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Research article

Varieties of a class of elementary subalgebras

Yang Pan[1] **and Yanyong Hong**[2,*]

[1] College of Artificial Intelligence and Big Data, Hefei University, Hefei 230601, China
[2] School of Science, Hangzhou Normal University, Hangzhou 310036, China

* **Correspondence:** Email: hongyanyong2008@yahoo.com.

Abstract: Let G be a connected standard simple algebraic group of type C or D over an algebraically closed field $\Bbbk$ of positive characteristic $p > 0$, and $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G. Motivated by the variety of $\mathbb{E}(r, \mathfrak{g})$ of r-dimensional elementary subalgebras of a restricted Lie algebra $\mathfrak{g}$, in this paper we characterize the irreducible components of $\mathbb{E}(\mathrm{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$ where the p-rank $\mathrm{rk}_p(\mathfrak{g})$ is defined to be the maximal dimension of an elementary subalgebra of $\mathfrak{g}$.

Keywords: elementary subalgebras; commuting roots; irreducible components
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1. Introduction

Let $(\mathfrak{g}, [p])$ be a finite dimensional restricted Lie algebra over an algebraically closed field $\Bbbk$ of positive characteristic $p > 0$. Following [3] we say that $\mathfrak{g}$ is *elementary*, provided $\mathfrak{g}$ is abelian and $[p] = 0$. Given a positive integer r, there is a mount of literatures investigating the set $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of dimension r. For instance, the very original paper [3] studies the geometric properties of $\mathbb{E}(r, \mathfrak{g})$, which shows that it is a projective variety. The structure of $\mathbb{E}(r, \mathfrak{g})$ is described there for simple algebraic Lie algebras of types A, C and $r = \mathrm{rk}_p(\mathfrak{g})$ being the p-rank of $\mathfrak{g}$. For a finite dimensional restricted Lie algebra $\mathfrak{g}$, the p-rank $\mathrm{rk}_p(\mathfrak{g})$ is defined as follows

$$\mathrm{rk}_p(\mathfrak{g}) := \max\{\, r \in \mathbb{N}_0 \,;\, \mathbb{E}(r, \mathfrak{g}) \neq \emptyset \,\}.$$

Later on, the first author explores the irreducible components of the variety $\mathbb{E}(r, \mathfrak{g})$ in [6] for simple algebraic Lie algebras of type A when r equals $\mathrm{rk}_p(\mathfrak{g}) - 1$.

We now assume that G is a simple algebraic $\Bbbk$-group with irreducible root system Φ. The interested reader may consult [1, 2, 4, 10] for the theory of algebraic groups. Let $\Delta := \{\, \alpha_1, \ldots, \alpha_n \,\}$ be the set of positive simple roots. For any $I \subset \Delta$ define the parabolic subgroup W_I and its corresponding root system Φ_I with Φ_I^+ being the set of positive roots. We recall two definitions in [9].

Definition 1. *We set* $S := \Delta \setminus I$ *and then define*

$$\Phi_S^{\mathrm{rad}} = \Phi^+ \setminus \Phi_I^+$$

to be the set of positive roots that cannot be written as a linear combination of the simple roots not in S*. If* $S = \{\alpha_i\}$*, then we simply write* Φ_i^{rad} *instead of* $\Phi_{\{\alpha_i\}}^{\mathrm{rad}}$*.*

Definition 2. *Let* α *and* β *be two roots of* Φ*. We say that* α *and* β *commute if* $\alpha + \beta$ *is not a root.*

Building on methods developed in [5, 7, 9], we find that the maximal elementary subalgebras of dimension $\mathrm{rk}_p(\mathfrak{g})$ are given by the combinatorics of the commuting roots of Φ. It is our aim in this paper to present the method of finding the maximal subsets of commuting positive roots of order $r_{\mathrm{smax}} := \mathrm{rk}_p(\mathfrak{g}) - 1$ for types C and D, and the geometric properties of the varieties $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$ for these two types. Let Φ be the root system of type C or D respectively, and $\mathrm{Max}(r, \Phi^+)$ be the set of maximal subsets of commuting positive roots of order r. We refer to [5] for his linear algebraic approach to sets of commuting roots for irreducible root systems, which enables our set $\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ to be more tractable. We compute $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$ under the assumption that G is standard, which means the derived subgroup of G is simply connected, p is a good prime for G and the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ admits a non-degenerate G-invariant symmetric bilinear form. We rely on the result of Premet (see Lemma 2.2, [8]) to show that any elementary subalgebras of $\mathfrak{g}$ can be conjugated into $\mathfrak{u} \subset \mathfrak{g}$, the Lie algebra of the unipotent radical U of the Borel subgroup $B \leq G$. We define $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$ as the set of maximal elementary subalgebras of dimension r_{smax} in $\mathfrak{u}$. The calculation of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$ then proceeds via two steps. First, we determine $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$ as a set. We define a map $\mathrm{Lie} : \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+) \to \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$ which sends a maximal subset of commuting positive roots of order r_{smax} to a maximal elementary subalgebra of dimension r_{smax} in $\mathfrak{u}$ and show that there is an inverse map $\mathrm{LT} : \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max} \to \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ which splits Lie. The map Lie is not necessarily surjective but we show that for types C and D except for small ranks it is surjective up to conjugation by U. Further, after giving the definition of ideals of the root system Φ, we effectively prove that the maximal elementary subalgebras of dimension r_{smax} in $\mathfrak{u}$ up to conjugation by G are given by the ideals of Φ. To finish the calculation of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$, we allow actions by the Weyl group $\mathscr{W} \leq G$ and determine the irreducible components of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$.

This paper is organized as follows: In section 2, we determine the set $\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ for types $C_n (n \geq 3)$ and $D_n (n \geq 5)$. Section 3 deals with the surjection of the map Lie up to conjugation by G for types C and D. Finally in section 4 we give the irreducible components of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$.

2. Maximal subsets of commuting positive roots

Throughout this section, if M and N are two subsets of Φ, we will use the symbol $[M, N] = 0$ to denote that roots in M and N commute. Let $r_{\max}$ be the maximal order of a subset of commuting positive roots. The result in [9] tells us that $\mathrm{rk}_p(\mathfrak{g})$ and $r_{\max}$ are equal. Since the p-rank $\mathrm{rk}_p(\mathfrak{g})$ should be clear for our consideration in this section, we list the related facts through Table 1.

Table 1. Maximal sets of commuting roots.

Type T	Rank	$\mathrm{Max}(r_{\max}, \Phi^+)$	$r_{\max}$
C_n	$n \geq 3$	Φ_n^{rad}	$\frac{1}{2}n(n+1)$
D_n	$n = 4$	$\Phi_1^{\mathrm{rad}}, \Phi_3^{\mathrm{rad}}, \Phi_4^{\mathrm{rad}}$	6
	$n \geq 5$	$\Phi_{n-1}^{\mathrm{rad}}, \Phi_n^{\mathrm{rad}}$	$\frac{1}{2}n(n-1)$

2.1. Type $C_n, n \geq 3$

Let G be a simple algebraic group with root system Φ which is of type C_n ($n \geq 3$), and $\mathfrak{g} :=$ Lie(G). Let $\Phi^+ = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq n \}$ be the set of positive roots and $\Phi = \Phi^+ \cup -\Phi^+$. Setting $\alpha_i = \epsilon_i - \epsilon_{i+1}$, where $1 \leq i \leq n-1$ and $\alpha_n = 2\epsilon_n$. It follows that $\Delta = \{ \alpha_1, \ldots, \alpha_n \}$ is a base of Φ.

Theorem 2.1. *Let Φ be of type C_n. The set* $\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ *consists only one element*

$$\Phi_{n-1,n}^C := \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j < n\} \cup \{ \epsilon_r - \epsilon_n \mid 1 \leq r < n \}.$$

Proof. Let J be a subset of $\{1, \ldots, n\}$ and $J' := \{1, \ldots, n\} \setminus J$ be its complement. A maximal subset of commuting roots of Φ is uniquely given by (see [9, A.3])

$$\phi(J) := \{ \epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j, -\epsilon_j - \epsilon_{j'} \mid i, i' \in J \text{ and } j, j' \in J' \}.$$

Notice that $\phi(J) \subset \Phi^+$ if and only if $J = \{ 1, \ldots, n \}$. It follows that

$$\Phi_n^{\mathrm{rad}} = \phi(\{1, \ldots, n\})$$

of order $\frac{1}{2}n(n+1)$ is the unique element of $\mathrm{Max}(r_{\max}, \Phi^+)$.

Let $M(C)$ be an element of $\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$. The fact above implies that $M(C)$ cannot be a maximal subset of commuting roots of Φ. It asserts that $M(C) \subsetneq \phi(J_0)$ for some J_0, and $|\phi(J_0)| = |M(C)| + 1$. We conclude that $J_0 = \{ 1, \ldots, n-1 \}$ and

$$M(C) = \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j < n\} \cup \{ \epsilon_r - \epsilon_n \mid 1 \leq r < n \}$$

which will be denoted by $\Phi_{n-1,n}^C$. □

2.2. Type $D_n, n \geq 4$

Let G be a simple algebraic group with root system Φ which is of type $D_n (n \geq 4)$, and $\mathfrak{g} :=$ Lie(G). Let

$$\Phi^+ = \{ \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n \}$$

be the set of positive roots of Φ. Defining $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$ together with $\alpha_n = \epsilon_{n-1} + \epsilon_n$, then $\Delta = \{ \alpha_1, \ldots, \alpha_n \}$ is a base of Φ.

