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Research article

Three-way decision based on canonical soft sets of hesitant fuzzy sets

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Abstract: The theory of three-way decision is built on the philosophy of thinking in threes. The essence of three-way decision is trisecting the whole and taking different strategies for different parts accordingly. The theory of three-way decision has been successfully implemented to diverse fields since it provides an elegant and efficient solution for solving complicated problems. In this paper, a useful representation for hesitant fuzzy sets is obtained by means of canonical soft sets. We also define unit interval parameterized soft sets and their derived hesitant fuzzy sets. Mutual representations and inner connections between hesitant fuzzy sets and soft sets are examined. With the help of soft rough sets, a generalized rough model based on hesitant fuzzy sets is established. A novel three-way decision method is presented for solving decision-making problems by means of hesitant fuzzy sets and canonical soft sets. Finally, a numerical example regarding peer review of research articles is given to illustrate the validity and efficacy of the proposed method.

Keywords: hesitant fuzzy set; three-way decision; rough set; soft set; canonical soft set
Mathematics Subject Classification: 03E72, 62A86, 68T35, 90B50, 06D72, 94D05

1. Introduction

Inspired by Pawlak's rough sets [37] and probabilistic rough sets [9, 48], Yao [49–51] proposed the theory of three-way decision (3WD) which provides reasonable semantic interpretation to the positive, negative and boundary regions of rough sets. The core of 3WD is trisecting the whole and taking different strategies for different parts. We can simplify complex problems by dividing the whole into three parts. 3WD provide us with a framework to solve the generalized uncertainty problem. It might be considered as part of a more complete decision-making strategy which in situations of e.g., large computational burden, should only refine the analysis of the alternatives belonging to the positive (and possibly the boundary) regions. The philosophy of thinking in threes includes not only splitting the set of alternatives in three parts, it can also evoke thinking in three views, three components, three categories, three levels or three dimensions [53]. Different interpretations of ternary options make 3WD become a useful and universal theoretical method to solve practical problems. For instance, in a social choice context, Alcantud and Laruelle [2] characterized a pragmatical model of voting with three actions. Recently, Laruelle [25] has argued that citizens in democratic countries tend to abstain because they cannot voice their dissatisfaction in their ballots. She reports the results of a field experiment that took place during the 2017 French presidential elections: Real voters were entirely satisfied with ternary voting [2], and understood well its aim and purpose. Many scholars have studied and expanded the three decision theories and applied them to many disciplines in the past decade. Other researchers have represented several classifications of methods based on the 3WD theory and favorably practiced in many fields. For example, Yu et al. [55] offered an effective automatic process by applying the decision-theoretic rough set model to clustering. Jia et al. [24] developed a mathematical model for optimal representation of the decision-theoretic rough set. Liang and Liu [28] proposed a risk decision model based on hesitant fuzzy rough sets. Liang et al. [29] presented a 3WD model with linguistic features for decision-making problems. Yao [52] established the TAO ("trisecting-acting-outcome") framework as a fundamental model for 3WD. Hu et al. [23] presented two kinds of 3WD approaches and examined their properties. Li et al. [26] introduced 3WD models based on subset-evaluation which generalize the original models. Liu et al. [30] proposed a new 3WD model with intuitionistic fuzzy numbers. Li et al. [27] generalized Pawlak's classical model and introduced rough set models on two universes. Yang and Yao [47] presented two semantics for interpreting soft sets and constructed two soft set-based 3WD models. Yao [54] investigated set-theoretic models of 3WD and illustrated the essential role of the TAO framework in 3WD.

In addition to rough set theory which we have mentioned above, there are other models of uncertain knowledge whose advantages are used in our study. They incorporate features related to indiscernibility or granularity (rough set theory), partial membership (fuzzy set theory and its extension under hesitancy), and parameterized descriptions (soft set theory).

In 1965, Zadeh [56] acquainted fuzzy set (FS) theory to manage the impreciseness in the data. After its presence, many scholars have spread them to L-fuzzy sets [21], IVFSs ("interval-valued fuzzy sets") [42] and IFSs ("Intuitionistic fuzzy sets") [5]. In practical application, when determining the membership degree of an element to a fuzzy set, people often hesitate between two or more possible values and find it difficult to choose. In order to solve this problem, Torra [41] proposed the concept of a hesitant fuzzy set (HFS). Compared with classical sets and FSs, HFSs contain all the possible membership values corresponding to each alternative, which if properly handled, can reduce the information loss in the decision-making process. Based on 3WD theory, Wang et al. [43] developed a novel approach to MADM ("multiple attribute decision making") under a hesitant fuzzy setting.

To build a more comprehensive theoretical structure for representing and administering uncertainty, the concept of a soft set (SS) was born from a different narrative that concerns parameterization [36].

As a new type of nonstandard sets, the SS involves not only the domain of objects, but also a parameter space related to the object domain. In recent years, the research on soft set theory has developed quickly. Maji et al. [34] applied the theory of SSs to solve a decision-making problem using rough mathematics. Maji et al. [33] proposed several operations of SSs. They also considered the fuzzy extension of SSs in [31].

Later on, Maji et al. [32] joined SSs with the features of IFSs and hence stated the notion of an IFSS ("intuitionistic fuzzy soft set"). Peng et al. [38] combined SSs with PFSs ("Pythagorean fuzzy sets") to put forward the concept of Pythagorean fuzzy soft sets. Athira et al. [6] proposed some new entropy measures for Pythagorean fuzzy soft sets. By virtue of Archimedean t-norm and t-conorm, Garg and Arora [20] generalized the MSM ("Maclaurin symmetric mean") aggregation operators to IFSSs. Chen et al. [8] developed a parameterization reduction method for SSs. Ali et al. [4] revisited and improved some basic operations such as "intersection", "union", "difference", between the pairs of SSs. Garg and Arora [19] developed a correlation coefficient-based TOPSIS method to address decision-making problems with intuitionistic fuzzy soft information. A concept of generalized fuzzy soft sets was stated and investigated by Majumdar and Samanta [35]. Feng et al. [17] extended the preference ranking organization method for enrichment evaluation with IFSSs. Agarwal et al. [1] defined the notion of GIFSSs ("generalized intuitionistic fuzzy soft sets") and considered practical applications of GIFSSs. Feng et al. [13] improved the GIFSS and simplified this notion as a combination of an IFSS over a universal set and an IFS in a parameter set. Garg and Arora [18] put forth group-based GIFSSs and discussed their applications to group decision-making problems.

Returning to the motivational topic of rough set (RS) theory, we recall that it was introduced by Pawlak in 1982, who attempted to analyze the granularity-based uncertainty during the decision process [37]. Driven by the features of FSs and RSs, Dubois and Prade [10] defined the notion of rough fuzzy sets and fuzzy rough sets. Raszikowaka and Kerre [39] further extended the concepts presented by Dubois and Prade to (I, T)-fuzzy rough sets. Using the intuitionistic fuzzy overlap function and its residual implication, Wen et al. [44] presented a new type of intuitionistic fuzzy rough sets. Given the important role of FSs and RSs, Feng et al. [14] combined SSs with them. Further, Feng et al. [16] put forth the concept of soft rough sets and demonstrated that Pawlak's RS model is a special case of the presented concept when the SS in a soft approximation space is a partition SS. By combining fuzzy soft sets with fuzzy rough sets, Sun and Ma [40] put forward soft fuzzy rough sets and considered their application to decision making problems. Xie and Gong [46] presented a notion called hesitant soft fuzzy rough set based on SSs and HFSs. They ascertained the relationship between hesitant fuzzy rough sets and hesitant soft fuzzy rough sets as well.

The main goal of our paper is to take advantage of a novel construction of SSs derived from HFSs, in order to justify a 3WD strategy that acts on hesitant fuzzy data. In passing, we study some theoretical facts that expand our knowledge about the new SS produced from an HFS, which we term as the canonical soft set. The rest of this article is organized as follows. Section 2 gives a review of the rudiments on FSs, HFSs, SSs, RSs and 3WD. In Section 3, the notions of canonical soft sets, unit interval parameterized SSs, and derived HFSs are introduced. The relationships between canonical soft sets and derived HFSs are discussed. In Section 4, a novel 3WD algorithm is proposed to address group decision-making problems by using canonical soft sets of HFSs. In addition, an example regarding paper review is given to illustrate the validity of the proposed algorithm in Section 5. A summary of this study is given in Section 6.

2. Preliminaries

Some basic concepts related to FSs, SSs, RSs and 3WD are briefly reviewed in this section. Through the paper, $\mho$ denotes the universal set and $I = [0, 1]$. For any nonempty set K, we use the following notations:

- $\mathfrak{P}(K)$ represents the "power set" of K;
- $\mathfrak{P}^*(K)$ represents the family of all nonempty subsets of K;
- $\mathfrak{F}^*(K)$ represents the family of all nonempty finite subsets of K;
- $\mathfrak{F}_n(K)$ represents the family of all finite subsets E of K whose cardinality satisfies $|E| = n$ $(n \geq 1)$.

2.1. Fuzzy sets and HFSs

Zadeh [56] suggested that a certain type of uncertainty of information can be quantified by partial membership values. Consequently, a fuzzy set $\mathcal{M}$ in $\mho$ is defined by a mapping $\mathcal{M} : \mho \to I$ which represents the *membership function* of $\mathcal{M}$.

This definition extends the usual notion of a characteristic function, which in set theory, is equivalent to a standard subset. Now when $u \in \mho$, the value $\mathcal{M}(u)$ is the degree to which u belongs to $\mathcal{M}$. Let $\mathcal{FS}(\mho)$ denote the class of all FSs on $\mho$. A number of relevant operations of fuzzy sets can be introduced. With the "min-max" system, Zadeh defined the *"intersection", "union",* and *"complement"* operations of FSs as follows:

$$(\mathcal{M} \cap \mathcal{N})(\tau) = \mathcal{M}(\tau) \wedge \mathcal{N}(\tau),$$
$$(\mathcal{M} \cup \mathcal{N})(\tau) = \mathcal{M}(\tau) \vee \mathcal{N}(\tau),$$
$$\mathcal{M}^c(\tau) = 1 - \mathcal{M}(\tau),$$

where $\mathcal{M}, \mathcal{N} \in \mathcal{FS}(\mho)$ and $\tau \in \mho$. When $\mathcal{M}(\tau) \leq \mathcal{N}(\tau)$ for all $\tau \in \mho$, then $\mathcal{M} \subseteq \mathcal{N}$. It is obvious that $\mathcal{M} = \mathcal{N}$ if and only if $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{N} \subseteq \mathcal{M}$.

When an object has various viable membership values, it is often tough to ascertain its precise membership grade. For this, a concept of HFSs ("hesitant fuzzy sets") was proposed by Torra [41]. In an HFS, the membership grades of objects are represented by sets containing several (maybe infinitely many) values in the unit interval I. In the following, we recall some rudiments regarding the theory of HFSs.

Definition 2.1. [41] An HFS on $\mho$ is a mapping $\mathfrak{h} : \mho \to \mathfrak{P}(I)$.

