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Research article

Study of the Atangana-Baleanu-Caputo type fractional system with a generalized Mittag-Leffler kernel

Mdi Begum Jeelani[1,*], Abeer S. Alnahdi[1], Mohammed A. Almalahi[2], Mohammed S. Abdo[3], Hanan A. Wahash[5] and M. A. Abdelkawy[1,4]

[1] Department of Mathematics, Imam Mohammad Ibn Saud Islamic University, Riyadh, Saudi Arabia
[2] Department of Mathematics, Hajjah University, Hajjah, Yemen
[3] Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen
[4] Department of Mathematics, Faculty of Science, Beni-Seuf University, Beni-Seuf, Egypt
[5] Department of Mathematics, Dr.Babasaheb Ambedkar Marathwada University, Aurangabad, India

* **Correspondence:** Email: mbshaikh@imamu.edu.sa.

Abstract: We devote our interest in this work to investigate the sufficient conditions for the existence, uniqueness, and Ulam-Hyers stability of solutions for a new fractional system in the frame of Atangana-Baleanu-Caputo fractional operator with multi-parameters Mittag-Leffler kernels investigated lately by Abdeljawad (Chaos: An Interdisciplinary J. Nonlinear Sci. Vol. 29, no. 2, (2019): 023102). Moreover, the continuous dependence of solution and δ-approximate solutions are analyzed to such a system. Our approach is based on Banach's and Schaefer's fixed point theorems and some mathematical techniques. In order to illustrate the validity of our results, an example is given.

Keywords: generalized Mittag-Leffler; generalized ABC fractional integrals and derivatives; existence; fixed point theorem; δ-approximate solution
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1. Introduction

Fractional differential equations have a profound physical background and rich theoretical connotations and have been particularly eye-catching in recent years. Several-order differential equations refer to equations that contain fractional derivatives or fractional integrals. Fractional order derivatives and integrals have a wide range of applications in many disciplines such as physics, biology, chemistry, etc., such as power with chaotic dynamic behavior systems, dynamics of quasi-chaotic systems, and complex materials or porous media, random walks with memory, etc. For more information see [1–3]. The approximate controllability of the fractional system can be found in [4–10]. Recently, some researchers have realized the importance of finding new fractional derivatives (FDs) with different singular or nonsingular kernels to meet the need to modeling more real-world problems in different fields of science and engineering. For instance, Caputo and Fabrizio [11] studied a new kind of FDs in the exponential kernel. Atangana and Baleanu (AB) [12] investigated a new type and interesting FD with Mittag-Leffler kernels. Abdeljawad in [13] extended this type for higher arbitrary order and formulated their associated integral operators. But the corresponding integral operators of AB derivative do not have a semigroup property, which makes dealing with them theoretically or mathematically somewhat complicated. Very recently, Abdeljawad in [14, 15], introduced a fractional derivative with nonsingular kernel in Atangana-Baleanu-Caputo (ABC) settings with multi-parametered Mittag-Leffler (ML) function and study their semigroup properties, its discrete version in [16]. This diversity of FDs has made the topic of fractional calculus attractive and allows researchers to choose the appropriate operator to obtain better results. For some theoretical works on ABC type FDEs, we refer the reader to the series papers [17–20]. On the other hand side, the study of systems involving FDEs is also important as such systems occur in various problems of applied nature. For some theoretical works on systems of FDEs, we refer to series of papers [21–23].

The topic of stability of systems is one of the most important qualitative characteristics of a solution, for more details about the stability of systems see [24–27].

Abdeljawad et al. [28] studied qualitative analyses of some logistic models in the settings of ABC fractional operators with multi-parameter ML kernels, described as follows:

$$\begin{cases} {}^{ABC}_{\theta_0}\mathbf{D}^{p,q,v} z(\theta) = mz(\theta)\left(1 - z(\theta)\right), \theta > \theta_0, \\ z(\theta_0) = z^0 \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} {}^{ABC}_{\theta_0}\mathbf{D}^{p,q,v} z(\theta) = mz(\theta)\left(1 - \frac{z(\theta)}{l}\right)\left(z(\theta) - n\right), \theta > \theta_0, \\ z(\theta_0) = z^0 \in \mathbb{R}, \end{cases}$$

where ${}^{ABC}_{\theta_0}\mathbf{D}^{p,q,v}$ is the generalized left ABC FD of order $p \in (0,1]$, $q, v > 0$ and $m, n, l > 0$.

