2)}{\kappa_2 - \kappa_1} - \frac{\eta + 1}{(\kappa_2 - \kappa_1)^2} \Upsilon(\mu + \nu - \kappa_2) \\
&\quad + \frac{\eta(\eta + 1)}{(\kappa_2 - \kappa_1)^{\eta+2}} \int_{\mu+\nu-\kappa_2}^{\mu+\nu-\kappa_1} [(\mu + \nu - \kappa_1) - u]^{\eta-1} \Upsilon(u) du.
\end{aligned}$$

Substituting the value of K_1 and K_2 in (3.4), we get (3.3). $\square$

Theorem 3.4. For a twice differentiable function Υ on $[\mu, \nu]$ such that $0 \leq \mu < \nu$ with $\Upsilon'' \in L_1[\mu, \nu]$ and $|\Upsilon''|$ is convex, then

$$\begin{aligned}
&\left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\
&\quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\
&\leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta + 2)} \left[\left\{ (|\Upsilon''(\mu)| + |\Upsilon''(\nu)|) \frac{1}{\eta + 2} - \left(\frac{|\Upsilon''(\kappa_1)|}{(\eta + 2)(\eta + 3)} + \frac{|\Upsilon''(\kappa_2)|}{\eta + 3} \right) \right\} \right. \\
&\quad \left. + \left\{ (|\Upsilon''(\mu)| + |\Upsilon''(\nu)|) \frac{\eta + 1}{\eta + 2} - \left(\frac{\eta + 1}{2(\eta + 3)} |\Upsilon''(\kappa_1)| + \frac{(\eta + 1)(\eta + 4)}{2(\eta + 2)(\eta + 3)} |\Upsilon''(\kappa_2)| \right) \right\} \right],
\end{aligned} \tag{3.5}$$

for all $\kappa_1, \kappa_2 \in [\mu, \nu]$, $\eta > 0$, $\zeta \in [0, 1]$ and $\Gamma(\cdot)$ is the Gamma function.

Proof. From the Lemma 3.2 and applying the modulus property along with Jensen-Mercer's inequality for the convex function $|\Upsilon''|$, we have

$$\begin{aligned}
&\left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\
&\quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\
&\leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta + 2)} \left[\int_0^1 \left| 1 - \zeta^{\eta+1} \right| \left| \Upsilon''(\mu + \nu - (\zeta \kappa_1 + (1 - \zeta) \kappa_2)) \right| d\zeta \right. \\
&\quad \left. + \int_0^1 \left| \zeta^{\eta+1} \right| \left| \Upsilon''(\mu + \nu - ((1 - \zeta) \kappa_1 + \zeta \kappa_2)) \right| d\zeta \right]
\end{aligned}$$

$$\leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\int_0^1 \left| 1 - \zeta^{\eta+1} \right| \left\{ \left| \Upsilon''(\mu) \right| + \left| \Upsilon''(\nu) \right| - \left(\zeta \left| \Upsilon''(\kappa_1) \right| + (1 - \zeta) \left| \Upsilon''(\kappa_2) \right| \right) \right\} d\zeta \right. \\ \left. + \left[\int_0^1 \left| \zeta^{\eta+1} \right| \left\{ \left| \Upsilon''(\mu) \right| + \left| \Upsilon''(\nu) \right| - \left((1 - \zeta) \left| \Upsilon''(\kappa_1) \right| + \zeta \left| \Upsilon''(\kappa_2) \right| \right) \right\} d\zeta \right]. \right.$$

After further simplifications, we get required result. $\square$

Corollary 3.1. *If we let $\eta = 1$ in Theorem 3.4, then we have the following inequality*

$$\left| \frac{\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2)}{2} - \frac{1}{\kappa_2 - \kappa_1} \int_{\mu + \nu - \kappa_2}^{\mu + \nu - \kappa_1} \Upsilon(u) du \right| \\ \leq \frac{(\kappa_2 - \kappa_1)^2}{4} \left[\left(\left| \Upsilon''(\mu) \right| + \left| \Upsilon''(\nu) \right| \right) - \left(\frac{1}{3} \left| \Upsilon''(\kappa_1) \right| + \frac{2}{3} \left| \Upsilon''(\kappa_2) \right| \right) \right].$$