Conventionally, an HFS $\mathfrak{h}$ on $\mho$ defined as a set of ordered pairs, namely $\mathfrak{h} = \{\langle \tau, \mathfrak{h}(\tau) \rangle : \tau \in \mho\}$. According to [45], an HFE ("*hesitant fuzzy element*") means a subset of I. That is, the power set $\mathfrak{P}(I)$ of I consists of all HFEs. Given an HFS $\mathfrak{h}$ and $\tau \in \mho$, the HFE $\mathfrak{h}(\tau)$ contains all the possible membership grades of τ to the HFS $\mathfrak{h}$. Henceforth, $\mathcal{HFS}(\mho)$ denotes the class of all HFSs on $\mho$.

Definition 2.2. An HFS $\mathfrak{h} : \mho \to \mathfrak{P}^*(I)$ is said to be a *regular HFS* (RHFS) on $\mho$.

In practical applications, it is justifiable to assume that the employed HFSs are regular. This means that at least one value in I must be assigned for each element of $\mho$ as its possible membership grade. Henceforth, the class of all RHFSs on $\mho$ is denoted by $\mathcal{RHFS}(\mho)$.

Definition 2.3. [7] A *typical hesitant fuzzy set* (THFS) on $\mho$ is a mapping $\mathfrak{h} : \mho \to \mathfrak{F}^*(I)$.

A nonempty finite subset of I is called a *typical hesitant fuzzy element* (THFE). It is clear that the family of all THFEs coincides with $\mathfrak{F}^*(I)$. Conventionally, the possible membership grades in a THFE E are listed in ascending order. In other words, E is written in the form $E = \{\tau_1, \tau_2, \cdots, \tau_n\} \subseteq I$ such that $\tau_1 < \tau_2 < \cdots < \tau_n$ and $1 \leq |E| = n < \infty$. Particularly, the THFE $E = \{1\}$ is said to be *full* and $E = \{0\}$ is referred to as the *empty* HFE. Accordingly, $\{\langle \tau, \{1\}\rangle : \tau \in \mho\}$ is called the *ideal* (or *full*) HFS on $\mho$, and $\{\langle \tau, \{0\}\rangle : \tau \in \mho\}$ is called the *anti-ideal* (or *empty*) HFS on $\mho$ In the following, we denote the class of all THFSs on $\mho$ by $\mathcal{THFS}(\mho)$.

Definition 2.4. [3] A THFS $\mathfrak{h}$ on $\mho$ is *n-typical* if there exists a positive integer n such that $|\mathfrak{h}(\tau)| = n$ for all $\tau \in \mho$.

To put it in another way, an n-typical hesitant fuzzy set is a mapping $\mathfrak{h} : \mho \to \mathfrak{F}_n(I)$ for some $n \geq 1$. For an HFS, there are multiple values in HFEs since there is more than one criterion for determining membership grades. This can be explained by a common application scenario. When multiple experts determine the membership grades of an object, there might be different values for the same object because different experts have distinguishing opinions.

2.2. *Soft sets*

In order to establish a theory framework to describe and deal with uncertainty, Molodtsov [36] proposed the important concept of SSs from the perspective of parameterization. Let $E_\mho$ be a set containing all the relevant parameters of objects in $\mho$. Conventionally, the set $E_\mho$ is called the *parameter space*. For convenience, we can simply denote $E_\mho$ by E. The pair $(\mho, E)$ is named as a soft universe. In practical applications, the parameter space E includes all attributes and characteristics of the object in $\mho$. The mathematical definition of SSs is as follows:

Definition 2.5. [36] Assume that $(\mho, E)$ is a soft universe. An ordered pair $\mathcal{T} = (G, B)$ is called a *soft set* over $\mho$. The subset $B \subseteq E$ is the *parameter set* of $\mathcal{T}$ and the *approximate function* of $\mathcal{T}$ is the mapping $G : B \to \mathfrak{P}(\mho)$.

Definition 2.6. [14] Let $\mathcal{T} = (G, B)$ be a soft set over $\mho$. Then $\mathcal{T}$ is said to be *serial* if $G(\varrho) \neq \emptyset$ for all $\varrho \in B$.

Definition 2.7. [14] Let $\mathcal{T} = (G, B)$ be a soft set over $\mho$. Then $\mathcal{T}$ is said to be *full* if $\bigcup_{\varrho \in B} G(\varrho) = \mho$.

A *covering* $\mathbb{C} = \{C^\alpha \mid \alpha \in \Lambda\}$ of $\mho$ is a family of non-empty subsets of $\mho$ such that for every $\tau \in \mho$, there exists an $\alpha \in \Lambda$ with $\tau \in C^\alpha$. A covering $\mathbb{C}$ of $\mho$ is said to be a *partition* of $\mho$ if

$$C^\alpha \neq C^\beta \Rightarrow C^\alpha \cap C^\beta = \emptyset$$

for all $\alpha, \beta \in \Lambda$.

Definition 2.8. [14] A soft set $\mathcal{T} = (G, B)$ over $\mho$ is called a *covering soft set* if it is full and serial.

Definition 2.9. [14] A soft set $\mathcal{T} = (G, B)$ over $\mho$ is called a *partition soft set* if $\{G(a) \mid a \in B\}$ forms a partition of $\mho$.

Definition 2.10. [15] A soft set $\mathcal{T} = (G, B)$ over $\mho$ is said to be *injective* if

$$a \neq b \Rightarrow G(a) \neq G(b)$$

for all $a, b \in B$.

Definition 2.11. A soft set $\mathcal{T} = (G, B)$ over $\mho$ is said to be *disjoint* if

$$a \neq b \Rightarrow G(a) \cap G(b) = \varnothing$$

for all $a, b \in B$.

Proposition 2.12. *Suppose that $\mathcal{T} = (G, B)$ is a soft set over $\mho$. Then the following are equivalent:*

(1) *$\mathcal{T}$ is an injective partition soft set;*

(2) *$\mathcal{T}$ is a disjoint covering soft set.*

Proof. Suppose that $\mathcal{T} = (G, B)$ is an injective partition soft set over $\mho$. By Definition 2.9, $\{G(a) \mid a \in B\}$ forms a partition of $\mho$. Thus $\mathcal{T}$ is a covering soft set such that

$$G(a) \neq G(b) \Rightarrow G(a) \cap G(b) = \varnothing$$

for all $a, b \in B$. Note also that $\mathcal{T}$ is an injective soft set. Thus for all $a, b \in B$, we have

$$a \neq b \Rightarrow G(a) \neq G(b).$$

It follows that $a \neq b$ implies $G(a) \cap G(b) = \varnothing$ for all $a, b \in B$. This shows that $\mathcal{T}$ is a disjoint covering soft set over $\mho$.

Conversely, assume that $\mathcal{T}$ is a disjoint covering soft set over $\mho$. According to Definition 2.8, $\mathcal{T}$ is both full and serial. Thus we have $\bigcup_{b \in B} G(b) = \mho$ and $G(b) \neq \varnothing$ for all $b \in B$. Since $\mathcal{T}$ is disjoint, we can deduce that $\mathcal{T}$ is an injective soft set such that $G(a) \neq G(b)$ implies $G(a) \cap G(b) = \varnothing$ for all $a, b \in B$. Thus $\{G(a) \mid a \in B\}$ forms a partition of $\mho$. This shows that $\mathcal{T}$ is an injective partition soft set over $\mho$. ◻

2.3. Rough sets and soft rough sets

Let Γ represent an equivalence relation on $\mho$. The pair $(\mho, \Gamma)$ is said to be a *Pawlak approximation space* [37]. For any $\tau_1, \tau_2 \in \mho$, if $(\tau_1, \tau_2) \in \Gamma$ we said that τ_1 and τ_2 are *Γ-indiscernible*. The equivalence class of $\tau_1 \in \mho$ is denoted by $[\tau_1]_\Gamma$. The quotient set of $\mho$ with respect to Γ is denoted by $\mho/\Gamma$.

Assume that Γ is an equivalence relation on $\mho$ and Υ is a subset of $\mho$. Then Υ can be characterized by means of the following two approximations:

$$\Gamma_* \Upsilon = \{\tau \in \mho : [\tau]_\Gamma \subseteq \Upsilon\},$$
$$\Gamma^* \Upsilon = \{\tau \in \mho : [\tau]_\Gamma \cap \Upsilon \neq \varnothing\}.$$

The sets $\Gamma_* \Upsilon$ and $\Gamma^* \Upsilon$ are called the *lower* and *upper* approximations of Υ with respect to the approximation space $(\mho, \Gamma)$. We say that the set Υ is *definable* if $\Gamma_* \Upsilon = \Gamma^* \Upsilon$; otherwise, Υ is said to be *rough* or *inexact*.

Furthermore, we can define the *positive region*, *boundary region* and *negative region* of Υ respectively as follows:

$$\mathrm{Pos}_{\Gamma}\Upsilon = \Gamma_*\Upsilon,$$
$$\mathrm{Bnd}_{\Gamma}\Upsilon = \Gamma^*\Upsilon - \Gamma_*\Upsilon,$$
$$\mathrm{Neg}_{\Gamma}\Upsilon = \mho - \Gamma^*\Upsilon.$$

Assume that $C = \Gamma_*\Upsilon$ and $D = \Gamma^*\Upsilon$ where $\Upsilon \subseteq \mho$. Sometimes, a *rough set* can also be represented by the pair $(C, D) \in \mathfrak{P}(\mho) \times \mathfrak{P}(\mho)$. If Υ is defined by a predicate $\mathfrak{H}$ and $\tau \in \mho$, we have the following:

- $\tau \in \Gamma_*\Upsilon$ means that τ certainly has property $\mathfrak{H}$;
- $\tau \in \Gamma^*\Upsilon$ means that τ possibly has property $\mathfrak{H}$;
- $\tau \in \mathrm{Neg}_{\Gamma}\Upsilon$ means that τ definitely does not have property $\mathfrak{H}$.

Feng et al. [16] considered the generalization of Pawlak's rough sets based on soft set theory, and put forward the concept of soft rough sets. Different from rough sets based on Pawlak approximation space, soft rough sets are based on the granular structure given by a soft approximation space (SAS). An SAS is an ordered pair $(\mho, \mathcal{T})$, where $\mathcal{T}$ is a soft set over $\mho$. More specifically, the soft lower and upper approximations are defined in the following way:

Definition 2.13. [16] Assume that $\mathcal{T} = (G, B)$ is a soft set over $\mho$, $H \subseteq \mho$ and $\mathcal{P} = (\mho, \mathcal{T})$ is a soft approximation space. The *soft lower approximation* $\underline{apr}_{\mathcal{P}}(H)$ and the *soft upper approximation* $\overline{apr}_{\mathcal{P}}(H)$ of H with respect to the SAS $\mathcal{P} = (\mho, \mathcal{T})$ are defined as:

$$\underline{apr}_{\mathcal{P}}(H) = \{\tau \in \mho : \exists \varrho \in B\,(\tau \in G(\varrho) \subseteq H)\},$$

$$\overline{apr}_{\mathcal{P}}(H) = \{\tau \in \mho : \exists \varrho \in B\,(\tau \in G(\varrho), G(\varrho) \cap H \neq \emptyset)\}.$$

When $\underline{apr}_{\mathcal{P}}(H) = \overline{apr}_{\mathcal{P}}(H)$, we say that the set H is *soft $\mathcal{P}$-definable*; otherwise, H is said to be a soft $\mathcal{P}$-rough set.