Motivated by the recent advancements of ABC operator, its applications, and by the above works, the aim of the current work is to investigate the existence, uniqueness, stability, and continuous dependence results, and discuss the δ-approximate solutions for a new model in the frame of generalized ABC fractional operators with multi-parameters ML kernels described as follows:

$$\begin{cases} {}^{ABC}_{0}\mathbf{D}^{p,q,v} z_1(\theta) = \mathcal{F}_1\left(\theta, z_1(\theta), ..., z_n(\theta)\right), \\ {}^{ABC}_{0}\mathbf{D}^{p,q,v} z_2(\theta) = \mathcal{F}_2\left(\theta, z_1(\theta), ..., z_n(\theta)\right), \\ \vdots \\ \vdots \\ {}^{ABC}_{0}\mathbf{D}^{p,q,v} z_n(\theta) = \mathcal{F}_n\left(\theta, z_1(\theta), ..., z_n(\theta)\right), \\ z_k(0) = z_k^0 \in \mathbb{R}, k = 1, 2, ..., n, \end{cases} \tag{1.1}$$

where ${}^{ABC}_{0}\mathbf{D}^{p,q,v}$ is the generalized ABC FD of order $p \in (0,1]$, $q, v > 0$. $\mathcal{F}_k \in C\left([0,T], \mathbb{R}^+\right)$ and satisfies some conditions described later in our analysis. Many researchers in different fields of science and engineering used ABC FD with one parameter ML kernel, but their corresponding AB integral operators do not have a semigroup property, which makes dealing with them theoretically or mathematically somewhat complicated. Nevertheless, in this work, we use a new operator containing interesting kernels, we believe that the qualitative properties of solutions for FDEs should be studied via this operator. This work aims to investigate some properties of solutions for the proposed model via a nonsingular FD in ABC settings with multi-parameter ML kernel introduced lately by [14, 15]. Due to the fractional derivative used in this work have semigroup property and recently proposed, the results obtained in this work are new and open the door for the researchers to study more real-world problems in different fields.

Notice that, the considered system is investigated under the generalized ML law. In the case of the ABC fractional operator, the requirement of the vanishing condition of the right hand side of the dynamic system to fulfill the initial data needs recuperation on the modeled population. However, the nature of the generalized ML kernel will enable the emancipation of any restrictions on the initial data.

The structure of our paper is as follows. In Section 2, we present notations, auxiliary lemmas and some basic definitions that are needed for our analysis. In Section 3, we discuss the existence and uniqueness results for the model (1.1). Ulam-Hyers stability results for the model (1.1) are discussed in Section 4. In Section 5, we study the continuous dependence of solution and δ-approximate solutions for the model (1.1). In Section 6, we provide an example to illustrate the validity of our results. The last section is devoted to concluding remarks about our results.

2. Preliminary results

In order to achieve our main purposes, we present here some definitions and basic auxiliary results that are required throughout our paper. Let $\mathcal{J} = [0, T] \subset \mathbb{R}^+$ and $\mathcal{X} = \{z(\theta) : z(\theta) \in C(\mathcal{J}, \mathbb{R}^+)\}$ be a space with the norm $\|z\| = \sup\{|z(\theta)| : \theta \in \mathcal{J}\}$. Clearly, $(\mathcal{X}, \|\cdot\|)$ is a Banach space. For our analysis, we need defined the product space $\mathcal{G} := \underbrace{\mathcal{X} \times \mathcal{X} \times \mathcal{X} \times ....\mathcal{X}}_{n-time}$. Undoubtedly that $\mathcal{G}$ is also a Banach space with the following norm

$$\|(z_1, z_2, ..., z_n)\| = \sum_{k=1}^{n} \|z_k\|.$$

For $0 \leq p < 1, q, v > 0$, we defined the space

$$C^{p,q,v}(\mathcal{J}, \mathbb{R}^+) = \left\{z(\theta) \in \mathcal{X} :{}_0^{ABC} \mathbf{D}^{p,q,v} z(\theta) \in \mathcal{X}\right\}.$$

Definition 2.1. *[3] Let $\lambda \in \mathbb{R}$ and $p, \beta, v, z \in \mathbb{C}$ with $Re(p) > 0$, the generalized ML functions $E^{v}_{p,\beta}(\lambda, z)$ are defined by*

$$E^{v}_{p,\beta}(\lambda, z) = \sum_{k=0}^{\infty} \frac{z^{kp+\beta-1} (v)_k}{\Gamma(\alpha k + \beta) k!}. \tag{2.1}$$

In the case of $\beta = v = 1$, Eq (2.1) reduced to

$$E_{\overline{p}}(\lambda, z) \triangleq E^{1}_{p,1}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{kp}}{\Gamma(pk + 1)},$$

where $(v)_k = v(v + 1) .... (v + k - 1)$. Since $(1)_k = k!$, then $E^{1}_{p,\beta}(\lambda, z) = E_{p,\beta}(\lambda, z)$.