Definition 3. *We define* $\Phi^{\text{rad}}_{\alpha_1,\alpha_2}$ *as a subset of* Φ *consisting of roots*

$$\epsilon_1 \pm \epsilon_i, \epsilon_2 \pm \epsilon_j, \textit{where } 2 \le i \le n \textit{ and } 3 \le j \le n.$$

Proposition 2.2. *Let* $\mathcal{R} \subset \Phi^{\text{rad}}_{\alpha_1,\alpha_2}$ *be a subset of commuting positive roots. We either have*

(1) $\mathcal{R} \subset \Phi^{\text{rad}}_1$*, or*
(2) $\mathcal{R} \subset \mathcal{S}_{ab} := \{ \epsilon_1 + \epsilon_2 \} \cup \mathcal{S}_a \cup \mathcal{S}_b$.

where $\mathcal{S}_a \subset \{ \epsilon_1 - \epsilon_r, \epsilon_2 + \epsilon_r \mid 3 \le r \le n \}$ *is a maximal subset having the property* $\epsilon_1 - \epsilon_r \in \mathcal{S}_a$ *if and only if* $\epsilon_2 + \epsilon_r \notin \mathcal{S}_a$ *and* $\mathcal{S}_b \subset \{ \epsilon_1 + \epsilon_r, \epsilon_2 - \epsilon_r \mid 3 \le r \le n \}$ *is a maximal subset with the property* $\epsilon_1 + \epsilon_r \in \mathcal{S}_b$ *if and only if* $\epsilon_2 - \epsilon_r \notin \mathcal{S}_b$.

Proof. If $\epsilon_1 - \epsilon_2 \in \mathcal{R}$, then $\epsilon_2 \pm \epsilon_j \notin \mathcal{R}$ for $3 \le j \le n$, which implies the inclusion $\mathcal{R} \subset \Phi^{\text{rad}}_1 = \{ \epsilon_1 \pm \epsilon_i \}_{2 \le i \le n}$. Alternatively, $\epsilon_1 - \epsilon_2 \notin \mathcal{R}$, it follows that $\mathcal{R} \subset \mathcal{S}_{ab}$. □

Notation 2.3. *We make a restriction on the rank by letting* $n = 4$*. There are four possibilities for* $\mathcal{S}_a$*, denoted by*

(1) $\mathcal{S}^1_a = \{ \epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4 \}$,
(2) $\mathcal{S}^2_a = \{ \epsilon_1 - \epsilon_3, \epsilon_2 + \epsilon_4 \}$,
(3) $\mathcal{S}^3_a = \{ \epsilon_2 + \epsilon_3, \epsilon_2 + \epsilon_4 \}$,
(4) $\mathcal{S}^4_a = \{ \epsilon_1 - \epsilon_4, \epsilon_2 + \epsilon_3 \}$.

Similarly, $\mathcal{S}_b$ *has the following four forms*

(1) $\mathcal{S}^1_b = \{ \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4 \}$,
(2) $\mathcal{S}^2_b = \{ \epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_4 \}$,
(3) $\mathcal{S}^3_b = \{ \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4 \}$,
(4) $\mathcal{S}^4_b = \{ \epsilon_1 + \epsilon_4, \epsilon_2 - \epsilon_3 \}$.

Lemma 2.4. *Let* Φ *be of type* D_4*. Then the elements of* $\text{Max}(5, \Phi^+)$ *are*

(1) $\mathcal{S}^{ij}_{ab} := \{ \epsilon_1 + \epsilon_2 \} \cup \mathcal{S}^i_a \cup \mathcal{S}^j_b$ *where* $1 \le i, j \le 4$ *and* $(i, j) \ne (1, 1), (3, 1), (4, 1)$,
(2) $\mathcal{S}_1 := \{ \epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_3 - \epsilon_4 \}$,
(3) $\mathcal{S}_2 := \{ \epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4, \epsilon_3 + \epsilon_4 \}$,
(4) $\mathcal{S}_3 := \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_3 \pm \epsilon_4\}$,
(5) $\mathcal{S}_4 := \{ \epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3, \epsilon_3 \pm \epsilon_4 \}$.

Proof. It is known that $r_{\text{smax}} = \text{rk}_p(\mathfrak{g}) - 1 = 5$ for D_4. Let $\Psi^+ = \Phi^+ \setminus \Phi^{\text{rad}}_{\alpha_1,\alpha_2} = \{ \epsilon_3 \pm \epsilon_4 \}$ be the complement of $\Phi^{\text{rad}}_{\alpha_1,\alpha_2}$ in Φ^+, and $M(D) \in \text{Max}(5, \Phi^+)$. Assume that $M(D) = M_a \cup M_b$, where $M_a \subset \Psi^+$ and $M_b \subset \Phi^{\text{rad}}_{\alpha_1,\alpha_2}$.

If $M_a = \emptyset$, then $|M_b| = 5$. By Proposition 2.2, $M(D) \subset \mathcal{S}_{ab}$. Since $M(D)$ is maximal, according to Notation 2.3, we have

$$M(D) = \mathcal{S}^{ij}_{ab} := \{ \epsilon_1 + \epsilon_2 \} \cup \mathcal{S}^i_a \cup \mathcal{S}^j_b$$

where $1 \le i, j \le 4$ and $(i, j) \ne (1, 1), (3, 1), (4, 1)$.

If $|M_a| = 1$, then $|M_b| = 4$. Since $M_a \subset \Psi^+$, it follows that $M_b \subsetneq \Phi_1^{\text{rad}}$, $M_b \subsetneq \{\epsilon_1 + \epsilon_2\} \cup \mathcal{S}_a^4 \cup \mathcal{S}_b^2$, or $M_b \subsetneq \{\epsilon_1 + \epsilon_2\} \cup \mathcal{S}_a^3 \cup \mathcal{S}_b^1$. But $M(D)$ is maximal, so $M_b = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_4\}$ when $M_a = \{\epsilon_3 - \epsilon_4\}$ and $M_b = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4\}$ when $M_a = \{\epsilon_3 + \epsilon_4\}$.

If $|M_a| = 2$, then $|M_b| = 3$. Then $M_a = \Psi^+$, and $M_b = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3\} \subsetneq \Phi_1^{\text{rad}}$, or $M_b = \{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3\} \subsetneq \mathcal{S}_{ab}$.

Summarizing here, there are 17 possibilities of $M(D)$, they are $\mathcal{S}_{ab}^{ij}$ for $1 \leq i, j \leq 4$ and $(i, j) \neq (1,1), (3,1), (4,1)$, $\mathcal{S}_1 := \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_3 - \epsilon_4\}$, $\mathcal{S}_2 =: \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4, \epsilon_3 + \epsilon_4\}$, $\mathcal{S}_3 := \{\epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_3 \pm \epsilon_4\}$, and $\mathcal{S}_4 := \{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3, \epsilon_3 \pm \epsilon_4\}$. □

Lemma 2.5. *Let Φ be of type D_5. Then the elements of* $\text{Max}(9, \Phi^+)$ *are of the forms*

$$\{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j \mid i \neq i', i, i' \in I_s \text{ and } j \in J_s\} \cap \Phi^+ \text{ for } s \in \{1, 2\},$$

where $I_1 = \{1, 2, 3\}$ with $J_1 = \{4, 5\}$ and $I_2 = \{1, 2, 3, 5\}$ with $J_2 = \{4\}$.

Proof. Note that $r_{\text{smax}} = \text{rk}_p(\mathfrak{g}) - 1 = 9$ for D_5. Let

$$\Psi^+ = \Phi^+ \setminus \Phi_{\alpha_1,\alpha_2}^{\text{rad}} = \{\epsilon_3 \pm \epsilon_4, \epsilon_3 \pm \epsilon_5, \epsilon_4 \pm \epsilon_5\},$$

and $M(D) \in \text{Max}(9, \Phi^+)$. Assume that $M(D) = M_a \cup M_b$, where $M_a \subset \Psi^+$, $M_b \subset \Phi_{\alpha_1,\alpha_2}^{\text{rad}}$.

If $\epsilon_3 - \epsilon_4 \in M_a$, then $\epsilon_4 \pm \epsilon_5 \notin M_a$ which gives $M_a \subset M_a^0 := \{\epsilon_3 \pm \epsilon_4, \epsilon_3 \pm \epsilon_5\}$. Alternatively, if $\epsilon_3 - \epsilon_4 \notin M_a$, then $M_a \subset M_a^1 := \{\epsilon_3 + \epsilon_4, \epsilon_3 + \epsilon_5, \epsilon_4 + \epsilon_5\}$, or $M_a \subset M_a^2 := \{\epsilon_3 + \epsilon_4, \epsilon_3 - \epsilon_5, \epsilon_4 - \epsilon_5\}$, or $M_a \subset M_a^3 := \{\epsilon_3 + \epsilon_4, \epsilon_3 \pm \epsilon_5\}$, or $M_a \subset M_a^4 := \{\epsilon_3 + \epsilon_4, \epsilon_4 \pm \epsilon_5\}$. So, we have $|M_a| \leq 4$. If $|M_b| = 8$, then M_b must be Φ_1^{rad}, which is maximal in Φ^+, it is a contradiction. Hence, $|M_a| \geq 2$.