Below the concepts given in Definition 2.13 are reformulated:

Proposition 2.14. [16] *Suppose that $\mathcal{T} = (G, B)$ is a soft set over $\mho$ and $\mathcal{P} = (\mho, \mathcal{T})$ is an SAS. Then*

$$\underline{apr}_{\mathcal{P}}(H) = \bigcup_{\varrho \in B}\{G(\varrho) : G(\varrho) \subseteq H\},$$
$$\overline{apr}_{\mathcal{P}}(H) = \bigcup_{\varrho \in B}\{G(\varrho) : G(\varrho) \cap H \neq \emptyset\},$$

where H is a subset of $\mho$.

Continuing with the framework of Definition 2.13, we recall some useful properties:

Theorem 2.15. [14] *Assume that $\mathcal{T} = (G, B)$ is a soft set over $\mho$ and $\mathcal{P} = (\mho, \mathcal{T})$ is an SAS. For $H, J \subseteq \mho$, we have*

(1) $\underline{apr}_{\mathcal{P}}(\emptyset) = \overline{apr}_{\mathcal{P}}(\emptyset) = \emptyset$;

(2) $\underline{apr}_{\mathcal{P}}(\mho) = \overline{apr}_{\mathcal{P}}(\mho) = \bigcup_{\varrho \in B} G(\varrho)$;

(3) $H \subseteq J \Rightarrow \underline{apr}_{\mathcal{P}}(H) \subseteq \underline{apr}_{\mathcal{P}}(J)$;

(4) $H \subseteq J \Rightarrow \overline{apr}_{\mathcal{P}}(H) \subseteq \overline{apr}_{\mathcal{P}}(J)$;

(5) $\underline{apr}_{\mathcal{P}}(H \cap J) \subseteq \underline{apr}_{\mathcal{P}}(H) \cap \underline{apr}_{\mathcal{P}}(J)$;

(6) $\underline{apr}_{\mathcal{P}}(H \cup J) \supseteq \underline{apr}_{\mathcal{P}}(H) \cup \underline{apr}_{\mathcal{P}}(J)$;

(7) $\overline{apr}_{\mathcal{P}}(H \cup J) = \overline{apr}_{\mathcal{P}}(H) \cup \overline{apr}_{\mathcal{P}}(J)$;

(8) $\overline{apr}_{\mathcal{P}}(H \cap J) \subseteq \overline{apr}_{\mathcal{P}}(H) \cap \overline{apr}_{\mathcal{P}}(J)$.

Theorem 2.16. [16] *Suppose that $\mathcal{T}$ is a soft set over $\mho$ and $\mathcal{P} = (\mho, \mathcal{T})$ is an SAS. Then the following are equivalent:*

(1) *$\mathcal{T}$ is a full soft set;*

(2) $\underline{apr}_{\mathcal{P}}(\mho) = \mho$;

(3) $\overline{apr}_{\mathcal{P}}(\mho) = \mho$;

(4) $\Upsilon \subseteq \overline{apr}_{\mathcal{P}}(\Upsilon)$ *for all* $\Upsilon \subseteq \mho$;

(5) $\overline{apr}_{\mathcal{P}}(\{\tau\}) \neq \emptyset$ *for all* $\tau \in \mho$.

2.4. Three-way decision

3WD is an uncertain framework based on theory of probability rough sets and decision rough sets. The basic philosophy of 3WD is thinking in threes which means to understand and process a whole through three different and related parts. Based on the trisecting-acting-outcome framework, Yao [52] has proposed a generalized 3WD model. Hereinafter, the framework trisecting-acting-outcome is simply named TAO. The TAO model of 3WD concentrates on the following three works:

(1) To divide the universe into three parts;

(2) To take actions on the three parts separately;

(3) To optimize the results and obtain a desirable outcome.

It is clear that the TAO model of 3WD is an interpretative case of thinking in threes. The fundamental notion of 3WD is trisecting the whole. Three parts of trisection can be seen as three different aspects, perspectives or components of the whole and their integration represents and covers the whole. Hence the three parts are both independent and interrelated.

Yao [51] introduced a general model for building a trisection with a single evaluation function. In this model, a linearly ordered set $(L, \leq)$ is used as the codomain of an evaluation function, which specifies the scale of the evaluation. Given two evaluation values $n, m \in L$, $n < m$ means $n \leq m$ and $n \neq m$. Moreover, we also write $n < m$ as $m > n$.

Definition 2.17. [51] Let $\mathcal{E} : \mho \longrightarrow (L, \leq)$ be an evaluation function on $\mho$. Given two thresholds $\lambda, \mu \in L$ with $\lambda < \mu$, a trisection of $\mho$ can be derived as follows:

$$\mathrm{Pos}^{[\mu,\cdot)}(\mathcal{E}) = \{\tau \in \mho : \mu \leq \mathcal{E}(\tau)\},$$

$$\mathrm{Bnd}^{(\lambda,\mu)}(\mathcal{E}) = \{\tau \in \mho : \lambda < \mathcal{E}(\tau) < \mu\},$$
$$\mathrm{Neg}^{(\cdot,\lambda]}(\mathcal{E}) = \{\tau \in \mho : \mathcal{E}(\tau) \leq \lambda\}.$$

Intuitively, we refer to elements in the disjoint sets $\mathrm{Pos}^{[\mu,\cdot)}(\mathcal{E})$, $\mathrm{Bnd}^{(\lambda,\mu)}(\mathcal{E})$ and $\mathrm{Neg}^{(\cdot,\lambda]}(\mathcal{E})$ as positive, neutral and negative objects, respectively. The above model plays a fundamental role in constructing set-theoretic models of 3WDs [51]. It should be noted that how to select easy-to-understand and meaningful scales is an issue of great importance. In many practical applications, it is sufficient to consider subsets of real numbers as evaluation scales. More specifically, as a special case of Definition 2.17, we have the following notion.

Definition 2.18. [54] Suppose that $\mathcal{E} : \mho \longrightarrow [b, a]$ is an evaluation on $\mho$ where $b < a$ are two real numbers. Given two thresholds γ, δ with $b \leq \gamma < \delta \leq a$, a trisection of $\mho$ can be derived as follows:

$$H^{[\delta,a]}(\mathcal{E}) = \{\tau \in \mho : \delta \leq \mathcal{E}(\tau) \leq a\},$$
$$M^{(\gamma,\delta)}(\mathcal{E}) = \{\tau \in \mho : \gamma < \mathcal{E}(\tau) < \delta\},$$
$$L^{[b,\gamma]}(\mathcal{E}) = \{\tau \in \mho : b \leq \mathcal{E}(\tau) \leq \gamma\}.$$

Elements in the disjoint sets $H^{[\delta,a]}(\mathcal{E})$, $M^{(\gamma,\delta)}(\mathcal{E})$ and $L^{[b,\gamma]}(\mathcal{E})$ are called objects with high, medium and low values, respectively.

To facilitate the reading of this article, the abbreviations of some frequently used terminologies are listed in Table 1.

Table 1. Abbreviations of terminologies.

Abbreviations	Full terminologies
3WD	three-way decision
CSS	canonical soft set
DHFS	derived hesitant fuzzy set
FS	fuzzy set
GIFSS	generalized intuitionistic fuzzy soft set
HFE	hesitant fuzzy element
HFS	hesitant fuzzy set
IFS	intuitionistic fuzzy set
IFSS	intuitionistic fuzzy soft set
IVFS	interval-valued fuzzy set
IVIFS	interval-valued intuitionistic fuzzy set
MADM	multi-attribute decision making
RHFS	regular hesitant fuzzy set
RS	rough set
SAS	soft approximation space
SS	soft set
TAO	trisecting-acting-outcome
THFE	typical hesitant fuzzy element
THFS	typical hesitant fuzzy set
UIPSS	unit interval parameterized soft set

3. Mutual representations between HFSs and SSs

In this section, we define some useful concepts including canonical soft sets, unit interval parameterized SSs and derived HFSs. The relationships among these notions are examined in detail.

Definition 3.1. Suppose that $\mathfrak{h} : \mho \to \mathfrak{P}(I)$ is an HFS. Then

$$V_{\mathfrak{h}} = \bigcup_{\tau \in \mho} \mathfrak{h}(\tau)$$

is called the *value space* of the HFS $\mathfrak{h}$.

In other words, the value space of an HFS is a set containing all possible membership values of objects in $\mho$.

Definition 3.2. Let $\mathfrak{h} : \mho \to \mathfrak{P}(I)$ be an HFS with its value space $V_{\mathfrak{h}}$. Then

$$\mathfrak{h}^{-1}(\kappa) = \{\tau \in \mho : \kappa \in \mathfrak{h}(\tau)\}$$

is called the *value coset* of $\kappa \in V_{\mathfrak{h}}$.

Definition 3.3. A soft set $\mathcal{T} = (G, B)$ over $\mho$ is called a *unit interval parameterized soft set* (UIPSS) if $B \subseteq I$.

Definition 3.4. [12] Let $\mathcal{T} = (G, B)$ be a soft set over $\mho$ and $\nu \in \mho$. Then

$$\mathrm{Co}_{\mathcal{T}}(\nu) = \{\varrho \in B : \nu \in G(\varrho)\}$$

is said to be the *parameter coset* of the alternative ν in G.

Note that $\mathrm{Co}_{\mathcal{T}}(\nu)$ contains all the parameters of the alternative ν, according to the information carried by $\mathcal{T}$.

Definition 3.5. Let $\mathcal{T} = (G, B)$ be a UIPSS over $\mho$ and $\nu \in \mho$. Then we can define an HFS $\mathfrak{h}_{\mathcal{T}} : \mho \to \mathfrak{P}(I)$ on $\mho$ by

$$\mathfrak{h}_{\mathcal{T}}(\nu) = \mathrm{Co}_{\mathcal{T}}(\nu) = \{\varrho \in B : \nu \in G(\varrho)\},$$

which is said to be the *derived hesitant fuzzy set* (DHFS) of the UIPSS $\mathcal{T}$.

Definition 3.6. Let $\mathfrak{h} : \mho \to \mathfrak{P}(I)$ be an HFS. A soft set $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ over $\mho$ is called the *canonical soft set* (CSS) of the HFS $\mathfrak{h}$, where $V_{\mathfrak{h}} = \bigcup_{\tau \in \mho} \mathfrak{h}(\tau)$ and $\hat{C}_{\mathfrak{h}}(\kappa) = \mathfrak{h}^{-1}(\kappa)$ for all $\kappa \in V_{\mathfrak{h}}$.