Theorem 3.5. *For a twice differentiable function Υ on $[\mu, \nu]$ such that $0 \leq \mu < \nu$ with $\Upsilon'' \in L_1[\mu, \nu]$ and $|\Upsilon''|^r$ is convex with $r > 1$, then*

$$\left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ \left. - \left[{}^{AB}_{(\mu + \nu - \kappa_2)} I_{(\mu + \nu - \kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu + \nu - \kappa_1)} I_{(\mu + \nu - \kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\left(\frac{1}{(\eta+1)s+1} \right)^{\frac{1}{s}} \left(\left| \Upsilon''(\mu) \right|^r + \left| \Upsilon''(\nu) \right|^r - \left(\frac{\left| \Upsilon''(\kappa_1) \right|^r + \left| \Upsilon''(\kappa_2) \right|^r}{2} \right) \right)^{\frac{1}{r}} \right. \\ \left. + \left(\frac{(\eta+1)s}{(\eta+1)s+1} \right)^{\frac{1}{s}} \left(\left| \Upsilon''(\mu) \right|^r + \left| \Upsilon''(\nu) \right|^r - \left(\frac{\left| \Upsilon''(\kappa_1) \right|^r + \left| \Upsilon''(\kappa_2) \right|^r}{2} \right) \right)^{\frac{1}{r}} \right], \quad (3.6)$$

for all $\kappa_1, \kappa_2 \in [\mu, \nu]$, $\eta > 0$, $\zeta \in [0, 1]$ and $\Gamma(\cdot)$ is the Gamma function.

Where $\frac{1}{s} + \frac{1}{r} = 1$.

Proof. Using the Hölder's integral inequality along with Jensen-Mercer's inequality for $|\Upsilon''|^r$ in Lemma 3.2, we have

$$\left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\ \left. - \left[{}^{AB}_{(\mu + \nu - \kappa_2)} I_{(\mu + \nu - \kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu + \nu - \kappa_1)} I_{(\mu + \nu - \kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\ \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\int_0^1 \left| 1 - \zeta^{\eta+1} \right| \left| \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right| d\zeta \right. \\ \left. + \int_0^1 \left| \zeta^{\eta+1} \right| \left| \Upsilon''(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) \right| d\zeta \right] \\ \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\left(\int_0^1 \left| 1 - \zeta^{\eta+1} \right|^s d\zeta \right)^{\frac{1}{s}} \left(\int_0^1 \left| \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right|^r d\zeta \right)^{\frac{1}{r}} \right. \\ \left. + \left(\int_0^1 \left| \zeta^{\eta+1} \right|^s d\zeta \right)^{\frac{1}{s}} \left(\int_0^1 \left| \Upsilon''(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) \right|^r d\zeta \right)^{\frac{1}{r}} \right]$$

$$\begin{aligned}
& \times \left(\int_0^1 \left| \Upsilon''(\mu + \nu - (\zeta \kappa_2 + (1 - \zeta) \kappa_1)) \right|^r d\zeta \right)^{\frac{1}{r}} \\
& \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\left(\int_0^1 |1 - \zeta^{\eta+1}|^s d\zeta \right)^{\frac{1}{s}} \left\{ \left| \Upsilon''(\mu) \right|^r \int_0^1 d\zeta \right. \right. \\
& \quad + \left| \Upsilon''(\nu) \right|^r \int_0^1 d\zeta - \left| \Upsilon''(\kappa_2) \right|^r \int_0^1 (1 - \zeta) d\zeta - \left| \Upsilon''(\kappa_1) \right|^r \int_0^1 \zeta d\zeta \left. \right\}^{\frac{1}{r}} \\
& \quad + \left(\int_0^1 \zeta^{s(\eta+1)} d\zeta \right)^{\frac{1}{s}} \left\{ \left| \Upsilon''(\mu) \right|^r + \left| \Upsilon''(\nu) \right|^r \right. \\
& \quad \left. - \left| \Upsilon''(\kappa_2) \right|^r \int_0^1 \zeta d\zeta - \left| \Upsilon''(\kappa_1) \right|^r \int_0^1 (1 - \zeta) d\zeta \right\}^{\frac{1}{r}} \Big].
\end{aligned}$$