If $|M_a| = 4$, then $|M_b| = 5$ and $M_a = M_a^0 = \{\epsilon_3 \pm \epsilon_4, \epsilon_3 \pm \epsilon_5\}$. But $[M_a, M_b] = 0$, which implies $|M_b| \leq 3$ whenever $M_b \subset \Phi_1^{\text{rad}}$ or $M_b \subset \mathcal{S}_{ab}$, it is a contradiction.

If $|M_a| = 3$, then $|M_b| = 6$. In this case, we first assume that $M_b \subset \Phi_1^{\text{rad}}$. Then $\epsilon_1 \pm \epsilon_i$ exist for at least one choice for i from the set $\{3, 4, 5\}$, this implies $|M_a| \leq 2$ by $[M_a, M_b] = 0$, it is a contradiction. Then we may assume $M_b \subset \mathcal{S}_{ab}$, it follows that $|M_b \cap (\mathcal{S}_a \cup \mathcal{S}_b)| \geq 5$. We list several possibilities to get a contradiction in this case: (a) $M_a = M_a^1$ or M_a^2, there is no M_b with $|M_b| = 6$ such that $M(D)$ is maximal; (b)$M_a = M_a^3$ or M_a^4, then $M_b \subset \{\epsilon_i + \epsilon_j\} \cup \{\epsilon_1 + \epsilon_2\}$ where $1 \leq i \leq 2$ and $3 \leq j \leq 4$, thus $|M_b \cap (\mathcal{S}_a \cup \mathcal{S}_b)| \leq 4$; (c)$M_a \subset M_a^0$, then $\epsilon_3 \pm \epsilon_i$ occurs in M_a for $i = 4$ or $i = 5$, which implies $|M_b| \leq 5$.

If $|M_a| = 2$, then $|M_b| = 7$. If $M_b \subset \Phi_1^{\text{rad}}$, then $M_a = \emptyset$ by $[M_a, M_b] = 0$, it is a contradiction. If $M_b \subset \mathcal{S}_{ab}$, then

$$M_b = \{\epsilon_1 + \epsilon_2\} \cup \{\epsilon_i + \epsilon_j, \epsilon_i - \epsilon_4 \mid i = 1, 2 \text{ and } j = 3, 5\}$$

with $M_a = \{\epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_5\}$, or

$$M_b = \{\epsilon_1 + \epsilon_2\} \cup \{\epsilon_i + \epsilon_3, \epsilon_i - \epsilon_j \mid i = 1, 2 \text{ and } j = 4, 5\}$$

with $M_a = \{\epsilon_3 - \epsilon_4, \epsilon_3 - \epsilon_5\}$.

Summarizing here, by taking $I_1 = \{1, 2, 3\}$ with $J_1 = \{4, 5\}$ and $I_2 = \{1, 2, 3, 5\}$ with $J_2 = \{4\}$ there are two possibilities for $M(D)$, that is

$$M(D) = \{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j \mid i \neq i', i, i' \in I_s \text{ and } j \in J_s\} \cap \Phi^+ \text{ for } s \in \{1, 2\}.$$

□

Proposition 2.6. *Suppose that Φ is of type D_n with $n \geq 6$. Let $\Psi^{\text{rad}}_{n-2} := \Psi^+ \cap \Phi^{\text{rad}}_n$ and $\Psi^{\text{rad}}_{n-3} := \Psi^+ \cap \Phi^{\text{rad}}_{n-1}$, where $\Psi = \Phi \setminus \pm\Phi^{\text{rad}}_{\alpha_1,\alpha_2}$ is a root system of D_{n-2}. Let $M(D) \in \text{Max}(\frac{n(n-1)}{2} - 1, \Phi^+)$, then there is no such a decomposition $M(D) = M_a \cup M_b$ with $M_a = \Psi^{\text{rad}}_{n-2}$ or Ψ^{rad}_{n-3} and $M_b \subset \Phi^{\text{rad}}_{\alpha_1,\alpha_2}$.*

Proof. We first know that $\frac{n(n-1)}{2} - 1$ is the rank r_{smax} of D_n, and $\frac{(n-2)(n-3)}{2}$ is the p-rank of D_{n-2}. Since Ψ is the root system of D_{n-2} and by Table 1

$$\text{Max}(\frac{n(n-1)}{2}, \Phi^+) = \{\Phi^{\text{rad}}_{n-1}, \Phi^{\text{rad}}_n\},$$

it follows that

$$\Psi^{\text{rad}}_{n-2} := \Psi^+ \cap \Phi^{\text{rad}}_n,$$

and

$$\Psi^{\text{rad}}_{n-3} := \Psi^+ \cap \Phi^{\text{rad}}_{n-1} \qquad (*)$$

are the only two elements of $\text{Max}(\frac{(n-2)(n-3)}{2}, \Psi^+)$.

Recall that $\Phi^{\text{rad}}_{\alpha_1,\alpha_2} = \{\, \epsilon_1 \pm \epsilon_i, \epsilon_2 \pm \epsilon_j \mid 2 \leq i \leq n \text{ and } 3 \leq j \leq n \,\}$. If $M_a = \Psi^{\text{rad}}_{n-2}$, then $\epsilon_i - \epsilon_j \notin M_b$ for $i = 1, 2$ and $3 \leq j \leq n$ by $[M_a, M_b] = 0$, this gives rise to

$$M_b \subset \{\, \epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_i \,\}_{3 \leq i \leq n},$$

or

$$M_b \subset \{\epsilon_1 + \epsilon_i, \epsilon_2 + \epsilon_j \mid 2 \leq i \leq n \text{ and } 3 \leq j \leq n\}.$$

Note that $|M_b| = |M(D)| - |M_a| = 2n - 4$ if $M(D)$ exists. An inspection of these two cases gives either $|M_b| \leq n < 2n - 4$ or $M(D) = M_a \cup M_b \subsetneq \Phi^{\text{rad}}_n$, from which we deduce that $M(D)$ does not exist. Alternatively, we assume that $M_a = \Psi^{\text{rad}}_{n-3}$. For this situation, we get

$$M_b \subset \{\, \epsilon_1 \pm \epsilon_2, \epsilon_1 - \epsilon_n, \epsilon_1 + \epsilon_j \,\}_{3 \leq j \leq n-1},$$

or

$$M_b \subset \{\, \epsilon_1 - \epsilon_n, \epsilon_1 + \epsilon_i \,\}_{2 \leq i \leq n-1} \cup \{\, \epsilon_2 - \epsilon_n, \epsilon_2 + \epsilon_j \,\}_{3 \leq j \leq n-1}.$$

Accordingly, we have either $|M_b| \leq n < 2n - 4$ or $M(D) = M_a \cup M_b \subsetneq \Phi^{\text{rad}}_{n-1}$, which also shows the non-existence of $M(D)$. □

Lemma 2.7. *Let $n = 6$. Keep the notations for $\Psi, M(D), M_a$ and M_b as above. Then the elements of $\text{Max}(14, \Phi^+)$ are*

$$\{\, \epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j \mid i \neq i', i, i' \in I_s \textit{ and } j \in J_s \,\} \cap \Phi^+ \textit{for } s \in \{\, 1, 2 \,\}$$

where $I_1 = \{\, 1, 2, 3, 4 \,\}$ with $J_1 = \{\, 5, 6 \,\}$ and $I_2 = \{\, 1, 2, 3, 4, 6 \,\}$ with $J_2 = \{\, 5 \,\}$.

Proof. Since $n = 6$, the p-rank $\mathrm{rk}_p(\mathfrak{g})$ is 15 and the rank $r_{\mathrm{smax}} = \mathrm{rk}_p(\mathfrak{g}) - 1$ is 14. Since Ψ is the root system of D_4, and the p-rank of D_4 is 6, it gives $|M_a| \leq 6$. By Proposition 2.2 and the maximality of Φ_1^{rad} in Φ^+, we have $|M_b| \leq 9$. Hence, there are only two possibilities for $M(D)$ if it exists: $|M_a| = 6$ and $|M_b| = 8$, or $|M_a| = 5$ and $|M_b| = 9$.