Example 3.7. Let $\mho = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ and $\mathfrak{h}_1$ be an HFS over $\mho$. Suppose that the HFEs of $\mathfrak{h}_1$ are given by $\mathfrak{h}_1(\tau_1) = \{0.2, 0.4, 0.5\}$, $\mathfrak{h}_1(\tau_2) = \{0.4, 0.5\}$, $\mathfrak{h}_1(\tau_3) = \{0.4, 0.7, 0.8\}$, $\mathfrak{h}_1(\tau_4) = \{0.5, 0.8\}$. By Definition 3.1, the value space of $\mathfrak{h}_1$ is $V_{\mathfrak{h}_1} = \{0.2, 0.4, 0.5, 0.7, 0.8\} = V_1$. According to Definition 3.2, the value cosets of $\mathfrak{h}_1$ can be computed. The obtained results are $\mathfrak{h}_1^{-1}(0.2) = \{\tau_1\}$, $\mathfrak{h}_1^{-1}(0.4) = \{\tau_1, \tau_2, \tau_3\}$, $\mathfrak{h}_1^{-1}(0.5) = \{\tau_1, \tau_2, \tau_4\}$, $\mathfrak{h}_1^{-1}(0.7) = \{\tau_3\}$ and $\mathfrak{h}_1^{-1}(0.8) = \{\tau_3, \tau_4\}$. The CSS $\mathfrak{C}_{\mathfrak{h}_1} = (\mathfrak{h}_1^{-1}, V_1)$ of the HFS $\mathfrak{h}_1$ is shown in Table 2.

Table 2. The CSS $\mathfrak{C}_{\mathfrak{h}_1} = (\mathfrak{h}_1^{-1}, V_1)$ of the HFS $\mathfrak{h}_1$.

$\mho$	0.2	0.4	0.5	0.7	0.8
τ_1	1	1	1	0	0
τ_2	0	1	1	0	0
τ_3	0	1	0	1	1
τ_4	0	0	1	0	1

The following result shows an explicit mutual connection between UIPSSs and HFSs:

Theorem 3.8. *Assume that $\mathcal{T} = (G, B)$ is a serial UIPSS over $\mho$ and $\mathfrak{h}_{\mathcal{T}} : \mho \to \mathfrak{P}(I)$ is the DHFS of $\mathcal{T}$. Then we have $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}} = \mathcal{T}$, where $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}}$ is the CSS of $\mathfrak{h}_{\mathcal{T}}$.*

Proof. Let $\mathcal{T} = (G, B)$ be a serial UIPSS over $\mho$ and $\mathfrak{h}_{\mathcal{T}} : \mho \to \mathfrak{P}(I)$ be the DHFS of $\mathcal{T}$. By Definition 3.5, the DHFS $\mathfrak{h}_{\mathcal{T}}$ of $\mathcal{T}$ is given by $\mathfrak{h}_{\mathcal{T}}(\nu) = \mathrm{Co}_{\mathcal{T}}(\nu) = \{\varrho \in B : \nu \in G(\varrho)\}$ with $\nu \in \mho$. Then according to Definition 3.6, the CSS of $\mathfrak{h}_{\mathcal{T}}$ is a UIPSS $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}} = (\hat{C}_{\mathfrak{h}_{\mathcal{T}}}, V_{\mathfrak{h}_{\mathcal{T}}})$ over $\mho$ and $\hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa) = \mathfrak{h}_{\mathcal{T}}^{-1}(\kappa)$ for all $\kappa \in V_{\mathfrak{h}_{\mathcal{T}}}$.

Firstly, we prove $V_{\mathfrak{h}_{\mathcal{T}}} = B$. By Definition 3.1, we have $V_{\mathfrak{h}_{\mathcal{T}}} = \bigcup_{\nu\in\mho} \mathfrak{h}_{\mathcal{T}}(\nu)$. For any $\kappa \in V_{\mathfrak{h}_{\mathcal{T}}}$, there exist a $\nu_0 \in \mho$ such that $\kappa \in \mathfrak{h}_{\mathcal{T}}(\nu_0)$. It is clear that $\kappa \in \mathrm{Co}_{\mathcal{T}}(\nu_0)$ such that $\kappa \in B$. Hence we have $V_{\mathfrak{h}_{\mathcal{T}}} \subseteq B$. Conversely, assume that $\mathcal{T} = (G, B)$ is a serial UIPSS. By Definition 2.6, we have $G(\varrho) \neq \varnothing$ for any $\varrho \in B$. This means that for any $\varrho \in B$ there exist a $\nu_0 \in \mho$ such that $\nu_0 \in G(\varrho)$. Then, by definition, we have $\varrho \in \mathrm{Co}_{\mathcal{T}}(\nu_0)$ and then $\varrho \in \mathfrak{h}_{\mathcal{T}}(\nu_0)$. It is clear that $\varrho \in V_{\mathfrak{h}_{\mathcal{T}}}$ for any $\varrho \in B$. Thus we have $B \subseteq V_{\mathfrak{h}_{\mathcal{T}}}$. We have proved $V_{\mathfrak{h}_{\mathcal{T}}} = B$.

Then we prove $\hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa) = G(\kappa)$ for any $\kappa \in V_{\mathfrak{h}_{\mathcal{T}}} = B$. Assume that $\nu_0 \in G(\kappa)$ for any $\kappa \in B$. According to Definition 3.5, we have $\kappa \in \mathfrak{h}_{\mathcal{T}}(\nu_0)$. Then, by Definition 3.2, we have $\nu_0 \in \mathfrak{h}_{\mathcal{T}}^{-1}(\kappa)$. So it is easy to see $\nu_0 \in \hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa)$. Hence we have proved that $G(\kappa) \subseteq \hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa)$ for any $\kappa \in B$. Conversely, Assume that $\nu_0 \in \hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa)$. From the definition of CSSs, we have $\nu_0 \in \mathfrak{h}_{\mathcal{T}}^{-1}(\kappa)$. Then by definition of value coset, we have $\kappa \in \mathfrak{h}_{\mathcal{T}}(\nu_0)$ and then $\kappa \in \mathrm{Co}_{\mathcal{T}}(\nu_0)$. So it is easy to see $\nu_0 \in G(\kappa)$. Thus we have proved that $\hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa) \subseteq G(\kappa)$ for any $\kappa \in B$. We have proved that $\hat{C}_{\mathfrak{h}_{\mathcal{T}}}(\kappa) = G(\kappa)$ established for any $\kappa \in V_{\mathfrak{h}_{\mathcal{T}}} = B$. Hence, we conclude that $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}} = \mathcal{T}$, which completes the proof. □

It should be noted that $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}} = \mathcal{T}$ might not hold when the UIPSS $\mathcal{T}$ is not serial. To demonstrate this, let us consider the following example.

Example 3.9. Let $\mho = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ and $\mathcal{T} = (G, B)$ be an UIPSS over $\mho$. The parameter set of $\mathcal{T}$ is $B = \{0.1, 0.5, 0.7, 0.9, 1\}$ and the approximate function of $\mathcal{T}$ is given by $G(0.1) = \{\tau_1, \tau_4\}$, $G(0.5) = \{\tau_1, \tau_2, \tau_3\}$, $G(0.7) = \{\tau_1, \tau_2, \tau_4\}$, $G(0.9) = \{\tau_3, \tau_4\}$ and $G(1) = \varnothing$. The tabular representation of $\mathcal{T}$ is shown in Table 3. By Definition 3.5, the DHFS $\mathfrak{h}_{\mathcal{T}}$ of $\mathcal{T}$ is an HFS with its HFEs given by: $\mathfrak{h}_{\mathcal{T}}(\tau_1) = \{0.1, 0.5, 0.7\}$, $\mathfrak{h}_{\mathcal{T}}(\tau_2) = \{0.5, 0.7\}$, $\mathfrak{h}_{\mathcal{T}}(\tau_3) = \{0.5, 0.9\}$ and $\mathfrak{h}_{\mathcal{T}}(\tau_4) = \{0.1, 0.7, 0.9\}$. According to Definition 3.6, we can obtain the CSS $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}} = (\hat{C}_{\mathfrak{h}_{\mathcal{T}}}, V_{\mathfrak{h}_{\mathcal{T}}})$ of the DHFS $\mathfrak{h}_{\mathcal{T}}$, which is shown in Table 4. As we know, the parameter set $V_{\mathfrak{h}_{\mathcal{T}}}$ of $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}}$ is the value space of the DHFS $\mathfrak{h}_{\mathcal{T}}$. It is clear that

$$V_{\mathfrak{h}_{\mathcal{T}}} = \{0.1, 0.5, 0.7, 0.9\} \neq \{0.1, 0.5, 0.7, 0.9, 1\} = B.$$

This shows that $\mathfrak{C}_{\mathfrak{h}_{\mathcal{T}}} \neq \mathcal{T}$ if $\mathcal{T}$ is not a serial UIPSS.

Table 3. The UIPSS $\mathcal{T} = (G, B)$ over $\mho$.

$\mho$	0.1	0.5	0.7	0.9	1
τ_1	1	1	1	0	0
τ_2	0	1	1	0	0
τ_3	0	1	0	1	0
τ_4	1	0	1	1	0

Table 4. The CSS $\mathfrak{C}_{\mathfrak{h}_\mathcal{T}} = (\hat{C}_{\mathfrak{h}_\mathcal{T}}, V_{\mathfrak{h}_\mathcal{T}})$ of the DHFS $\mathfrak{h}_\mathcal{T}$.

$\mho$	0.1	0.5	0.7	0.9
τ_1	1	1	1	0
τ_2	0	1	1	0
τ_3	0	1	0	1
τ_4	1	0	1	1

Theorem 3.10. *Suppose that $\mathfrak{h}$ is an HFS on $\mho$ and $\mathfrak{C}_\mathfrak{h} = (\hat{C}_\mathfrak{h}, V_\mathfrak{h})$ is the CSS of $\mathfrak{h}$. Then we have $\mathfrak{h}_{\mathfrak{C}_\mathfrak{h}} = \mathfrak{h}$, where $\mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}$ is the DHFS of $\mathfrak{C}_\mathfrak{h}$.*

Proof. Let $\mathfrak{h} : \mho \to \mathfrak{P}(I)$ be an HFS on $\mho$ and $\mathfrak{C}_\mathfrak{h} = (\hat{C}_\mathfrak{h}, V_\mathfrak{h})$ be the CSS of $\mathfrak{h}$. By Definition 3.6, $\mathfrak{C}_\mathfrak{h}$ is an UIPSS and $\hat{C}_\mathfrak{h}(a) = \mathfrak{h}^{-1}(a)$ for all $a \in V_\mathfrak{h}$. Let $\mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}$ be the DHFS of $\mathfrak{C}_\mathfrak{h}$ and be given by $\mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}(v) = \mathrm{Co}_{\mathfrak{C}_\mathfrak{h}}(v) = \{a \in V_\mathfrak{h} : v \in \hat{C}_\mathfrak{h}(a)\}$.

Assume that $a \in \mathfrak{h}(v_0)$ for any $v_0 \in \mho$. By Definition 3.2, it is clear that $v_0 \in \mathfrak{h}^{-1}(a)$. Then we have $v_0 \in \hat{C}_\mathfrak{h}(a)$. From the definition of DHFS, it is easy to see $a \in \mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}(v_0)$. Thus we have $\mathfrak{h}(v_0) \subseteq \mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}(v_0)$ for any $v_0 \in \mho$. Conversely, assume that $a \in \mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}(v_0)$. By definition of DHFS, we have $v_0 \in \hat{C}_\mathfrak{h}(a)$ which also show that $v_0 \in \mathfrak{h}^{-1}(a)$. From the definition of value cosets, it is easy to see that $a \in \mathfrak{h}(v_0)$. Thus we have $\mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}(v_0) \subseteq \mathfrak{h}(v_0)$ for any $v_0 \in \mho$. We have proved that $\mathfrak{h}_{\mathfrak{C}_\mathfrak{h}}(v_0) = \mathfrak{h}(v_0)$ for any $v_0 \in \mho$. □

The following result presents some characterizations of RHFSs in terms of their CSSs.