After simplification we get 3.6. □

Corollary 3.2. *If we let $\eta = 1$ in Theorem 3.5, then we have the following inequality*

$$\begin{aligned}
& \left| \frac{\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2)}{2} - \frac{1}{\kappa_2 - \kappa_1} \int_{\mu+\nu-\kappa_2}^{\mu+\nu-\kappa_1} \Upsilon(u) du \right| \\
& \leq \frac{(\kappa_2 - \kappa_1)^2}{4} \left[\left(\frac{1}{2s+1} \right)^{\frac{1}{s}} \left(\left| \Upsilon''(\mu) \right|^r + \left| \Upsilon''(\nu) \right|^r - \left(\frac{\left| \Upsilon''(\kappa_1) \right|^r + \left| \Upsilon''(\kappa_2) \right|^r}{2} \right) \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \left(\frac{2s}{2s+1} \right)^{\frac{1}{s}} \left(\left| \Upsilon''(\mu) \right|^r + \left| \Upsilon''(\nu) \right|^r - \left(\frac{\left| \Upsilon''(\kappa_1) \right|^r + \left| \Upsilon''(\kappa_2) \right|^r}{2} \right) \right)^{\frac{1}{r}} \right].
\end{aligned}$$

Theorem 3.6. *For a twice differentiable function Υ on $[\mu, \nu]$ such that $0 \leq \mu < \nu$ with $\Upsilon'' \in L_1[\mu, \nu]$ and $|\Upsilon''|^r$ is convex, then*

$$\begin{aligned}
& \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\
& \quad \left. - \left[{}^{AB}_{(\mu+\nu-\kappa_2)} I_{(\mu+\nu-\kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + {}^{AB}_{(\mu+\nu-\kappa_1)} I_{(\mu+\nu-\kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\
& \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\left(\frac{\eta+1}{\eta+2} \right)^{1-\frac{1}{r}} \left(\frac{\eta+1}{\eta+2} \left| \Upsilon''(\mu) \right|^r \right. \right. \\
& \quad + \frac{\eta+1}{\eta+2} \left| \Upsilon''(\nu) \right|^r - \left(\frac{(\eta+1)(\eta+4)}{2(\eta+3)(\eta+2)} \right) \left| \Upsilon''(\kappa_2) \right|^r - \left(\frac{\eta+1}{2(\eta+3)} \right) \left| \Upsilon''(\kappa_1) \right|^r \left. \right)^{\frac{1}{r}} \\
& \quad + \left(\frac{1}{\eta+2} \right)^{1-\frac{1}{r}} \left(\frac{1}{\eta+2} \left| \Upsilon''(\mu) \right|^r \right. \\
& \quad \left. + \frac{1}{\eta+2} \left| \Upsilon''(\nu) \right|^r - \frac{1}{\eta+3} \left| \Upsilon''(\kappa_2) \right|^r - \frac{1}{(\eta+2)(\eta+3)} \left| \Upsilon''(\kappa_1) \right|^r \right)^{\frac{1}{r}} \Big], \tag{3.7}
\end{aligned}$$

for all $\kappa_1, \kappa_2 \in [\mu, \nu]$, $\eta > 0$, $\zeta \in [0, 1]$ and $\Gamma(\cdot)$ is the Gamma function. Where $\frac{1}{s} + \frac{1}{r} = 1$.