Definition 2.2. *[14, 15] Let $p \in (0,1)$, $Re(q) > 0, v \in \mathbb{R}$ and $\lambda = -\frac{p}{1-p}$. The generalized left ABC-FD and ABR-FD of a function z are defined by*

$$
{}_{0}^{ABC}\mathbf{D}^{p,q,v}z(\theta) = \frac{B(p)}{1-p}\int_{0}^{\theta} E_{p,q}^{v}(\lambda, \theta - s)z'(s)\,ds, \theta \geq 0,
$$

and

$$
{}_{0}^{ABR}\mathbf{D}^{p,q,v}z(\theta) = \frac{B(p)}{1-p}\frac{d}{d\theta}\int_{0}^{\theta} E_{p,q}^{v}(\lambda, \theta - s)z(s)\,ds, \theta \geq 0,
$$

repspectively, where $B(p) > 0$ is a normalizing function with $B(0) = B(1) = 1$ and $E_{p,q}^{v}(\lambda, \theta - s)$ is generalized ML functions.

Definition 2.3. *[15] Let $p \in (0,1]$, $q, v > 0$ and $z(\theta)$ be a function defined on $[0,T]$. Then, the left generalized AB fractional integral ${}_{0}^{AB}\mathbf{I}^{p,q,v}z(\theta)$ is given by*

$$
{}_{0}^{AB}\mathbf{I}^{p,q,v}z(\theta) = \sum_{i=0}^{v}\begin{pmatrix} v \\ i \end{pmatrix}\frac{p^{i}}{B(p)(1-p)^{i-1}}I_{0}^{pi-q+1}z(s)ds.
$$

If $z(\theta)$ is continuous function at 0 and ${}_{0}^{ABR}\mathbf{D}^{p,q,v}z(\theta)$ exists, then, we have from [14] that

$$
{}_{0}^{AB}\mathbf{I}^{p,q,v}\,{}_{0}^{ABR}\mathbf{D}^{p,q,v}z(\theta) = z(\theta),
$$

and

$$
{}_{0}^{ABR}\mathbf{D}^{p,q,v}\,{}_{0}^{AB}\mathbf{I}^{p,q,v}z(\theta) = z(\theta).
$$

Lemma 2.4. *[14, 15] For $p \in (0,1)$, $q > 0$, $v \in \mathbb{C}$ and $\lambda = -\frac{p}{1-p}$, we have*

$$
{}_{0}^{AB}\mathbf{I}^{p,q,v}\,{}_{0}^{ABC}\mathbf{D}^{p,q,v}z(\theta) = z(\theta) - z(0).
$$

Lemma 2.5. *[14, 15] For any $p \in (0,1)$, $q > 0$, $v \in \mathbb{R}$, and $\lambda = -\frac{p}{1-p}$, we have*

$$
{}_{0}^{ABC}\mathbf{D}^{p,q,v}z(\theta) = {}_{0}^{ABR}\mathbf{D}^{p,q,v}z(\theta) - \frac{B(p)}{1-p}z(0)E_{p,q}^{v}(\lambda, \theta).
$$

Theorem 2.6. *[29] Let $\mathcal{X}$ be a Banach space. The operator $\Phi : \mathcal{G} \to \mathcal{G}$ is Lipschitzian if there exists a constant $0 < L < 1$ such that i.e., $\|\Phi(z) - \Phi(z^*)\| \leq L\|z - z^*\|$ for all $z, z^* \in \mathcal{G}$. Then Φ is a contraction.*

Theorem 2.7. *[29] Let $\Phi : \mathcal{G} \to \mathcal{G}$ be an operator satisfies*
(1) Φ is completely continuous operator.
(2) The set $\xi(\Phi) = \{z \in \Phi : z = \delta\Phi(z), \delta \in [0,1]\}$ is bounded.
Then Φ has a fixed point in $\mathcal{G}$.