Theorem 3.11. *Assume that $\mathfrak{h}$ is an HFS on $\mho$ with its CSS $\mathfrak{C}_\mathfrak{h} = (\hat{C}_\mathfrak{h}, V_\mathfrak{h})$. Then the following are equivalent:*

(1) *$\mathfrak{h}$ is an RHFS;*

(2) *$\mathfrak{C}_\mathfrak{h}$ is a full soft set;*

(3) *$\mathfrak{C}_\mathfrak{h}$ is a covering soft set.*

Proof. Firstly, we show that (1) implies (2). Assume that $\mathfrak{h}$ is an RHFS. For every $u \in \mho$, $\mathfrak{h}(u)$ is non-empty, whence there exists $\alpha \in \mathfrak{h}(u)$. That is, $u \in \hat{C}_\mathfrak{h}(\alpha)$ for some $\alpha \in V_\mathfrak{h}$. Thus $\mho \subseteq \bigcup_{\alpha \in V_\mathfrak{h}} \hat{C}_\mathfrak{h}(\alpha)$. Clearly, we also have $\bigcup_{\alpha \in V_\mathfrak{h}} \hat{C}_\mathfrak{h}(\alpha) \subseteq \mho$. Thus $\bigcup_{\alpha \in V_\mathfrak{h}} \hat{C}_\mathfrak{h}(\alpha) = \mho$. Therefore, $\mathfrak{C}_\mathfrak{h} = (\hat{C}_\mathfrak{h}, V_\mathfrak{h})$ is a full soft set over $\mho$.

Next, assume that $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ is a full soft set. To prove that $\mathfrak{C}_{\mathfrak{h}}$ is a covering soft set, we only need to show that $\hat{C}_{\mathfrak{h}}(\alpha) \neq \varnothing$ for all $\alpha \in V_{\mathfrak{h}}$. In fact, let $\alpha \in V_{\mathfrak{h}}$. Then we have $\alpha \in \mathfrak{h}(u)$ for some $u \in \mho$. Hence, $\hat{C}_{\mathfrak{h}}(\alpha)$ is non-empty since $u \in \hat{C}_{\mathfrak{h}}(\alpha)$. This shows that (2) implies (3).

Lastly, we show that (3) implies (1). Assume that $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ is a covering soft set. By Definition 2.8, $\mathfrak{C}_{\mathfrak{h}}$ is a full soft set over $\mho$. Thus we have $\bigcup_{\alpha \in V_{\mathfrak{h}}} \hat{C}_{\mathfrak{h}}(\alpha) = \mho$. Therefore, there exists $\alpha \in V_{\mathfrak{h}}$ with $u \in \hat{C}_{\mathfrak{h}}(\alpha)$ for all $u \in \mho$. It follows that $\mathfrak{h}(u)$ is non-empty since $\alpha \in \mathfrak{h}(u)$. That is, $\mathfrak{h}$ is an RHFS. □

At the end of this section, we give several interesting characterizations of fuzzy sets by virtue of CSSs of HFSs.

Theorem 3.12. *Let $\mathfrak{h}$ be an HFS on $\mho$ with its CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$. Then the following are equivalent:*

(1) $\mathfrak{h}$ *is a fuzzy set;*

(2) $\mathfrak{h}(\tau)$ *is a singleton for all* $\tau \in \mho$*;*

(3) $\mathfrak{C}_{\mathfrak{h}}$ *is a disjoint full soft set;*

(4) $\mathfrak{C}_{\mathfrak{h}}$ *is a disjoint covering soft set;*

(5) $\mathfrak{C}_{\mathfrak{h}}$ *is an injective partition soft set.*

Proof. Note first that $\mathfrak{h}$ is a fuzzy set if and only if $|\mathfrak{h}(\tau)| = 1$ for all $\tau \in \mho$. Thus it is clear that (1) and (2) are equivalent.

Next, assume that $\mathfrak{h}(\tau)$ is a singleton for all $\tau \in \mho$. It is obvious that $\mathfrak{h}$ is an RHFS since $|\mathfrak{h}(\tau)| = 1$ for all $\tau \in \mho$. According to Theorem 3.11, it follows that the CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ of the HFS $\mathfrak{h}$ is a full soft set. We can also show that $\mathfrak{C}_{\mathfrak{h}}$ is a disjoint soft set. In fact, assume that there exist $\kappa_1, \kappa_2 \in V_{\mathfrak{h}}$ such that $\kappa_1 \neq \kappa_2$ and $\mathfrak{h}^{-1}(\kappa_1) \cap \mathfrak{h}^{-1}(\kappa_2) \neq \varnothing$. Take $\tau_0 \in \mathfrak{h}^{-1}(\kappa_1) \cap \mathfrak{h}^{-1}(\kappa_2)$. Then we have $\{\kappa_1, \kappa_2\} \subseteq \mathfrak{h}(\tau_0)$. Note also that $\kappa_1 \neq \kappa_2$. It follows that $|\mathfrak{h}(\tau_0)| \geq 2$, which leads to a contradiction. Therefore, $\mathfrak{C}_{\mathfrak{h}}$ is a disjoint full soft set. Conversely, suppose that $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ is a disjoint full soft set. From Definition 2.11, it follows that $\kappa_1 \neq \kappa_2$ implies $\mathfrak{h}^{-1}(\kappa_1) \cap \mathfrak{h}^{-1}(\kappa_2) = \varnothing$ for all $\kappa_1, \kappa_2 \in V_{\mathfrak{h}}$. This means that $|\mathfrak{h}(\tau)| \leq 1$ for all $\tau \in \mho$. Since $\mathfrak{C}_{\mathfrak{h}}$ is also a full soft set, it follows from Theorem 3.11 that $\mathfrak{h}$ is an RHFS. Hence, $|\mathfrak{h}(\tau)| \geq 1$ for all $\tau \in \mho$. Therefore, $\mathfrak{h}(\tau)$ is a singleton for all $\tau \in \mho$. This shows that (2) and (3) are equivalent.

According to Theorem 3.11, it can be seen that the CSS $\mathfrak{C}_{\mathfrak{h}}$ is a full soft set if and only if it is a covering soft set. This implies that (3) and (4) are equivalent.

In addition, it follows from Proposition 2.12 that (4) and (5) are equivalent. □

4. A novel 3WD model

The purpose of this section is to state a novel 3WD algorithm with potential to help us in decision-making with hesitant fuzzy information. This strategy will benefit from the novel idea of CSSs associated with HFSs.

4.1. Soft approximations based on HFSs

Suppose that $\mathfrak{h}$ is an HFS and $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ is its CSS. Then $P = (\mho, \mathfrak{C}_{\mathfrak{h}})$ forms an SAS. According to Definition 2.13, we state the soft lower and upper approximations as:

$$\underline{\Upsilon}_{\mathfrak{h}} = \bigcup_{\kappa \in V_{\mathfrak{h}}} \{\hat{C}_{\mathfrak{h}}(\kappa) \mid \hat{C}_{\mathfrak{h}}(\kappa) \subseteq \Upsilon\}, \tag{4.1}$$

$$\overline{\Upsilon}_{\mathfrak{h}} = \bigcup_{\kappa\in V_{\mathfrak{h}}}\{\hat{C}_{\mathfrak{h}}(\kappa) \mid \hat{C}_{\mathfrak{h}}(\kappa) \cap \Upsilon \neq \varnothing\}, \tag{4.2}$$

where Υ is a subset of $\mho$.

Furthermore, the positive, boundary and the negative regions of Υ can be defined from $\mathfrak{h}$ as follows:

$$\begin{aligned}
\mathrm{Pos}_{\mathfrak{h}}(\Upsilon) &= \underline{\Upsilon}_{\mathfrak{h}},\\
\mathrm{Bnd}_{\mathfrak{h}}(\Upsilon) &= \overline{\Upsilon}_{\mathfrak{h}} - \underline{\Upsilon}_{\mathfrak{h}},\\
\mathrm{Neg}_{\mathfrak{h}}(\Upsilon) &= \mho - \overline{\Upsilon}_{\mathfrak{h}}.
\end{aligned}$$

Example 4.1. Suppose that a transport company intends to deliver a batch of relief supplies to the disaster area. There are four available transport modes $\mho = \{\tau_1, \tau_2, \tau_3, \tau_4\}$. The meaning of the objects in $\mho$ is given as below:

- τ_1: highway transportation;
- τ_2: air transportation;
- τ_3: railway transportation;
- τ_4: sea transportation.

The transportation company invites several experts to evaluate the four methods of transportation and take into account factors such as time, cost and weather. Based on the evaluation results, an HFS $\mathfrak{h}_2$ is constructed. Assume that the $\mathfrak{h}_2$ is given by $\mathfrak{h}_2(\tau_1) = \{0.4, 0.8\}$, $\mathfrak{h}_2(\tau_2) = \{0.6, 0.8, 1\}$, $\mathfrak{h}_2(\tau_3) = \{0.2, 0.4, 0.8\}$, $\mathfrak{h}_2(\tau_4) = \{0, 0.2\}$. It is easy to see that

$$V_{\mathfrak{h}_2} = \mathfrak{h}_2(\tau_1) \cup \mathfrak{h}_2(\tau_2) \cup \mathfrak{h}_2(\tau_3) \cup \mathfrak{h}_2(\tau_4) = \{0, 0.2, 0.4, 0.6, 0.8, 1\} = V_2$$

is the value space of $\mathfrak{h}_2$. By Definition 3.6, we can establish the CSS $\mathfrak{C}_{\mathfrak{h}_2} = (\mathfrak{h}_2^{-1}, V_2)$ of HFS $\mathfrak{h}_2$. The obtained results are shown in Table 5.

Table 5. The CSS $\mathfrak{C}_{\mathfrak{h}_2} = (\mathfrak{h}_2^{-1}, V_2)$ of HFS $\mathfrak{h}_2$.