Proof. Applying the inequality along with Jensen-Mercer's inequality for $|\Upsilon''|^r$ in Lemma 3.2, we have

$$\begin{aligned}
& \left| \left(\frac{(\kappa_2 - \kappa_1)^\eta}{B(\eta)\Gamma(\eta)} + \frac{1 - \eta}{B(\eta)} \right) \left[\Upsilon(\mu + \nu - \kappa_1) + \Upsilon(\mu + \nu - \kappa_2) \right] \right. \\
& \quad \left. - \left[\frac{{}^{AB}}{(\mu + \nu - \kappa_2)} I_{(\mu + \nu - \kappa_1)}^\eta \{ \Upsilon(\mu + \nu - \kappa_1) \} + \frac{{}^{AB}}{(\mu + \nu - \kappa_1)} I_{(\mu + \nu - \kappa_2)}^\eta \{ \Upsilon(\mu + \nu - \kappa_2) \} \right] \right| \\
& \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\int_0^1 \left| 1 - \zeta^{\eta+1} \right| \left| \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right| d\zeta \right. \\
& \quad \left. + \int_0^1 \left| \zeta^{\eta+1} \right| \left| \Upsilon''(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) \right| d\zeta \right] \\
& \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\left(\int_0^1 [1 - \zeta^{\eta+1}] d\zeta \right)^{1-\frac{1}{r}} \left(\int_0^1 [1 - \zeta^{\eta+1}] \left| \Upsilon''(\mu + \nu - (\zeta\kappa_1 + (1 - \zeta)\kappa_2)) \right|^r d\zeta \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \left(\int_0^1 \zeta^{\eta+1} d\zeta \right)^{1-\frac{1}{r}} \left(\int_0^1 \zeta^{\eta+1} \left| \Upsilon''(\mu + \nu - ((1 - \zeta)\kappa_1 + \zeta\kappa_2)) \right|^r d\zeta \right)^{\frac{1}{r}} \right] \\
& \leq \frac{\eta(\kappa_2 - \kappa_1)^{\eta+2}}{B(\eta)\Gamma(\eta+2)} \left[\left(\int_0^1 [1 - \zeta^{\eta+1}] d\zeta \right)^{1-\frac{1}{r}} \left\{ \left| \Upsilon''(\mu) \right|^r \int_0^1 [1 - \zeta^{\eta+1}] d\zeta \right. \right. \\
& \quad \left. + \left| \Upsilon''(\nu) \right|^r \int_0^1 [1 - \zeta^{\eta+1}] d\zeta - \left| \Upsilon''(\kappa_2) \right|^r \int_0^1 [1 - \zeta^{\eta+1}] (1 - \zeta) d\zeta \right. \\
& \quad \left. - \left| \Upsilon''(\kappa_1) \right|^r \int_0^1 [1 - \zeta^{\eta+1}] \zeta d\zeta \right\}^{\frac{1}{r}} \\
& \quad \left. + \left(\int_0^1 \zeta^{\eta+1} d\zeta \right)^{1-\frac{1}{r}} \left\{ \left| \Upsilon''(\mu) \right|^r \int_0^1 \zeta^{\eta+1} d\zeta + \left| \Upsilon''(\nu) \right|^r \int_0^1 \zeta^{\eta+1} d\zeta \right. \right. \\
& \quad \left. - \left| \Upsilon''(\kappa_2) \right|^r \int_0^1 \zeta^{\eta+1} \zeta d\zeta - \left| \Upsilon''(\kappa_1) \right|^r \int_0^1 \zeta^{\eta+1} (1 - \zeta) d\zeta \right\}^{\frac{1}{r}} \right].
\end{aligned}$$

After simplification, we get inequality 3.7. □

4. Applications

Consider the following two special means for $\kappa_1 \neq \kappa_2$, with $0 < \kappa_1 < \kappa_2$.