If $|M_a| = 6$, then $M_a = \Psi_3^{\mathrm{rad}}, \Psi_4^{\mathrm{rad}}$ or Ψ_1^{rad} b Table 1. By Proposition 2.6, one only needs to check the case when $M_a = \Psi_1^{\mathrm{rad}}$. Recall that

$$\Psi_1^{\mathrm{rad}} = \{\, \epsilon_3 \pm \epsilon_i \,\}_{4 \leq i \leq 6}\,,$$

it gives

$$M_b \subset \{\, \epsilon_1 \pm \epsilon_2, \epsilon_1 + \epsilon_3 \,\}$$

or

$$M_b \subset \{\, \epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3 \,\}\,,$$

there is no M_b with $|M_b| = 8$. If $|M_a| = 5$, then $|M_b| = 9$ if $M(D)$ exists. If $M_b \subset \Phi_1^{\mathrm{rad}}$, then every element in Ψ^+ cannot commute with the elements of M_b, so $M(D)$ does not exist. Otherwise, we let $M_b \subset \mathcal{S}_{ab}$, then it has to be $M_b = \mathcal{S}_{ab}$ by comparing their orders. If M_a is not maximal in Ψ^+, then $M_a \subset \Psi_1^{\mathrm{rad}}, \Psi_3^{\mathrm{rad}}$ or Ψ_4^{rad}. We discuss these three possibilities to get a contradiction:

(a) $M_a \subset \Psi_1^{\mathrm{rad}}$, then there exists i_0 where $4 \leq i_0 \leq 6$ such that $\epsilon_3 \pm \epsilon_{i_0} \in M_a$. But $\epsilon_3 \pm \epsilon_{i_0}$ cannot commute with elements of $\mathcal{S}_{ab}$ from each of the sets

$$\{\, \epsilon_1 + \epsilon_{i_0}, \epsilon_2 - \epsilon_{i_0} \,\}$$

and

$$\{\, \epsilon_1 - \epsilon_{i_0}, \epsilon_2 + \epsilon_{i_0} \,\}\,;$$

(b) $M_a \subset \Psi_3^{\mathrm{rad}}$, then $\mathcal{S}_{ab}$ must be

$$\{\, \epsilon_1 + \epsilon_i, \epsilon_2 + \epsilon_i \,\}_{3 \leq i \leq 5} \cup \{\, \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_6, \epsilon_2 - \epsilon_6 \,\}\,,$$

but now $M_a \cup M_b \subsetneq \Phi_5^{\mathrm{rad}}$;

(c) $M_a \subset \Psi_4^{\mathrm{rad}}$, then $\mathcal{S}_{ab}$ must be

$$\{\, \epsilon_1 + \epsilon_i \,\}_{2 \leq i \leq 6} \cup \{\, \epsilon_2 + \epsilon_r \,\}_{3 \leq r \leq 6}\,,$$

but again

$$M_a \cup M_b \subsetneq \Phi_6^{\mathrm{rad}}.$$

Now the consideration is left only for M_a being maximal with $|M_a| = 5$. Since the rank r_{smax} is 5 for D_4, we have $M_a \in \mathrm{Max}(5, \Psi^+)$. By Lemma 2.4 there are only two are suitable here, they are

$$M_a = \{\, \epsilon_3 + \epsilon_4, \epsilon_i - \epsilon_5, \epsilon_i - \epsilon_6 \,\}_{3 \leq i \leq 4}\,,$$

or

$$M_a = \{\, \epsilon_3 + \epsilon_4, \epsilon_i - \epsilon_5, \epsilon_i + \epsilon_6 \,\}_{3 \le i \le 4} \,.$$

Given by these, the corresponding M_b is

$$M_b = \{\, \epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_s \,\}_{1 \le i \le 2, 3 \le r \le 4, 5 \le s \le 6} \,,$$

or

$$M_b = \{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_5, \epsilon_i + \epsilon_6\}_{1 \le i \le 2, 3 \le r \le 4}.$$

Summarizing here, by denoting $I_1 = \{\,1,2,3,4\,\}$ with $J_1 = \{\,5,6\,\}$ and $I_2 = \{\,1,2,3,4,6\,\}$ with $J_2 = \{\,5\,\}$ we have

$$M(D) = \{\, \epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j \mid i \ne i', i, i' \in I_s \text{ and } j \in J_s \,\} \cap \Phi^+ \text{for } s \in \{\,1,2\,\}.$$

□

Notation 2.8. *Let J be a subset of $\{\,1,\dots,n\,\}$. We denote by*

$$\phi(J) := \{\, \epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j, -\epsilon_j - \epsilon_{j'} \mid i \ne i', i, i' \in J \textit{ and } j \ne j', j, j' \notin J \,\}$$

a set of commuting roots and consider the following sets

$$\Phi^D_{n-2,n-1} := \phi(J) \cap \Phi^+ \textit{when } J = \{\,1,\dots,n-2\,\},$$

$$\Phi^D_{n-1,n} := \phi(J) \cap \Phi^+ \textit{when } J = \{1,\dots,n-2,n\}.$$

Theorem 2.9. *Suppose that Φ is of type $D_n (n \ge 5)$. Let $M(D) \in \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$, then $M(D)$ is either of the form $\Phi^D_{n-2,n-1}$ or of the form $\Phi^D_{n-1,n}$.*

Proof. We prove the above statement by induction. The statement is clear for $n = 5$ and $n = 6$ by Lemmas 2.5 and 2.7. Assume it is proved for $2m-1$ and $2m$, $m \ge 3$. We prove the statement is true for $2m+1$ and $2m+2$.

Keep the notation for Ψ, M_a and M_b as above again. Let $M(D) \in \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ and Φ be of type D_n, where $n \in \{\,2m+1, 2m+2\,\}$. Assume that $M(D) = M_a \cup M_b$ where $M_a \subset \Psi^+$, $M_b \subset \Phi^{\mathrm{rad}}_{\alpha_1,\alpha_2}$. Then we get the upper bounds for $|M_a|$ and $|M_b|$:

$$|M_a| \le \frac{(n-2)(n-3)}{2}, |M_b| \le 2n-2$$

which are constrained by $|M_a| + |M_b| = \frac{n(n-1)}{2} - 1$. By Proposition 2.6, we have $|M_a| \ne \frac{(n-2)(n-3)}{2}$. By the maximality of Φ^{rad}_1, we have $|M_b| \ne 2n-2$ (otherwise $M_b = \Phi^{\mathrm{rad}}_1$ but $2n-2 < \frac{n(n-1)}{2} - 1$ when $n \ge 7$). The only case left for our consideration is when $|M_a| = \frac{(n-2)(n-3)}{2} - 1$ and $|M_b| = 2n-3$.

Note that $\frac{(n-2)(n-3)}{2} - 1$ is the rank r_{smax} for Ψ of D_{n-2}. If M_a is maximal in Ψ^+, then by induction hypothesis

$$M_a = \phi(\hat{J}) \cap \Psi^+ \text{ for } \hat{J} = \{\,3,\dots,n-2\,\} \text{or } \hat{J} = \{\,3,\dots,n-2,n\,\}.$$

According to this, we get $M_b = \mathcal{S}_{ab}$. More precisely,

$$M_b = \{ \epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_{n-1}, \epsilon_i - \epsilon_n \}_{1 \le i \le 2, 3 \le r \le n-2}$$

when $\hat{J} = \{ 3, \dots, n-2 \}$, or

$$M_b = \{\epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_{n-1}, \epsilon_i + \epsilon_n\}_{1 \le i \le 2, 3 \le r \le n-2}$$

when $\hat{J} = \{ 3, \dots, n-2, n \}$. Both of them give rise to

$$M(D) = \phi(J) \cap \Phi^+$$

where $J = \hat{J} \cup \{1, 2\}$(i.e. $J = \{ 1, \dots, n-2 \}$ or $J = \{ 1, \dots, n-2, n \}$).

Afterwards, let us turn to the case when M_a is not maximal. Since

$$\mathrm{Max}(r_{\max}, \Psi^+) = \{ \Psi^{\mathrm{rad}}_{n-3}, \Psi^{\mathrm{rad}}_{n-2} \}$$

by $(*)$, this gives $M_a \subset \Psi^{\mathrm{rad}}_{n-3}$ or $M_a \subset \Psi^{\mathrm{rad}}_{n-2}$. If $M_a \subsetneq \Psi^{\mathrm{rad}}_{n-3}$, then

$$M_b = \{ \epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r, \epsilon_i - \epsilon_n \}_{1 \le i \le 2, 3 \le r \le n-1}$$

by $[M_a, M_b] = 0$, but then $M_a \cup M_b \subsetneq \Phi^{\mathrm{rad}}_{n-1}$ which is not maximal, so $M(D)$ does not exist. If $M_a \subsetneq \Psi^{\mathrm{rad}}_{n-2}$, then

$$M_b = \{ \epsilon_1 + \epsilon_2, \epsilon_i + \epsilon_r \}_{1 \le i \le 2, 3 \le r \le n}$$

by $[M_a, M_b] = 0$, but $M_a \cup M_b \subsetneq \Phi^{\mathrm{rad}}_n$, so there is no $M(D)$ by the same reason. □

3. Expressions of maximal elements of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})$

3.1. LT

(Sect. 3.1 of [9]). We have to choose a total ordering $\geq$ on Φ^+ which respects addition of positive roots, that is, if $\beta, \gamma, \lambda, \beta + \lambda, \gamma + \lambda \in \Phi^+$ and $\beta \geq \gamma$ then $\beta + \lambda \geq \gamma + \lambda$. We note that the standard ordering $\geq$ on Φ respects addition, as does a reverse lexicographical ordering with respect to any ordering of the simple roots. This ordering will define the extraspecial pairs in our root system and consequently the signs in the structure constants of the chevalley basis.

Let $\mathcal{E} \subset \mathfrak{u}$ be an elementary subalgebra. The ordering $\geq$ on Φ^+ gives an ordering on the basis elements x_β of $\mathfrak{u}$. Choose the unique basis of $\mathcal{E}$ which is in reduced echelon form with respect to this ordering and let $\mathrm{LT}(\mathcal{E})$ be the set of roots β such that the corresponding x_β are the leading term in this reduced basis. Observe that if x_β and x_γ are the leading terms of $b_1 = x_\beta + <$ lower terms $>$ and $b_2 = x_\gamma + <$ lower terms $>$ respectively, and if $\beta + \gamma \in \Phi^+$ then $[x_\beta, x_\gamma] = N_{\beta,\gamma} x_{\beta+\gamma}$ is the leading term of $[b_1, b_2]$. Thus if $[b_1, b_2] = 0$ then β and γ commute. This proves that $\mathrm{LT}(\mathcal{E})$ is a set of commuting roots.