$\mho$	0	0.2	0.4	0.6	0.8	1
τ_1	0	0	1	0	1	0
τ_2	0	0	0	1	1	1
τ_3	0	1	1	0	1	0
τ_4	1	1	0	0	0	0

Let $P = (\mho, \mathfrak{C}_{\mathfrak{h}_2})$ be the corresponding SAS. For $\Upsilon = \{\tau_1, \tau_2\} \subseteq \mho$, the soft lower approximation of Υ is

$$\underline{\Upsilon}_{\mathfrak{h}_2} = \bigcup_{\kappa\in V_2}\{\hat{C}_{\mathfrak{h}_2}(\kappa) \mid \hat{C}_{\mathfrak{h}_2}(\kappa) \subseteq \Upsilon\} = \mathfrak{h}_2^{-1}(0.6) \cup \mathfrak{h}_2^{-1}(1) = \{\tau_2\}$$

and the soft upper approximation of Υ is

$$\overline{\Upsilon}_{\mathfrak{h}_2} = \bigcup_{\kappa\in V_2}\{\hat{C}_{\mathfrak{h}_2}(\kappa) \mid \hat{C}_{\mathfrak{h}_2}(\kappa) \cap \Upsilon \neq \varnothing\} = \mathfrak{h}_2^{-1}(0.4) \cup \mathfrak{h}_2^{-1}(0.6) \cup \mathfrak{h}_2^{-1}(0.8) \cup \mathfrak{h}_2^{-1}(1) = \{\tau_1, \tau_2, \tau_3\}.$$

Moreover, we can obtain the positive region $\mathrm{Pos}_{\mathfrak{h}_2}(\Upsilon) = \underline{\Upsilon}_{\mathfrak{h}_2} = \{\tau_2\}$, the negative region $\mathrm{Neg}_{\mathfrak{h}_2}(\Upsilon) = \mho - \overline{\Upsilon}_{\mathfrak{h}_2} = \{\tau_4\}$ and the boundary region $\mathrm{Bnd}_{\mathfrak{h}_2}(\Upsilon) = \overline{\Upsilon}_{\mathfrak{h}_2} - \underline{\Upsilon}_{\mathfrak{h}_2} = \{\tau_1, \tau_3\}$. These results indicate that the air transportation is the best choice for this task while the sea transportation should not be considered.

The next result simplifies the computation of the soft upper approximations of singletons. Notice that their soft lower approximations are quite trivial: $\forall \tau_i \in \mho$, $\underline{\{\tau_i\}} = \{\tau_i\}$ if and only if $\hat{C}_{\mathfrak{h}}(\kappa) = \{\tau_i\}$ for some $\kappa \in V_{\mathfrak{h}}$; otherwise, $\underline{\{\tau_i\}} = \varnothing$.

Proposition 4.2. *Let* $\mathfrak{h}$ *be an HFS on* $\mho$ *and* $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ *be the CSS of* $\mathfrak{h}$ *. For every* $\tau_i \in \mho$*, the soft upper approximation of* $\{\tau_i\}$ *can be calculated by*

$$\overline{\{\tau_i\}}_{\mathfrak{h}} = \bigcup_{\kappa \in \mathfrak{h}(\tau_i)} \mathfrak{h}^{-1}(\kappa). \tag{4.3}$$

Proof. According to Eq (4.2), we immediately have $\overline{\{\tau_i\}}_{\mathfrak{h}} = \bigcup_{\kappa \in V_{\mathfrak{h}}} \{\hat{C}_{\mathfrak{h}}(\kappa) \mid \hat{C}_{\mathfrak{h}}(\kappa) \cap \{\tau_i\} \neq \varnothing\}$. By Definition 3.2 and Definition 3.6, we can deduce that

$$\overline{\{\tau_i\}}_{\mathfrak{h}} = \bigcup_{\kappa \in V_{\mathfrak{h}}} \{\mathfrak{h}^{-1}(\kappa) \mid \mathfrak{h}^{-1}(\kappa) \cap \{\tau_i\} \neq \varnothing\} = \bigcup_{\kappa \in V_{\mathfrak{h}}} \{\mathfrak{h}^{-1}(\kappa) \mid \tau_i \in \mathfrak{h}^{-1}(\kappa)\} = \bigcup_{\kappa \in V_{\mathfrak{h}}} \{\mathfrak{h}^{-1}(\kappa) \mid \kappa \in \mathfrak{h}(\tau_i)\}.$$

It follows that $\overline{\{\tau_i\}}_{\mathfrak{h}} = \bigcup_{\kappa \in \mathfrak{h}(\tau_i)} \mathfrak{h}^{-1}(\kappa)$, which completes the proof. □

4.2. A 3WD method with CSSs of HFSs

Suppose there is a director who is in charge of a group of specialists. All the specialists are requested to evaluate each alternative in a universe of discourse $\mho = \{\tau_1, \tau_2, \cdots, \tau_n\}$. For the sake of simplicity, we assume that the evaluation scores given by the experts are chosen from the unit interval [0,1], and a higher score means a more positive evaluation on the concerned alternative. Based on these evaluation results, an HFS $\mathfrak{h}$ can be constructed on $\mho$, which stores the decision information collected from the group of specialists. After checking the evaluation from the expert group, the director select several alternatives in $\mho$ to form a pre-decision set R. In this 3WD process, the main purpose is to derive a trisection of $\mho$ as a general consensus among all the specialists together with the director. In the following, we present a novel approach for solving 3WD problems by virtue of HFSs and their CSSs.

Firstly, we can transform the HFS $\mathfrak{h}$ into its CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$. Using the CSS $\mathfrak{C}_{\mathfrak{h}}$, we construct an SAS $\mathcal{P} = (\mho, \mathfrak{C}_{\mathfrak{h}})$. Based on the SAS $\mathcal{P}$, we can calculate the soft upper approximations $\overline{\{\tau_i\}}_{\mathfrak{h}}$ of each alternative $\tau_i \in \mho$ $(i = 1, 2, \cdots, n)$ according to Eq (4.3). It should be noted that $\overline{\{\tau_i\}}_{\mathfrak{h}}$ can be interpreted as the set of alternatives in $\mho$ which have at least one evaluation score in common with the alternative τ_i.

Secondly, we should select an appropriate evaluation function $\mathcal{E} : \mho \longrightarrow [0, 1]$ which plays a core role in 3WD process. As pointed out by Hu [22], a cogent evaluation function should satisfy the monotonicity axiom. In other words, an alternative with higher value of evaluation function is considered to be more positive than an alternative with lower value. Actually, this can also be seen from the trisection as given in Definition 2.17. Furthermore, it is more reasonable and flexible for us to consider the following two cases when addressing decision-making problems in real-world scenarios:

- The director would like to select favorable alternatives in $\mho$ to form a positive pre-decision set R;

- The director would like to select rejective alternatives in $\mho$ to form a negative pre-decision set R.

In light of the above-mentioned issues, we define the evaluation function as follows:

$$\mathcal{E}(\tau_i) = \begin{cases} \dfrac{|\overline{\{\tau_i\}}_{\mathfrak{h}} \cap R|}{|\overline{\{\tau_i\}}_{\mathfrak{h}}|}, & \text{if } R \text{ is positive}, \\ \dfrac{|\overline{\{\tau_i\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_i\}}_{\mathfrak{h}}|}, & \text{if } R \text{ is negative}, \end{cases} \tag{4.4}$$

where $\tau_i \in \mho$ $(i = 1, 2, \cdots, n)$, "$\setminus$" denotes the set difference operation and R is the pre-decision set.

Finally, given two thresholds γ, δ with $0 \leq \gamma < \delta \leq 1$, we can obtain the trisection of all alternatives in $\mho$ as follows:

$$H^{[\delta,1]}(\mathcal{E}) = \{\tau_i \in \mho : \delta \leq \mathcal{E}(\tau_i) \leq 1\},$$

$$M^{(\gamma,\delta)}(\mathcal{E}) = \{\tau_i \in \mho : \gamma < \mathcal{E}(\tau_i) < \delta\},$$

$$L^{[0,\gamma]}(\mathcal{E}) = \{\tau_i \in \mho : 0 \leq \mathcal{E}(\tau_i) \leq \gamma\}.$$

The alternatives in high value region $H^{[\delta,1]}(\mathcal{E})$ should be accepted. The alternatives in medium value region $M^{(\gamma,\delta)}(\mathcal{E})$ need further consideration. The alternatives in low value region $L^{[0,\gamma]}(\mathcal{E})$ should be rejected. The complete process of the proposed method is summarized as Algorithm 1 in Table 6.

Table 6. An HFS-CSS based 3WD approach.

Algorithm 1. An HFS-CSS based 3WD approach.
Input: An HFS $\mathfrak{h}$ on $\mho = \{\tau_1, \tau_2, \cdots, \tau_n\}$ and a pre-decision set R; **Output:** The three regions $H^{[\delta,1]}(\mathcal{E})$, $M^{(\gamma,\delta)}(\mathcal{E})$ and $L^{[0,\gamma]}(\mathcal{E})$; **Step 1:** Transform the HFS $\mathfrak{h}$ into its CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ by Definition 3.6; **Step 2:** Construct the SAS $\mathcal{P} = (\mho, \mathfrak{C}_{\mathfrak{h}})$ and compute the soft upper approximations $\overline{\{\tau_i\}}_{\mathfrak{h}}$ of every $\tau_i \in \mho$ by Eq (4.3); **Step 3:** Determine the evaluation function $\mathcal{E} : \mho \longrightarrow [0, 1]$ and compute the evaluation value $\mathcal{E}(\tau_i)$ of every $\tau_i \in \mho$ by Eq (4.4); **Step 4:** Specify a pair of low and high thresholds γ, δ with $0 \leq \gamma < \delta \leq 1$; **Step 5:** Compute the regions $H^{[\delta,1]}(\mathcal{E})$, $M^{(\gamma,\delta)}(\mathcal{E})$ and $L^{[0,\gamma]}(\mathcal{E})$; **Step 6:** Return $H^{[\delta,1]}(\mathcal{E})$, $M^{(\gamma,\delta)}(\mathcal{E})$ and $L^{[0,\gamma]}(\mathcal{E})$ for taking action accordingly.

5. An illustrative example

To confirm the reliability of the proposed 3WD method with CSSs of HFSs, we consider an example regarding peer review of research papers given as follows.

In the peer-review process of academic papers, each manuscript is reviewed by a group of referees who are professional in the field of their expertise. The reviewers are usually invited by an Associate Editor of an academic journal. Based on the comments from all the reviewers and the Associate Editor, the Editor-in-Chief can make a final decision to accept, revise or reject the manuscript.

Suppose that an Associate Editor of an academic journal is in charge of the review process of seven research papers. These articles constitute the universe of discourse $\mho = \{\tau_1, \tau_2, \cdots, \tau_7\}$. In this context, it is cleat that the Associate Editor serves as the director of the expert group constituted by several reviewers. For simplicity, we assume that the Associate Editor invites three referees to review each paper in $\mho$ and give their evaluation scores valued in the unit interval [0,1]. Clearly, a higher score reflects a better evaluation on the reviewed paper. Based on the evaluation scores given by three referees, we can construct an HFS $\mathfrak{h}$ on $\mho$, which is a 3-THFS by Definition 2.4. Assume that the HFEs of $\mathfrak{h}$ are given by $\mathfrak{h}(\tau_1) = \{0.1, 0.3, 0.5\}$, $\mathfrak{h}(\tau_2) = \{0.5, 0.6, 0.7\}$, $\mathfrak{h}(\tau_3) = \{0.1, 0.2, 0.4\}$, $\mathfrak{h}(\tau_4) = \{0.4, 0.5, 0.7\}$, $\mathfrak{h}(\tau_5) = \{0.3, 0.6, 0.7\}$, $\mathfrak{h}(\tau_6) = \{0.6, 0.7, 0.9\}$ and $\mathfrak{h}(\tau_7) = \{0.6, 0.8, 0.9\}$. The membership grades in the HFE $\mathfrak{h}(\tau_i)$ $(i = 1, 2, \cdots, 7)$ represent the scores given by three reviewers. After checking all these manuscripts and related review comments, the Associate Editor chooses three papers in $\mho$ to form a negative pre-decision set $R = \{\tau_1, \tau_3, \tau_4\}$. This means that the Associate Editor intends to reject papers τ_1, τ_3 and τ_4.