The arithmetic mean:

$$A(\kappa_1, \kappa_2) = \frac{\kappa_1 + \kappa_2}{2}.$$

The Harmonic mean:

$$H(\kappa_1, \kappa_2) = \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}.$$

The logarithmic-mean:

$$L(\kappa_1, \kappa_2) = \frac{\kappa_2 - \kappa_1}{\log \kappa_2 - \log \kappa_1}.$$

The generalized logarithmic-mean:

$$L_m(\kappa_1, \kappa_2) = \left[\frac{\kappa_2^{m+1} - \kappa_1^{m+1}}{(m+1)(\kappa_2 - \kappa_1)} \right]^{\frac{1}{m}}; \quad m \in \mathbb{R} \setminus \{-1, 0\}.$$

Proposition 4.1. Suppose $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $0 < \kappa_1 < \kappa_2$, 0 does not belongs to $[\kappa_1, \kappa_2]$ and $m \in \mathbb{Z}$, $|m| \geq 2$. Then the following inequality

$$\begin{aligned} & \left| A((\mu + \nu - \kappa_1)^m, (\mu + \nu - \kappa_2)^m) - L_m^m(\mu + \nu - \kappa_1, \mu + \nu - \kappa_2) \right| \\ & \leq \frac{m(m-1)}{4} (\kappa_2 - \kappa_1)^2 \left[2A(|\mu|^{m-2}, |\nu|^{m-2}) - \frac{1}{3} (|\kappa_1|^{m-2} + 2|\kappa_2|^{m-2}) \right]. \end{aligned}$$

Proof. Applying the Corollary 3.1 for the $\Upsilon(x) = x^m$. one can obtain the result immediately. $\square$

Proposition 4.2. Suppose $\kappa_1, \kappa_2 \in \mathbb{R} \setminus 0$ such that $0 < \kappa_1 < \kappa_2$, 0 does not belongs to $[\kappa_1, \kappa_2]$. Then the following inequality

$$\begin{aligned} & \left| H^{-1}(\mu + \nu - \kappa_1, \mu + \nu - \kappa_2) - L^{-1}(\mu + \nu - \kappa_1, \mu + \nu - \kappa_2) \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)^2}{4} \left[4A(|\mu|^{-3}, |\nu|^{-3}) - \frac{1}{3} (2|\kappa_1|^{-3} + 4|\kappa_2|^{-3}) \right]. \end{aligned}$$

Proof. Applying the Corollary 3.1 for the $\Upsilon(x) = \frac{1}{x}$. one can obtain the result immediately. $\square$

Proposition 4.3. Suppose $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $0 < \kappa_1 < \kappa_2$, 0 does not belongs to $[\kappa_1, \kappa_2]$ and $m \in \mathbb{Z}$, $|m| \geq 2$ with $r > 1$. Then the following inequality

$$\begin{aligned} & \left| A((\mu + \nu - \kappa_1)^m, (\mu + \nu - \kappa_2)^m) - L_m^m(\mu + \nu - \kappa_1, \mu + \nu - \kappa_2) \right| \\ & \leq \frac{m(m-1)(\kappa_2 - \kappa_1)^2}{4} \left[\left(\frac{1}{2s+1} \right)^{\frac{1}{s}} \left\{ 2A(|\mu|^{m-2}|^r, |\nu|^{m-2}|^r) - A(|\kappa_1|^{m-2}|^r, |\kappa_2|^{m-2}|^r) \right\}^{\frac{1}{r}} \right. \\ & \quad \left. + \left(\frac{2s}{2s+1} \right)^{\frac{1}{s}} \left\{ 2A(|\mu|^{m-2}|^r, |\nu|^{m-2}|^r) - A(|\kappa_1|^{m-2}|^r, |\kappa_2|^{m-2}|^r) \right\}^{\frac{1}{r}} \right]. \end{aligned}$$

Proof. Applying the Corollary 3.2 for the $\Upsilon(x) = x^m$. one can obtain the result immediately. $\square$

4.1. q-digamma function

Suppose $0 < \mathbf{q} < 1$, the $\mathbf{q}$ -digamma function $\varphi_{\mathbf{q}}$, is the $\mathbf{q}$ -analogue of the digamma function φ defined by (see in [30]).