3.2. Elementary subalgebras

We concentrate on G being a connected simple algebraic $\Bbbk$-group of type C or D with $\mathfrak{g} := \mathrm{Lie}(G)$ and p is a good prime. Let Φ be the root system of G with positive roots Φ^+. Let $\mathfrak{u} := \mathrm{Lie}(U)$ be the Lie algebra of the unipotent radical U of the Borel subgroup $B \le G$, and $\{ x_\alpha : \alpha \in \Phi^+ \}$ be a basis of $\mathfrak{u}$. Since p is good for G, we have $[x_\alpha, x_\beta] = 0$ if and only if $\alpha + \beta \notin \Phi$ for $\alpha, \beta \in \Phi$. Recall that $x_\alpha^{[p]} = 0$ for $\alpha \in \Phi$, one does have an elementary subalgebra

$$\mathrm{Lie}(R) := \mathrm{Span}_{\Bbbk}\{x_\alpha \ ; \ \alpha \in R\}$$

when R is a subset of commuting roots.

In section 2, we have determined all the elements of the set $\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$. In virtue of LT, for $R \in \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$, $\mathrm{Lie}(R)$ is not properly contained in any elementary subalgebra $\mathcal{E} \subset \mathfrak{u}$. If there were such $\mathcal{E}$, then we would have $R \subsetneq \mathrm{LT}(\mathcal{E})$, violating the maximality of R. We present the result through the following corollary:

Corollary 3.1. *Suppose p is a good prime. Let G be a connected simple algebraic group of type $C_n (n \ge 3)$ or $D_n (n \ge 5)$ over an algebraically closed field $\Bbbk$ of positive characteristic $p > 0$ and $\mathfrak{u} := \mathrm{Lie}(U)$ be the Lie algebra of the unipotent radical U of the Borel subgroup $B \le G$. Then the assignment*

$$R \mapsto \mathrm{Lie}(R)$$

induces an injective map

$$\mathrm{Lie} : \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+) \longrightarrow \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$$

where $\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ *is summarized by Table 2.*

Table 2. Maximal subsets of commuting roots of order r_{smax}.

Type T	Rank	$\mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$	r_{smax}
C_n	$n \ge 3$	$\Phi^C_{n-1,n}$	$\frac{1}{2}n(n+1) - 1$
D_n	$n \ge 5$	$\Phi^D_{n-2,n-1}, \Phi^D_{n-1,n}$	$\frac{1}{2}n(n-1) - 1$

3.3. Type C_n

Suppose that G is of type $C_n (n \ge 3)$. Let $\succeq$ be the reverse lexicographic ordering given by $\alpha_n \prec \alpha_{n-1} \prec \cdots \prec \alpha_1$.

Lemma 3.2. *Suppose that G is of type C_n with $n \ge 3$. If $\mathcal{E} \in \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$, then $\mathrm{LT}(\mathcal{E}) \in \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ with respect to $\succeq$.*

Proof. If $\mathrm{LT}(\mathcal{E}) \notin \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$, then $\mathrm{LT}(\mathcal{E}) \subsetneq \Phi_n^{\mathrm{rad}}$ by Table 1. Note that $\Phi^+ \setminus \Phi_n^{\mathrm{rad}} \succ \Phi_n^{\mathrm{rad}}$, it follows that all terms of basis vectors correspond to the roots lying in Φ_n^{rad}. Hence, $\mathcal{E}$ is contained in the elementary subalgebra $\mathrm{Lie}(\Phi_n^{\mathrm{rad}})$. Notice that $\dim \mathcal{E} < \dim \mathrm{Lie}(\Phi_n^{\mathrm{rad}})$, the containment is proper which contradicts maximality. □

In what follows, we will refine sets Φ^+ and $\Phi^C_{n-1,n}$. We define for $1 \le i \le n$

$$\Phi_i^+ := \{\, \epsilon_1 + \epsilon_i, \epsilon_2 + \epsilon_i, \ldots, 2\epsilon_i \,\},$$

and for $1 < i \le n$

$$\Phi_i^- := \{\, \epsilon_1 - \epsilon_i, \epsilon_2 - \epsilon_i, \ldots, \epsilon_{i-1} - \epsilon_i \,\}.$$

Then the elements of Φ_i^+ and Φ_i^- satisfy

$$\begin{aligned} 2\epsilon_i &\ge \epsilon_{i-1} + \epsilon_i \ge \cdots \ge \epsilon_1 + \epsilon_i, \\ \epsilon_{i-1} - \epsilon_i &\ge \epsilon_{i-2} - \epsilon_i \ge \cdots \ge \epsilon_1 - \epsilon_i \end{aligned}$$

and Φ^+ is the union of the following subsets:

$$\Phi_2^- \succ \cdots \succ \Phi_{n-1}^- \succ \Phi_n^- \succ \Phi_n^+ \succ \Phi_{n-1}^+ \succ \cdots \succ \Phi_1^+.$$

Recall the definition of $\Phi^C_{n-1,n}$ for C_n in Section 2.1, we get

$$\Phi^C_{n-1,n} = \Phi_n^- \cup \bigcup_{i=1}^{n-1} \Phi_i^+.$$

Theorem 3.3. *Suppose that G is of type C_n with $n \ge 3$. If $\mathcal{E} \in \mathbb{E}(r_{\rm smax}, \mathfrak{u})$ satisfies* $\mathrm{LT}(\mathcal{E}) = \Phi^C_{n-1,n}$ *then* $\mathcal{E} = \mathrm{Lie}(\Phi^C_{n-1,n})^{\exp(\mathrm{ad}(a x_{\alpha_n}))}$ *for some a.*

Proof. If $\mathrm{LT}(\mathcal{E}) = \Phi^C_{n-1,n}$, the reduced echelon form basis of $\mathcal{E}$ is

$$x_{ij} = x_{\epsilon_i+\epsilon_j}, 1 \le i \le j < n,$$

and

$$y_i = x_{\epsilon_i-\epsilon_n} + \sum_{s=1}^{n} a_{is} x_{\epsilon_s+\epsilon_n}, 1 \le i < n.$$

Step I. We prove that $a_{11} = 0$. Let $\exp(\mathrm{ad}(-a_{11} N^{-1}_{2\epsilon_n,\epsilon_1-\epsilon_n} x_{\alpha_n}))$ be the conjugation acting on $\mathcal{E}$, which is lower triangular with respect to $\succeq$. Therefore,

$$\mathrm{LT}(\exp(\mathrm{ad}(-a_{11} N^{-1}_{2\epsilon_n,\epsilon_1-\epsilon_n} x_{\alpha_n}))(\mathcal{E})) = \mathrm{LT}(\mathcal{E})$$

and the term $x_{\epsilon_1+\epsilon_n}$ in $\exp(\mathrm{ad}(-a_{11} N^{-1}_{2\epsilon_n,\epsilon_1-\epsilon_n} x_{\alpha_n}))(y_1)$ is eliminated.

Step II. We prove that all $a_{is} = 0$. For $j > 1$, we have

$$[y_1, y_j] = \sum_{s=1}^{n} N_{\epsilon_1-\epsilon_n,\epsilon_s+\epsilon_n} a_{js} x_{\epsilon_1+\epsilon_s} + \sum_{s=2}^{n} N_{\epsilon_s+\epsilon_n,\epsilon_j-\epsilon_n} a_{1s} x_{\epsilon_s+\epsilon_j}.$$

The coefficient of $x_{\epsilon_s+\epsilon_j}$ in $[y_1, y_j]$ is $N_{\epsilon_s+\epsilon_n,\epsilon_j-\epsilon_n} a_{1s}$, so $a_{1s} = 0$; the coefficient of $x_{\epsilon_1+\epsilon_s}$ in $[y_1, y_j]$ is $N_{\epsilon_1-\epsilon_n,\epsilon_s+\epsilon_n} a_{js}$, so $a_{js} = 0$. Thus $\mathcal{E} = \mathrm{Lie}(\Phi^C_{n-1,n})^{\exp(\mathrm{ad}(a x_{\alpha_n}))}$ for $a = a_{11} N^{-1}_{2\epsilon_n,\epsilon_1-\epsilon_n}$. □

3.4. Type D_n

Suppose that G is of type D_n. Let $\geq$ be the reverse lexicographic ordering given by

$$\alpha_{n-2} \succ \cdots \alpha_2 \succ \alpha_1 \succ \alpha_{n-1} \succ \alpha_n.$$

One can compute that if $i < j < n$, then $N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n} = N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n} = 1$.