In the following, we demonstrate how to help the Editor-in-Chief to achieve a general consensus among all the reviewers and the Associate Editor so as to make final decisions on seven papers by using Algorithm 1.

Step 1. Based on Definition 3.6, we first transform the 3-THFS $\mathfrak{h}$ into its CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ as shown in Table 7. It is easy to see that

$$V_{\mathfrak{h}} = \bigcup_{\tau_i \in \mho} \mathfrak{h}(\tau_i) = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}.$$

Table 7. The CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$ of the 3-THFS $\mathfrak{h}$.

$\mho$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
τ_1	1	0	1	0	1	0	0	0	0
τ_2	0	0	0	0	1	1	1	0	0
τ_3	1	1	0	1	0	0	0	0	0
τ_4	0	0	0	1	1	0	1	0	0
τ_5	0	0	1	0	0	1	1	0	0
τ_6	0	0	0	0	0	1	1	0	1
τ_7	0	0	0	0	0	1	0	1	1

By Definition 3.2, we can compute the value cosets of all evaluation scores in $V_{\mathfrak{h}}$. The obtained results are as follows: $\mathfrak{h}^{-1}(0.1) = \{\tau_1, \tau_3\}$, $\mathfrak{h}^{-1}(0.2) = \{\tau_3\}$, $\mathfrak{h}^{-1}(0.3) = \{\tau_1, \tau_5\}$, $\mathfrak{h}^{-1}(0.4) = \{\tau_3, \tau_4\}$, $\mathfrak{h}^{-1}(0.5) = \{\tau_1, \tau_2, \tau_4\}$, $\mathfrak{h}^{-1}(0.6) = \{\tau_2, \tau_5, \tau_6, \tau_7\}$, $\mathfrak{h}^{-1}(0.7) = \{\tau_2, \tau_4, \tau_5, \tau_6\}$, $\mathfrak{h}^{-1}(0.8) = \{\tau_7\}$ and $\mathfrak{h}^{-1}(0.9) = \{\tau_6, \tau_7\}$. It should be noted that the approximation function of the CSS $\mathfrak{C}_{\mathfrak{h}}$ is specified by these value cosets.

Step 2. Using the CSS $\mathfrak{C}_{\mathfrak{h}} = (\hat{C}_{\mathfrak{h}}, V_{\mathfrak{h}})$, we can construct an SAS $\mathcal{P} = (\mho, \mathfrak{C}_{\mathfrak{h}})$. By Eq (4.3), the soft upper approximations $\overline{\{\tau_i\}}_{\mathfrak{h}}$ of each paper $\tau_i \in \mho$ $(i = 1, 2, \cdots, 7)$ with respect to the SAS $\mathcal{P} = (\mho, \mathfrak{C}_{\mathfrak{h}})$ can be calculated as follows:

$$\overline{\{\tau_1\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.1) \cup \mathfrak{h}^{-1}(0.3) \cup \mathfrak{h}^{-1}(0.5) = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\},$$

$$\overline{\{\tau_2\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.5) \cup \mathfrak{h}^{-1}(0.6) \cup \mathfrak{h}^{-1}(0.7) = \{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\},$$
$$\overline{\{\tau_3\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.1) \cup \mathfrak{h}^{-1}(0.2) \cup \mathfrak{h}^{-1}(0.4) = \{\tau_1, \tau_3, \tau_4\},$$
$$\overline{\{\tau_4\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.4) \cup \mathfrak{h}^{-1}(0.5) \cup \mathfrak{h}^{-1}(0.7) = \{\tau_1, \tau_2, \tau_3, \tau_4, , \tau_5, \tau_6\},$$
$$\overline{\{\tau_5\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.3) \cup \mathfrak{h}^{-1}(0.6) \cup \mathfrak{h}^{-1}(0.7) = \{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\},$$
$$\overline{\{\tau_6\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.6) \cup \mathfrak{h}^{-1}(0.7) \cup \mathfrak{h}^{-1}(0.9) = \{\tau_2, \tau_4, \tau_5, \tau_6, \tau_7\},$$
$$\overline{\{\tau_7\}}_{\mathfrak{h}} = \mathfrak{h}^{-1}(0.6) \cup \mathfrak{h}^{-1}(0.8) \cup \mathfrak{h}^{-1}(0.9) = \{\tau_2, \tau_5, \tau_6, \tau_7\}.$$

The first equality $\overline{\{\tau_1\}}_{\mathfrak{h}} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ indicates that the papers τ_1, τ_2, τ_3, τ_4, τ_5 have at least one evaluation score in common with the paper τ_1. Also, it can be seen that the papers τ_1, τ_2, τ_4, τ_5, τ_6, τ_7 have at least one evaluation score in common with the paper τ_2, since we have $\overline{\{\tau_2\}}_{\mathfrak{h}} = \{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\}$. The rest of the above results can be explained in a similar fashion.

Step 3. Now, let us consider the evaluation function $\mathcal{E} : \mho \longrightarrow [0, 1]$ in this particular case. It is clear that the pre-decision set $R = \{\tau_1, \tau_3, \tau_4\}$ is negative since it contains the papers that the Associate Editor intends to reject. According to Eq (4.4), we calculate the evaluation values $\mathcal{E}(\tau_i)$ of each paper $\tau_i \in \mho$ $(i = 1, 2, \cdots, 7)$ as follows:

$$\mathcal{E}(\tau_1) = \frac{|\overline{\{\tau_1\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_1\}}_{\mathfrak{h}}|} = \frac{|\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}|} = 0.4,$$
$$\mathcal{E}(\tau_2) = \frac{|\overline{\{\tau_2\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_2\}}_{\mathfrak{h}}|} = \frac{|\{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\}|} = 0.67,$$
$$\mathcal{E}(\tau_3) = \frac{|\overline{\{\tau_3\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_3\}}_{\mathfrak{h}}|} = \frac{|\{\tau_1, \tau_3, \tau_4\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_1, \tau_3, \tau_4\}|} = 0,$$
$$\mathcal{E}(\tau_4) = \frac{|\overline{\{\tau_4\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_4\}}_{\mathfrak{h}}|} = \frac{|\{\tau_1, \tau_2, \tau_3, \tau_4, , \tau_5, \tau_6\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_1, \tau_2, \tau_3, \tau_4, , \tau_5, \tau_6\}|} = 0.5,$$
$$\mathcal{E}(\tau_5) = \frac{|\overline{\{\tau_5\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_5\}}_{\mathfrak{h}}|} = \frac{|\{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_7\}|} = 0.67,$$
$$\mathcal{E}(\tau_6) = \frac{|\overline{\{\tau_6\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_6\}}_{\mathfrak{h}}|} = \frac{|\{\tau_2, \tau_4, \tau_5, \tau_6, \tau_7\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_2, , \tau_4, \tau_5, \tau_6, \tau_7\}|} = 0.8,$$
$$\mathcal{E}(\tau_7) = \frac{|\overline{\{\tau_7\}}_{\mathfrak{h}} \setminus R|}{|\overline{\{\tau_7\}}_{\mathfrak{h}}|} = \frac{|\{\tau_2, \tau_5, \tau_6, \tau_7\} \setminus \{\tau_1, \tau_3, \tau_4\}|}{|\{\tau_2, \tau_5, \tau_6, \tau_7\}|} = 1.$$

In this context, the evaluation value $\mathcal{E}(\tau_i)$ can be interpreted as the proportion of papers in $\overline{\{\tau_i\}}_{\mathfrak{h}}$ which are not rejected by the Associate Editor. For instance, the evaluation value $\mathcal{E}(\tau_1) = 0.4$ shows that 40% of the papers which have at least one evaluation score in common with τ_1 are not rejected by the Associate Editor. Based on the evaluation values, we can rank τ_i and τ_j as follows:

- $\tau_i \succ \tau_j$ if and only if $\mathcal{E}(\tau_i) > \mathcal{E}(\tau_j)$;
- $\tau_i \prec \tau_j$ if and only if $\mathcal{E}(\tau_i) < \mathcal{E}(\tau_j)$;
- $\tau_i \approx \tau_j$ if and only if $\mathcal{E}(\tau_i) = \mathcal{E}(\tau_j)$.

Specifically, we have the following ranking of all the papers:

$$\tau_7 \succ \tau_6 \succ \tau_5 \approx \tau_2 \succ \tau_4 \succ \tau_1 \succ \tau_3.$$

Step 4. Next, we need to select a pair of thresholds γ, δ from the unit interval [0,1] with $\gamma < \delta$. These thresholds are used to attain the trisection of the universe of discourse. In this problem, we choose $\gamma = 0.4$ and $\delta = 0.7$ as the low and high thresholds, respectively.

Step 5. Based on the evaluation values and the thresholds, we can obtain the trisection of all the papers in $\mho$ as follows:

(1) High value region: $H^{[0.7,1]}(\mathcal{E}) = \{\tau_6, \tau_7\}$;

(2) Medium value region: $M^{(0.4,0.7)}(\mathcal{E}) = \{\tau_2, \tau_4, \tau_5\}$;

(3) Low value region: $L^{[0,0.4]}(\mathcal{E}) = \{\tau_1, \tau_3\}$.

Step 6. It can be seen that the above results are obtained by taking into account both the evaluation scores given by three referees and the pre-decision provided by the Associate Editor. According to the obtained trisection, the Editor-in-Chief is advised to make the final decision regarding papers in $\mho$ as follows:

(1) The papers τ_6, τ_7 in high value region $H^{[0.7,1]}(\mathcal{E})$ should be accepted;

(2) The papers τ_2, τ_4, τ_5 in medium value region $M^{(0.4,0.7)}(\mathcal{E})$ should be revised;

(3) The papers τ_1, τ_3 in low value region $L^{[0,0.4]}(\mathcal{E})$ should be rejected.

6. Conclusions

We defined the unit interval parameterized SS, CSS and derived HFS in this study. With the proposed notions, mutual representations between HFSs and SSs have been investigated. By transforming an HFS into its CSS, we developed a soft rough model based on HFSs. By virtue of this soft rough model, a novel 3WD method has been proposed for solving decision-making problems with hesitant fuzzy information. A numerical example was presented to illustrate the validity and feasibility of our 3WD method. It has shown that the proposed approach achieved a general consensus between the expert group and the director. In the future, we will further explore other potential applications of CSSs and examine more properties of HFSs from the perspective of SSs. In addition, it would be interesting to discover inner connections between probabilistic HFSs [57] and probabilistic SSs [11] by using similar techniques. These links might prompt new 3WD techniques in the context of group decision-making with partial memberships, hesitation, and probabilistic information.