$$\begin{aligned} \varphi_{\mathbf{q}} &= -\ln(1 - \mathbf{q}) + \ln \mathbf{q} \sum_{j=0}^{\infty} \frac{\mathbf{q}^{j+v}}{1 - \mathbf{q}^{j+v}} \\ &= -\ln(1 - \mathbf{q}) + \ln \mathbf{q} \sum_{j=0}^{\infty} \frac{\mathbf{q}^{jv}}{1 - \mathbf{q}^{jv}}. \end{aligned}$$

For $\mathbf{q} > 1$ and $v > 0$, $\mathbf{q}$ -digamma function $\varphi_{\mathbf{q}}$ defined by

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(\mathbf{q} - 1) + \ln \mathbf{q} \left[v - \frac{1}{2} - \sum_{j=0}^{\infty} \frac{\mathbf{q}^{-(j+v)}}{1 - \mathbf{q}^{-(j+v)}} \right] \\ &= -\ln(\mathbf{q} - 1) + \ln \mathbf{q} \left[v - \frac{1}{2} - \sum_{j=0}^{\infty} \frac{\mathbf{q}^{-jv}}{1 - \mathbf{q}^{-jv}} \right].\end{aligned}$$

Proposition 4.4. Let $0 < \mathbf{q} < 1$ and $\mu, \nu, \mathbf{q}, r, s$ be real numbers with $0 < \mu < \nu, r > 1$. Then

$$\begin{aligned}& \left| \frac{\varphi'_{\mathbf{q}}(\mu + \nu - \kappa_1) + \varphi'_{\mathbf{q}}(\mu + \nu - \kappa_2)}{2} - \frac{\varphi_{\mathbf{q}}(\mu + \nu - \kappa_1) + \varphi_{\mathbf{q}}(\mu + \nu - \kappa_2)}{\kappa_2 - \kappa_1} \right| \\ & \leq \frac{(\kappa_2 - \kappa_1)^2}{4} \left[\left(\frac{1}{2s+1} \right)^{\frac{1}{s}} \left(|\varphi_{\mathbf{q}}^{(3)}(\mu)|^r + |\varphi_{\mathbf{q}}^{(3)}(\nu)|^r - \left(\frac{|\varphi_{\mathbf{q}}^{(3)}(\kappa_1)|^r + |\varphi_{\mathbf{q}}^{(3)}(\kappa_2)|^r}{2} \right)^{\frac{1}{r}} \right) \right. \\ & \quad \left. + \left(\frac{2s}{2s+1} \right)^{\frac{1}{s}} \left(|\varphi_{\mathbf{q}}^{(3)}(\mu)|^r + |\varphi_{\mathbf{q}}^{(3)}(\nu)|^r - \left(\frac{|\varphi_{\mathbf{q}}^{(3)}(\kappa_1)|^r + |\varphi_{\mathbf{q}}^{(3)}(\kappa_2)|^r}{2} \right)^{\frac{1}{r}} \right) \right].\end{aligned}$$

holds true for all $\nu > 0$ and $s = \frac{r}{r-1}$.

Proof. Observe that the $\mathbf{q}$ -trigamma function $\Upsilon(\nu) = \varphi'_{\mathbf{q}}(\nu)$ is completely monotonically on $(0, \infty)$. That implies the function $\varphi_{\mathbf{q}}^{(3)}(\nu)$ is also completely monotonically on $(0, \infty)$ for each $\mathbf{q} \in (0, 1)$. (see in [31], p. 167). The result follows from Corollary 3.2. $\square$

5. Conclusions

The study dealt with investigating new Hermite-Hadamard-Mercer inequalities for AB-fractional integral operators. Several new integral identities are obtained involving first and second-order derivatives. Using new fractional operators, some achievements have been obtained in the study of Hermite-Hadamard type integral inequalities. We also extend the study of Hermite-Hadamard-Mercer type inequalities via AB-fractional integral operators for differentiable mapping whose derivatives in the absolute values are convex. Finally, we have applied our findings to give estimations of inequalities pertaining to special functions. All these integral inequalities are open to being investigated for other classes of generalized convex functions like s -convex, Strong convex and F -convex functions.

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Conflict of interest

The authors declare no conflict of interest.

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