Let $\mathcal{R} = \{ \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n-1 \}$. Then

$$\Phi_n^{\mathrm{rad}} = \mathcal{R} \cup \{ \epsilon_r + \epsilon_n \mid 1 \leq r < n \},$$
$$\Phi_{n-1}^{\mathrm{rad}} = \mathcal{R} \cup \{ \epsilon_r - \epsilon_n \mid 1 \leq r < n \}.$$

According to $\geq$, it gives rise to an refinement of Φ^+:

$$\{ \epsilon_i - \epsilon_j \mid 2 \leq i < j < n \} \succ \{ \epsilon_1 - \epsilon_j \mid 2 \leq j < n \} \succ \Phi_{n-1}^{\mathrm{rad}} \setminus \mathcal{R} \succ \Phi_n^{\mathrm{rad}} \setminus \mathcal{R} \succ \mathcal{R}.$$

Lemma 3.4. *Suppose that G is of type D_n with $n \geq 5$. If $\mathcal{E} \in \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$, then $\mathrm{LT}(\mathcal{E}) \in \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$ with respect to $\geq$.*

Proof. We prove it by assuming $\mathrm{LT}(\mathcal{E}) \notin \mathrm{Max}(r_{\mathrm{smax}}, \Phi^+)$. Then $\mathrm{LT}(\mathcal{E}) \subsetneq \Phi_n^{\mathrm{rad}}$, or $\mathrm{LT}(\mathcal{E}) \subsetneq \Phi_{n-1}^{\mathrm{rad}}$ by Table 1.

Case 1. $\mathrm{LT}(\mathcal{E}) \subsetneq \Phi_n^{\mathrm{rad}}$. If $\Phi_n^{\mathrm{rad}} \setminus \mathrm{LT}(\mathcal{E}) = \{ \epsilon_s + \epsilon_n \}$ for some $1 \leq s < n$, then the reduced echelon form basis of $\mathcal{E}$ consists of

$$x_{ij} = x_{\epsilon_i+\epsilon_j}, \ 1 \leq i < j < n,$$
$$y_i = x_{\epsilon_i+\epsilon_n} + a_{is} x_{\epsilon_s+\epsilon_n}, \ 1 \leq i < n, i \neq s \text{ and } a_{is} = 0 \text{ when } i < s.$$

Alternatively, we have $\Phi_n^{\mathrm{rad}} \setminus \mathrm{LT}(\mathcal{E}) = \{ \epsilon_s + \epsilon_t \}$ for $1 \leq s < t \leq n-1$. And then the reduced echelon form basis of $\mathcal{E}$ for $1 \leq i < j < n$ and $i \neq s, j \neq t$

$$x_{ij} = x_{\epsilon_i+\epsilon_j} + a_{ij} x_{\epsilon_s+\epsilon_t}, \ a_{ij} = 0 \text{ when } i < s \text{ or } i = s, j < t,$$
$$y_i = x_{\epsilon_i+\epsilon_n} + b_i x_{\epsilon_s+\epsilon_t}, 1 \leq i < n.$$

One can easily see that, both of them yield $\mathcal{E} \subsetneq \mathrm{Lie}(\Phi_n^{\mathrm{rad}})$, it is a contradiction.

Case 2. $\mathrm{LT}(\mathcal{E}) \subsetneq \Phi_{n-1}^{\mathrm{rad}}$. If $\Phi_{n-1}^{\mathrm{rad}} \setminus \mathrm{LT}(\mathcal{E}) = \{ \epsilon_s - \epsilon_n \}$, then there is the reduced echelon form basis of $\mathcal{E}$

$$x_{ij} = x_{\epsilon_i+\epsilon_j}, \ 1 \leq i < j \leq n-1,$$
$$y_i = x_{\epsilon_i-\epsilon_n} + a_{is} x_{\epsilon_s-\epsilon_n} + \sum_{t=1}^{n-1} b_{it} x_{\epsilon_t+\epsilon_n}, 1 \leq i < n, i \neq s \text{ and } a_{is} = 0 \text{ when } i < s.$$

Note that $\exp(\mathrm{ad}(-a_{is}N^{-1}_{\epsilon_s-\epsilon_i,\epsilon_i-\epsilon_n}x_{\epsilon_s-\epsilon_i}))$ for $i > s$ will rule out the term $a_{is}x_{\epsilon_s-\epsilon_n}$ in y_i and fix $a_{js}x_{\epsilon_s-\epsilon_n}$ in y_j if $j \neq i$. Let $\lambda_i = -a_{is}N^{-1}_{\epsilon_s-\epsilon_i,\epsilon_i-\epsilon_n}$, then conjugation by $b := \exp(\mathrm{ad}(\lambda_{n-1}x_{\epsilon_s-\epsilon_{n-1}})) \circ \cdots \circ \exp(\mathrm{ad}(\lambda_{s+1}x_{\epsilon_s-\epsilon_{s+1}}))$ on $\mathcal{E}$ yields the final reduced basis

$$x'_{ij} = x_{\epsilon_i+\epsilon_j},\ 1 \le i < j \le n-1,$$
$$y'_i = x_{\epsilon_i-\epsilon_n} + \sum_{t=1}^{n-1} b'_{it}x_{\epsilon_t+\epsilon_n},\ 1 \le i \neq s < n,$$

where $x'_{ij} = b.x_{ij}$ and $y'_i = b.y_i$. As $n \ge 5$, the proof in [9, Theorem 3.6] shows that all $b'_{it} = 0$. Consequently $\mathcal{E} \subsetneq \mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1})^{b^{-1}}$, which is not maximal. Alternatively, we get $\Phi^{\mathrm{rad}}_{n-1} \setminus \mathrm{LT}(\mathcal{E}) = \{\epsilon_s + \epsilon_t\}$ for $1 \le s < t < n$ and the reduced echelon form basis of $\mathcal{E}$

$$x_{ij} = x_{\epsilon_i+\epsilon_j} + a_{ij}x_{\epsilon_s+\epsilon_t},\ a_{ij} = 0 \text{ when } i < s \text{ or } i = s, j < t,$$
$$y_i = x_{\epsilon_i-\epsilon_n} + \sum_{r=1}^{n-1} b_{ir}x_{\epsilon_r+\epsilon_n} + d_i x_{\epsilon_s+\epsilon_t}, 1 \le i < n.$$

If $i, j, r < n$ are distinct, then the coefficient of $x_{\epsilon_j+\epsilon_r}$ in $[y_i, y_j]$ is $N_{\epsilon_r+\epsilon_n,\epsilon_j-\epsilon_n}b_{ir}$. As $n \ge 5$, we have $b_{ir} = 0$ for all $r \neq i$. Now for $i \neq j$ the coefficient of $x_{\epsilon_i+\epsilon_j}$ in $[y_i, y_j]$ is $N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n}b_{jj} + N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n}b_{ii}$. Thus if $i < j < t < n$ we have a system of equations

$$N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n}b_{jj} + N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n}b_{ii} = b_{jj} + b_{ii} = 0$$
$$N_{\epsilon_i-\epsilon_n,\epsilon_t+\epsilon_n}b_{tt} + N_{\epsilon_i+\epsilon_n,\epsilon_t-\epsilon_n}b_{ii} = b_{tt} + b_{ii} = 0$$
$$N_{\epsilon_j-\epsilon_n,\epsilon_t+\epsilon_n}b_{tt} + N_{\epsilon_j+\epsilon_n,\epsilon_t-\epsilon_n}b_{jj} = b_{tt} + b_{jj} = 0$$

whose solution is $b_{ii} = b_{jj} = b_{tt} = 0$. This gives $b_{ii} = 0$ for all i. Therefore, we have $\mathcal{E} \subsetneq \mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1})$, it is a contradiction. □

Theorem 3.5. *Suppose that G is of type D_n with $n \ge 6$. If $\mathcal{E} \in \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})$ satisfies $\mathrm{LT}(\mathcal{E}) = \Phi^D_{n-1,n}$ or $\Phi^D_{n-2,n-1}$ then $\mathcal{E} = \mathrm{Lie}(\Phi^D_{n-1,n})^{\exp(\mathrm{ad}(ax_{\alpha_{n-1}}))}$ or $\mathcal{E} = \mathrm{Lie}(\Phi^D_{n-2,n-1})^{\exp(\mathrm{ad}(ax_{\alpha_n}))}$ for some a.*

Proof. **Case 1.** $\mathrm{LT}(\mathcal{E}) = \Phi^D_{n-1,n}$. Then the reduced echelon form basis of $\mathcal{E}$ is

$$x_{ij} = x_{\epsilon_i+\epsilon_j} + \sum_{h=1}^{i-1} a_{ijh}x_{\epsilon_h+\epsilon_{n-1}}, 1 \le i < j < n-1,$$
$$y_i = x_{\epsilon_i+\epsilon_n} + \sum_{r=1}^{n-2} b_{ir}x_{\epsilon_r+\epsilon_{n-1}}, 1 \le i < n-1$$
$$z_i = x_{\epsilon_i-\epsilon_{n-1}} + \sum_{v=1}^{i-1}\sum_{t=v+1}^{n-2} c_{ivt}x_{\epsilon_v-\epsilon_t} + \sum_{r=1}^{n-1} d_{ir}x_{\epsilon_r-\epsilon_n} + k_i x_{\epsilon_{n-1}+\epsilon_n} + \sum_{s=1}^{n-2} \ell_{is}x_{\epsilon_s+\epsilon_{n-1}}, 1 \le i < n-1.$$

Step I. We prove that $c_{ivt} = a_{ijh} = 0$. If $i \ge 3$ and $v \ge 2$, the coefficient of $x_{\epsilon_1+\epsilon_v}$ in $[x_{1t}, z_i]$ is $N_{\epsilon_1+\epsilon_t,\epsilon_v-\epsilon_t}c_{ivt}$, so $c_{ivt} = 0$. Then for all $i \ge 2$, we have

$$z_i = x_{\epsilon_i-\epsilon_{n-1}} + \sum_{t=2}^{n-2} c_{i1t}x_{\epsilon_1-\epsilon_t} + \sum_{r=1}^{n-1} d_{ir}x_{\epsilon_r-\epsilon_n} + k_i x_{\epsilon_{n-1}+\epsilon_n} + \sum_{s=1}^{n-2} \ell_{is}x_{\epsilon_s+\epsilon_{n-1}}.$$

Consider the bracket for $i \geq 2$

$$0 = [x_{2t}, z_i] = c_{i12} N_{\epsilon_2+\epsilon_t, \epsilon_1-\epsilon_2} x_{\epsilon_1+\epsilon_t} + c_{i1t} N_{\epsilon_2+\epsilon_t, \epsilon_1-\epsilon_t} x_{\epsilon_1+\epsilon_2} + a_{2t1} N_{\epsilon_1+\epsilon_{n-1}, \epsilon_i-\epsilon_{n-1}} x_{\epsilon_1+\epsilon_i}.$$

Then $a_{2t1} = 0$ (It is possible since $n \geq 6$), and finally $c_{i1t} = 0$. Now for $i \geq 2$, the coefficient of $x_{\epsilon_h+\epsilon_i}$ in $[x_{ij}, z_i]$ is $N_{\epsilon_h+\epsilon_{n-1}, \epsilon_i-\epsilon_{n-1}} a_{ijh}$, thus $a_{ijh} = 0$ for all $h < i$.