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. Agarwal, K. K. Biswas, M. Hanmandlu, Generalized intuitionistic fuzzy soft sets with applications in decision-making, *Appl. Soft Comput.*, **13** (2013), 3552–3566. doi: 10.1016/j.asoc.2013.03.015.
2. J. C. R. Alcantud, A. Laruelle, Dis&approval voting: A characterization, *Soc. Choice Welf.*, **43** (2014), 1–10. doi: 10.1007/s00355-013-0766-7.
3. J. C. R. Alcantud, V. Torra, Decomposition theorems and extension principles for hesitant fuzzy sets, *Inf. Fusion*, **41** (2018), 48–56. doi: 10.1016/j.inffus.2017.08.005.
4. M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. doi: 10.1016/j.camwa.2008.11.009.
5. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96.
6. T. M. Athira, J. J. Sunil, H. Garg, A novel entropy measure of Pythagorean fuzzy soft sets, *AIMS Math.*, **5** (2020), 1050–1061. doi: 10.3934/math.20200073.
7. B. Bedregal, R. Reiser, H. Bustince, C. López-Molina, V. Torra, Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms, *Inf. Sci.*, **255** (2014), 82–99. doi: 10.1016/j.ins.2013.08.024.
8. D. G. Chen, E. C. C. Tsang, D. S. Yeung, X. Z. Wang, The parameterization reduction of soft sets and its application, *Comput. Math. Appl.*, **49** (2005), 757–763. doi: 10.1016/j.camwa.2004.10.036.
9. X. Deng, Y. Yao, Decision-theoretic three-way approximations of fuzzy sets, *Inf. Sci.*, **279** (2014), 702–715. doi: 10.1016/j.ins.2014.04.022.
10. D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *Int. J. Gen. Syst.*, **17** (1990), 191–209. doi: 10.1080/03081079008935107.
11. F. Fatimah, D. Rosadi, R. F. Hakim, J. C. R. Alcantud, Probabilistic soft sets and dual probabilistic soft sets in decision-making, *Neural Comput. Applic.*, **31** (2019), 397–407. doi: 10.1007/s00521-017-3011-y.
12. F. Feng, J. Cho, W. Pedrycz, H. Fujita, T. Herawan, Soft set based association rule mining, *Knowl.-Based Syst.*, **111** (2016), 268–282. doi: 10.1016/j.knosys.2016.08.020.
13. F. Feng, H. Fujita, M. I. Ali, R. R. Yager, X. Liu, Another view on generalized intuitionistic fuzzy soft sets and related multiattribute decision making methods, *IEEE Trans. Fuzzy Syst.*, **27** (2019) 474-488. doi: 10.1109/TFUZZ.2018.2860967.
14. F. Feng, C. X. Li, B. Davvaz, M. I. Ali, Soft sets combined with fuzzy sets and rough sets: A tentative approach, *Soft Comput.*, **14** (2010), 899–911. doi: 10.1007/s00500-009-0465-6.
15. F. Feng, Y. M. Li, Soft subsets and soft product operations, *Inf. Sci.*, **232** (2013) 44–57. doi: 10.1016/j.ins.2013.01.001.

16. F. Feng, X. Y. Liu, V. L. Fotea, Y. B. Jun, Soft sets and soft rough sets, *Inf. Sci.*, **181** (2011), 1125–1137. doi: 10.1016/j.ins.2010.11.004.
17. F. Feng, Z. Xu, H. Fujita, M. Q. Liang, Enhancing PROMETHEE method with intuitionistic fuzzy soft sets, *Int. J. Intell. Syst.*, **35** (2020), 1071–1104. doi: 10.1002/int.22235.
18. H. Garg, R. Arora, Generalized and group-based generalized intuitionistic fuzzy soft sets with applications in decision-making, *Appl. Intell.*, **48** (2018), 343–356. doi: 10.1007/s10489-017-0981-5.
19. H. Garg, R. Arora, TOPSIS method based on correlation coefficient for solving decision-making problems with intuitionistic fuzzy soft set information, *AIMS Math.*, **5** (2020), 2944–2966. doi: 10.3934/math.2020190.
20. H. Garg, R. Arora, Generalized Maclaurin symmetric mean aggregation operators based on Archimedean t-norm of the intuitionistic fuzzy soft set information, *Artif. Intell. Rev.*, **54** (2021), 3173–3213. doi: 10.1007/s10462-020-09925-3.
21. J. Goguen, L-fuzzy sets, *J. Math. Anal. Appl.*, **18** (1967), 145–147.
22. B. Q. Hu, Three-way decisions space and three-way decisions, *Inf. Sci.*, **281** (2014), 21–52. doi: 10.1016/j.ins.2014.05.015.
23. B. Q. Hu, H. Wong, K. C. Yiu, On two novel types of three-way decisions in three-way decision spaces, *Int. J. Approx. Reason.*, **82** (2017), 285–306. doi: 10.1016/j.ijar.2016.12.007.
24. X. Y. Jia, Z. M. Tang, W. H. Liao, L. Shang, On an optimization representation of decision-theoretic rough set model, *Int. J. Approx. Reason.*, **55** (2014), 156–166. doi: 10.1016/j.ijar.2013.02.010.
25. A. Laruelle, "Not this one": Experimental use of the approval and disapproval ballot, *Homo Oecon.*, 2021. doi: 10.1007/s41412-021-00110-7.
26. X. N. Li, B. Z. Sun, Y. H. She, Generalized matroids based on three-way decision models, *Int. J. Approx. Reason.*, **90** (2017), 21–52. doi: 10.1016/j.ijar.2017.07.012.
27. X. N. Li, Q. Q. Sun, H. M. Chen, H. J. Yi, Three-way decision on two universes, *Inf. Sci.*, **515** (2020), 263–279. doi: 10.1016/j.ins.2019.12.020.
28. D. C. Liang, D. Liu, A novel risk decision making based on decision-theoretic rough sets under hesitant fuzzy information, *IEEE Trans. Fuzzy Syst.*, **23** (2015), 192–207. doi: 10.1109/TFUZZ.2014.2310495.
29. D. C. Liang, W. Pedrycz, D. Liu, P. Hu, Three-way decisions based on decision-theoretic rough sets under linguistic assessment with the aid of group decision making, *Appl. Soft. Comput.*, **29** (2015), 256–269. doi: 10.1016/j.asoc.2015.01.008.
30. P. D. Liu, Y. M. Wang, F. Jia, H. Fujita, A multiple attribute decision making three-way model for intuitionistic fuzzy numbers, *Int. J. Approx. Reason.*, **119** (2020), 177–203. doi: 10.1016/j.ijar.2019.12.020.
31. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.
32. P. K. Maji, R. Biswas, A. R. Roy, Intuitionistic fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 677–692.
33. P. K. Maji, R. Biswas, A. R. Roy, Soft set throry, *Comput. Math. Appl.*, **45** (2003) 555–562. doi: 10.1016/S0898-1221(03)00016-6.

34. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077–1083.

35. P. Majumdar, S. K. Samanta, Generalised fuzzy soft sets, *Comput. Math. Appl.*, **59** (2010), 1425–1432. doi: 10.1016/j.camwa.2009.12.006.

36. D. A. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. doi: 10.1016/S0898-1221(99)00056-5.

37. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. doi: 10.1007/BF01001956.

38. X. D. Peng, Y. Yang, J. Song, Y. Jiang, Pythagoren fuzzy soft set and its application, *Comput. Eng.*, **41** (2015), 224–229.

39. A. M. Raszikowaka, E. E. Kerre, A comparative study of fuzzy rough sets, *Fuzzy Sets Syst.*, **126** (2002), 137–155. doi: 10.1016/S0165-0114(01)00032-X.

40. B. Z. Sun, W. Ma, Soft fuzzy rough sets and its application in decision making, *Artif. Intell. Rev.*, **41** (2014), 67–80. doi: 10.1007/s10462-011-9298-7.

41. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 529–539. doi: 10.1002/int.20418.

42. I. B. Turksen, Interval valued fuzzy sets based on normal forms, *Fuzzy Sets Syst.*, **20** (1986), 191–210. doi: 10.1016/0165-0114(86)90077-1.

43. J. J. Wang, X. L. Ma, Z. S. Xu, J. M. Zhan, Three-way multi-attribute decision making under hesitant fuzzy environments, *Inf. Sci.*, **552** (2021), 328–351. doi: 10.1016/j.ins.2020.12.005.

44. X. F. Wen, X. H. Zhang, T. Lei, Intuitionistic fuzzy (IF) overlap functions and IF-rough sets with applications, *Symmetry*, **13** (2021), 1494. doi: 10.3390/sym13081494.

45. M. M. Xia, Z. S. Xu, Hesitant fuzzy information aggregation in decision making, *Int. J. Approx. Reason.*, **52** (2011), 395–407. doi: 10.1016/j.ijar.2010.09.002.

46. T. Xie, Z. T. Gong, A hesitant soft fuzzy rough set and its applications, *IEEE Access*, **7** (2019), 167766–167783. doi: 10.1109/ACCESS.2019.2954179.

47. J. L. Yang, Y. Y. Yao, Semantics of soft sets and three-way decision with soft sets, *Knowl.-Based Syst.*, **194** (2020), 105538. doi: 10.1016/j.knosys.2020.105538.

48. Y. Y. Yao, Probabilistic approaches to rough sets, *Expert Syst.*, **20** (2003), 287–297. doi: 10.1111/1468-0394.00253.

49. Y. Y. Yao, Three-way decisions with probabilistic rough sets, *Inf. Sci.*, **180** (2010), 341–353. doi: 10.1016/j.ins.2009.09.021.

50. Y. Y. Yao, The superiority of three-way decisions in probabilistic rough set models, *Inf. Sci.*, **181** (2011), 1080–1096. doi: 10.1016/j.ins.2010.11.019.

51. Y. Y. Yao, An outline of a theory of three-way decisions, In: J. T. Yao, Y. Yang, R. Słowiński, S. Greco, H. X. Li, S. Mitra, L. Polkowski, *Rough sets and current trends in computing*, Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, **7413** (2012), 1–17. doi: 10.1007/978-3-642-32115-3_1.

52. Y. Y. Yao, Three-way decision and granular computing, *Int. J. Approx. Reason.*, **103** (2018), 107–123. doi: 10.1016/j.ijar.2018.09.005.

53. Y. Y. Yao, Tri-level thinking: Models of three-way decision, *Int. J. Mach. Learn. Cyber.*, **11** (2020), 947–959. doi: 10.1007/s13042-019-01040-2.

54. Y. Y. Yao, Set-theoretic models of three-way decision, *Granul. Comput.*, **6** (2021), 133–148. doi: 10.1007/s41066-020-00211-9.

55. H. Yu, Z. G. Liu, G. Y. Wang, An automatic method to determine the number of clusters using decision-theoretic rough set, *Int. J. Approx. Reason.*, **55** (2013), 101–115. doi: 10.1016/j.ijar.2013.03.018.

56. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. doi: 10.1016/S0019-9958(65)90241-X.

57. S. Zhang, Z. S. Xu, Y. He, Operations and integrations of probabilistic hesitant fuzzy information in decision making, *Inf. Fusion*, **38** (2017), 1–11. doi: 10.1016/j.inffus.2017.02.001.