Step II. We prove that $b_{ir} = d_{ir} = 0$. Let $\lambda = -d_{11} N^{-1}_{\epsilon_{n-1}-\epsilon_n, \epsilon_1-\epsilon_{n-1}}$. Using conjugation by $\exp(\mathrm{ad}(\lambda x_{\epsilon_{n-1}-\epsilon_n}))$, we may assume that $d_{11} = 0$. If i, j, r are distinct, then we have

$$[y_i, z_j] = \sum_{r=1}^{n-1} N_{\epsilon_i+\epsilon_n, \epsilon_r-\epsilon_n} d_{jr} x_{\epsilon_i+\epsilon_r} + \sum_{r=1}^{n-2} N_{\epsilon_r+\epsilon_{n-1}, \epsilon_j-\epsilon_{n-1}} b_{ir} x_{\epsilon_j+\epsilon_r}.$$

This gives $d_{ir} = b_{ir} = 0$ for all $r \neq i$. Now for $i \neq j$, the coefficient of $x_{\epsilon_i+\epsilon_j}$ in $[y_i, z_j]$ is $N_{\epsilon_i+\epsilon_n, \epsilon_j-\epsilon_n} d_{jj} + N_{\epsilon_i+\epsilon_{n-1}, \epsilon_j-\epsilon_{n-1}} b_{ii} = d_{jj} + b_{ii} = 0$. As $n \geq 6$ ($n \geq 5$ is enough), this gives $d_{ii} = b_{ii} = 0$ for all i.

Step III. We prove that $k_i = \ell_{is} = 0$. For $i \neq j$, the coefficient of $x_{\epsilon_j+\epsilon_n}$ in $[z_i, z_j]$ is $N_{\epsilon_{n-1}+\epsilon_n, \epsilon_j-\epsilon_{n-1}} k_i$, so all $k_i = 0$. If $i, j, s < n-1$ are distinct, then the coefficient of $x_{\epsilon_s+\epsilon_j}$ in $[z_i, z_j]$ is $N_{\epsilon_s+\epsilon_{n-1}, \epsilon_j-\epsilon_{n-1}} \ell_{is}$. As $n \geq 6$ ($n \geq 5$ is enough), this gives $\ell_{is} = 0$ for all $s \neq i$. Now for $i \neq j$, the coefficient of $x_{\epsilon_i+\epsilon_j}$ in $[z_i, z_j]$ is $N_{\epsilon_i-\epsilon_{n-1}, \epsilon_j+\epsilon_{n-1}} \ell_{jj} + N_{\epsilon_i+\epsilon_{n-1}, \epsilon_j-\epsilon_{n-1}} \ell_{ii}$. Thus if $i < j < t < n-1$ are distinct, we have a system of equations

$$\begin{aligned} N_{\epsilon_i-\epsilon_{n-1}, \epsilon_j+\epsilon_{n-1}} \ell_{jj} + N_{\epsilon_i+\epsilon_{n-1}, \epsilon_j-\epsilon_{n-1}} \ell_{ii} &= \ell_{jj} + \ell_{ii} = 0 \\ N_{\epsilon_i-\epsilon_{n-1}, \epsilon_t+\epsilon_{n-1}} \ell_{tt} + N_{\epsilon_i+\epsilon_{n-1}, \epsilon_t-\epsilon_{n-1}} \ell_{ii} &= \ell_{tt} + \ell_{ii} = 0 \\ N_{\epsilon_j-\epsilon_{n-1}, \epsilon_t+\epsilon_{n-1}} \ell_{tt} + N_{\epsilon_j+\epsilon_{n-1}, \epsilon_t-\epsilon_{n-1}} \ell_{jj} &= \ell_{tt} + \ell_{jj} = 0 \end{aligned}$$

with unique solution $\ell_{ii} = \ell_{jj} = \ell_{tt} = 0$. This gives $\ell_{ii} = 0$ for all i and finally yields $\mathcal{E} = \mathrm{Lie}(\Phi^D_{n-1,n})^{\exp(\mathrm{ad}(a x_{\alpha_{n-1}}))}$.

Case 2. $\mathrm{LT}(\mathcal{E}) = \Phi^D_{n-2,n-1}$. Then the reduced echelon form basis of $\mathcal{E}$ is

$$\begin{aligned} x_{ij} &= x_{\epsilon_i+\epsilon_j} + \sum_{h=1}^{i-1} a_{ijh} x_{\epsilon_h+\epsilon_{n-1}}, \ 1 \leq i < j < n-1, \\ y_i &= x_{\epsilon_i-\epsilon_n} + \sum_{p=1}^{n-1} b_{ip} x_{\epsilon_p+\epsilon_n} + \sum_{q=1}^{n-2} c_{iq} x_{\epsilon_q+\epsilon_{n-1}}, \ 1 \leq i < n-1, \\ z_i &= x_{\epsilon_i-\epsilon_{n-1}} + \sum_{v=1}^{i-1} \sum_{t=v+1}^{n-2} d_{ivt} x_{\epsilon_v-\epsilon_t} + k_i x_{\epsilon_{n-1}-\epsilon_n} + \sum_{r=1}^{n-1} f_{ir} x_{\epsilon_r+\epsilon_n} + \sum_{s=1}^{n-2} g_{is} x_{\epsilon_s+\epsilon_{n-1}}, \ 1 \leq i < n-1. \end{aligned}$$

If i, j, p are distinct, then the coefficient of $x_{\epsilon_p+\epsilon_j}$ in $[y_i, y_j]$ is $N_{\epsilon_p+\epsilon_n, \epsilon_j-\epsilon_n} b_{ip}$, it follows that $b_{ip} = 0$ for all $p \neq i$. If $i \neq j$, the coefficient of $x_{\epsilon_i+\epsilon_j}$ in $[y_i, y_j]$ is $N_{\epsilon_i-\epsilon_n, \epsilon_j+\epsilon_n} b_{jj} + N_{\epsilon_i+\epsilon_n, \epsilon_j-\epsilon_n} b_{ii}$. Thus if $i < j < t < n-1$ are distinct, we have a system of equations

$$\begin{aligned} N_{\epsilon_i-\epsilon_n, \epsilon_j+\epsilon_n} b_{jj} + N_{\epsilon_i+\epsilon_n, \epsilon_j-\epsilon_n} b_{ii} &= b_{jj} + b_{ii} = 0 \\ N_{\epsilon_i-\epsilon_n, \epsilon_t+\epsilon_n} b_{tt} + N_{\epsilon_i+\epsilon_n, \epsilon_t-\epsilon_n} b_{ii} &= b_{tt} + b_{ii} = 0 \\ N_{\epsilon_j-\epsilon_n, \epsilon_t+\epsilon_n} b_{tt} + N_{\epsilon_j+\epsilon_n, \epsilon_t-\epsilon_n} b_{jj} &= b_{tt} + b_{jj} = 0 \end{aligned}$$

with $b_{ii} = b_{jj} = b_{tt} = 0$. This implies $b_{ii} = 0$ for all i. Then the calculation for the other coefficients is similar to Case 1. We conclude that there exists some a such that $\mathcal{E} = \mathrm{Lie}(\mathrm{LT}(\mathcal{E}))^{\exp(\mathrm{ad}(a x_{\alpha_n}))}$. □

4. Summarizing for unipotent case

Summarizing the discussions for G in section 3, we are to give the main result except for some small ranks for each type. Before doing this, we recall the definition of an ideal of Φ^+. We say $R \subseteq \Phi^+$ is an ideal if $\alpha + \beta \in R$ whenever $\alpha \in \Phi^+, \beta \in R$ and $\alpha + \beta \in \Phi^+$; see [9, Definition 2.10]. A prototypical example for such an ideal arises from Φ_i^{rad}, where α_i is a simple root. In the sequel, the move to ideals helps to establish that both $G.\text{Lie}(R)$ and $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(H))$ are closed.

Lemma 4.1. *If α_i is a simple root, then $\Phi_i^{\text{rad}} \setminus \{\alpha_i\}$ is an ideal.*

Proof. Suppose that $\alpha \in \Phi^+, \beta \in \Phi_i^{\text{rad}} \setminus \{\alpha_i\}$ and $\alpha + \beta \in \Phi^+$. Since Φ_i^{rad} is an ideal, it follows that $\alpha + \beta \in \Phi_i^{\text{rad}}$. As α_i is a simple root, then $\alpha + \beta \neq \alpha_i$, which gives $\alpha + \beta \in \Phi_i^{\text{rad}} \setminus \{\alpha_i\}$, so $\Phi_i^{\text{rad}} \setminus \{\alpha_i\}$ is an ideal. □

Theorem 4.2. *Suppose that G is of type $C_n (n \geq 3)$ or $D_n (n \geq 6)$. Then*

$$\mathbb{E}(r_{\text{smax}}, \mathfrak{u})_{\max} \subseteq G.\text{Lie}(R),$$

where R is the ideal listed in the third column of the following Table 3.

Table 3. Ideals for Theorem 4.2.

Type	Rank	Ideal R
C_n	$n \geq 3$	$\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$
D_n	$n \geq 6$	$\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$

Proof. Theorems 3.3 and 3.5 ensure that

$$\mathbb{E}(r_{\text{smax}}, \mathfrak{u})_{\max} \subseteq \bigcup_{I \in \text{Max}(r_{\text{smax}}, \Phi^+)} U.\text{Lie}(I).$$

where I is the set of commuting roots showing in Table 4.

Table 4. sets for Theorem 4.2.

Type	Rank	Set I
C_n	$n \geq 3$	$\Phi_{n-1,n}^{\text{C}}$
D_n	$n \geq 6$	$\Phi_{n-1,n}^{\text{D}}, \Phi_{n-2,n-1}^{\text{D}}$

Let $\dot{w} \in N_G(T)$ be a representative of an element w in Weyl group $\mathscr{W}$ and I be a set of commuting roots. It is clear that $\dot{w}.\text{Lie}(I) = \text{Lie}(w.I)$. We are to show that each I of $\text{Max}(r_{\text{smax}}, \Phi^+)$ can be $\mathscr{W}$-conjugated to an ideal R. For type C, the simple reflection s_n acts by negating ϵ_n and fixing the remaining ϵ_i therefore any representative $\dot{s}_n \in N_G(T)$ conjugates $\text{Lie}(\Phi_{n-1,n}^{\text{C}})$ to $\text{Lie}(\Phi_n^{\text{rad}} \setminus \{\alpha_n\})$.

For type D, the Weyl group is a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ and S_n, and its action on roots is induced from the action on the set $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ where we may take S_n to permute the indices of the ϵ_i and the j^{th} generator of $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ to send ϵ_j to $-\epsilon_j$, ϵ_{j+1} to $-\epsilon_{j+1}$, and fix all other ϵ_i. So by conjugation we may assume our elementary subalgebras are of the form $\mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1} \setminus \{\alpha_{n-1}\})$. Finally we use Lemma 4.1 to prove that $\Phi^{\mathrm{rad}}_n \setminus \{\alpha_n\}$ and $\Phi^{\mathrm{rad}}_{n-1} \setminus \{\alpha_{n-1}\}$ are ideals and are done. □

Corollary 4.3. *Let G be a standard simple algebraic $\mathbb{k}$-group with root system $C_n (n \geq 3)$ or $D_n (n \geq 6)$. Then*

$$\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g}) = G.\mathrm{Lie}(R) \cup \bigcup_{H \text{ an ideal}} G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(H))$$

is the union of irreducible closed subsets, where ideals R, H are listed in the Table 5.

Table 5. Ideals for Corollary 4.3.

Type	Rank	Ideal R	Ideal H
C_n	$n \geq 3$	$\Phi^{\mathrm{rad}}_n \setminus \{\alpha_n\}$	Φ^{rad}_n
D_n	$n \geq 6$	$\Phi^{\mathrm{rad}}_{n-1} \setminus \{\alpha_{n-1}\}$	$\Phi^{\mathrm{rad}}_{n-1}, \Phi^{\mathrm{rad}}_n$

Proof. Theorem 4.2 gives the set $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$. We need to consider the elements of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u}) \setminus \mathbb{E}(r_{\mathrm{smax}}, \mathfrak{u})_{\max}$. In accordance with [9, Corollary 3.9] and [8, Lemma 2.2], we arrive at the equality. In viewing of the proof of Corollary 3.7 of [6], the right hand is a union of irreducible closed subsets. □

Theorem 4.4. *Let G be a standard simple algebraic $\mathbb{k}$-group with root system $C_n (n \geq 3)$ or $D_n (n \geq 6)$. Then the irreducible components of $\mathbb{E}(r_{\mathrm{smax}}, \mathfrak{g})$ for each type can be characterized; see Table 6.*

Table 6. Irreducible components for Theorem 4.4.

Type	Rank	Irreducible components
C_n	$n \geq 3$	$G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(\Phi^{\mathrm{rad}}_n))$
D_n	$n \geq 6$	$G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1})), G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(\Phi^{\mathrm{rad}}_n))$

Proof. By Corollary 4.3, it suffices to check the maximality of each irreducible closed subvariety. For type C_n, it is clear that $G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(\Phi^{\mathrm{rad}}_n))$ is the unique irreducible component. For type D_n, it is clear that $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1} \setminus \{\alpha_{n-1}\})$ is not maximal, so it suffices to check the maximality of $G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(R))$ for $R = \Phi^{\mathrm{rad}}_{n-1}$ or $R = \Phi^{\mathrm{rad}}_n$. We may assume

$$G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1})) \subseteq G.\mathbb{E}(r_{\mathrm{smax}}, \mathrm{Lie}(\Phi^{\mathrm{rad}}_n)).$$

Then we have $\mathrm{Lie}(\Phi^{\mathrm{rad}}_{n-1} \setminus \{\alpha_{n-1}\}) = g.\mathrm{Lie}(\Phi^{\mathrm{rad}}_n \setminus \{\gamma\})$ for some $g \in G$ and $\gamma \in \Phi^{\mathrm{rad}}_n$. By Lemma 3.8 of [6], we have $\Phi^{\mathrm{rad}}_{n-1} \setminus \{\alpha_{n-1}\}$ and $\Phi^{\mathrm{rad}}_n \setminus \{\gamma\}$ are $\mathscr{W}$-conjugate. On the other hand, one can easily check that $\Phi_n^{\text{rad}} \setminus \{\gamma\}$ and $\Phi_n^{\text{rad}} \setminus \{\alpha_n\}$ are $\mathscr{W}$-conjugate, so there is some $w \in \mathscr{W}$ such that $w.\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\} = \Phi_n^{\text{rad}} \setminus \{\alpha_n\}$. Notice that

$$\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\} = \{\epsilon_i + \epsilon_j \mid 1 \le i < j \le n-1\} \cup \{\epsilon_i - \epsilon_n \mid 1 \le i < n-1\},$$
$$\Phi_n^{\text{rad}} \setminus \{\alpha_n\} = \{\epsilon_i + \epsilon_j \mid 1 \le i < j \le n-1\} \cup \{\epsilon_i + \epsilon_n \mid 1 \le i < n-1\}.$$

Let

$$\mathcal{A} := \{\epsilon_i + \epsilon_n \mid 1 \le i < n-1\}.$$

We now consider $w^{-1}.\mathcal{A}$ in $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$. By the order of set $\mathcal{A}$, one can verify that

$$w^{-1}.\mathcal{A} = \mathcal{B} := \{\epsilon_i + \epsilon_{n-1} \mid 1 \le i < n-1\}$$

or

$$w^{-1}.\mathcal{A} = \mathcal{C} := \{\epsilon_i - \epsilon_n \mid 1 \le i < n-1\}.$$

If $w^{-1}.\mathcal{A} = \mathcal{B}$, then by observing the action of w on $\Phi_{n-1}^{\text{rad}} \setminus \{\alpha_{n-1}\}$, we have $w.\mathcal{C} = \{\epsilon_i - \epsilon_{n-1} \mid 1 \le i < n-1\}$. Since $\{\epsilon_i - \epsilon_{n-1} \mid 1 \le i < n-1\} \notin \Phi_n^{\text{rad}} \setminus \{\alpha_n\}$, it is impossible. If $w^{-1}.\mathcal{A} = \mathcal{C}$, then we find that $w.\mathcal{B} = \{\epsilon_i - \epsilon_{n-1} \mid 1 \le i < n-1\}$. This is also impossible by the same reason. Hence, the closed subset $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_{n-1}^{\text{rad}}))$ is maximal. The maximality of $G.\mathbb{E}(r_{\text{smax}}, \text{Lie}(\Phi_n^{\text{rad}}))$ is verified in a similar way and is omitted. □

Remark. In [9] the authors show that $\mathbb{E}(r_{\max}, \mathfrak{g})$ is a finite disjoint union of partial flag varieties, which differs from the above result.

5. Conclusions

In this paper we characterize the irreducible components of the variety $\mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$, where $\mathfrak{g} := \text{Lie}(G)$ is the Lie algebra of a connected standard simple algebraic group G of type C or D. The results show that $\mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$ is not a finite disjoint union of partial flag varieties, which differs from $\mathbb{E}(\text{rk}_p(\mathfrak{g}), \mathfrak{g})$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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