Lemma 2.8. *[15] For $p \in (0,1)$, $q > 0$, $v \in \mathbb{R}$, $\lambda = -\frac{p}{1-p}$ and let $h(\theta)$ be a continuous functions such that $h(0) = 0$ for $q = 1$. Then, the following problem*

$$
\begin{cases} {}_{0}^{ABC}\mathbf{D}^{p,q,v}z(\theta) = h(\theta), \\ \quad z(0) = z^{0} \in \mathbb{R}, \end{cases} \tag{2.2}
$$

is equivalent to the following fractional integral

$$
z(\theta) = z^{0} + \sum_{i=0}^{v}\begin{pmatrix} v \\ i \end{pmatrix}\frac{p^{i}}{B(p)(1-p)^{i-1}}I_{0}^{pi-q+1}h(\theta). \tag{2.3}
$$

Definition 2.9. *If* $(z_1, z_2, ..., z_n) \in \mathcal{G}$, *then* $(z_1, z_2, ..., z_n)$ *is said to be a solution of (1.1), if*

(1) $z_k(0) = z_k^0 \in \mathbb{R}$ *for* $k = 1, 2, ..., n$.

(2) $(z_1, z_2, ..., z_n)$ *satisfied the following integral equation*

$$z_k(\theta) = z_k^0 + \sum_{i=0}^{v} \left(\begin{array}{c} v \\ i \end{array} \right) \frac{p^i}{B(p)\,(1-p)^{i-1}} I_0^{pi-q+1} \mathcal{F}_k\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right).$$

3. Existence and uniqueness of solutions

We devoted this section to derive the equivalent fractional integral equations for the model (1.1). First of all, by using fixed point technique and mathematical techniques, we prove the existence and uniqueness of solution for model (1.1).

In view of Lemma 2.8, the equivalent fractional integral of model (1.1) is given as follows

$$\begin{cases} z_1(\theta) = z_1^0 + {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_1\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right), \\ z_2(\theta) = z_2^0 + {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_2\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right), \\ \quad\quad . \\ \quad\quad . \\ z_n(\theta) = z_n^0 + {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_n\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right). \end{cases} \tag{3.1}$$

Let us consider the continuous operator $\Phi : \mathcal{G} \to \mathcal{G}$ defined by

$$\Phi\left(z_1, z_2, ..., z_n\right)(\theta) = \left(\Phi_1\left(z_1, z_2, ..., z_n\right)(\theta), ..., \Phi_n\left(z_1, z_2, .., z_n\right)(\theta)\right), \tag{3.2}$$

where

$$\begin{cases} \Phi_1\left(z_1, z_2, ..., z_n\right)(\theta) = z_1^0 + {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_1\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right), \\ \Phi_2\left(z_1, z_2, ..., z_n\right)(\theta) = z_2^0 + {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_2\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right), \\ \quad\quad . \\ \quad\quad . \\ \Phi_n\left(z_1, z_2, ..., z_n\right)(\theta) = z_n^0 + {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_n\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right). \end{cases}$$

Notice that the model (1.1) has a solution $(z_1, z_2, ..., z_n)$ if Φ has a fixed point. To achieve our results, the following hypothesis must be hold.

(H_1) : Let $\mathcal{F}_k : \mathcal{J} \times \mathcal{G} \to \mathbb{R}, (k = 1, 2, ..., n)$ be a continuous functions and there exist constants numbers $\lambda_k, \varepsilon_k^1, \varepsilon_k^2, ..., \varepsilon_k^n > 0$, such that

$$\|\mathcal{F}_k\left(\theta, z_1, z_2, ..., z_n\right)\| \leq \lambda_k + \varepsilon_k^1 \|z_1\| + \varepsilon_k^2 \|z_2\| + ... + \varepsilon_k^n \|z_n\|, \tag{3.3}$$

for all $(\theta, z_1, z_2, ..., z_n) \in \mathcal{J} \times \mathcal{G}$.

(H_2) : The kernels $\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))$ satisfies the following Lipschitz condition

$$\left\|\mathcal{F}_k(\theta, z_1, z_2, ..., z_n) - \mathcal{F}_k(\theta, z_1^*, z_2^*, ..., z_n^*)\right\| \leq L_k \sum_{j=1}^{n} \left\|z_j - z_j^*\right\|, \tag{3.4}$$

such that $0 \leq L_k < 1, (z_1, z_2, ..., z_n), (z_1^*, z_2^*, ..., z_n^*) \in \mathcal{G}$.

To simplify our analysis, we set

$$\Lambda_1 = \frac{\max\left\{\sum_{k=1}^{n}\varepsilon_k^1, \sum_{k=1}^{n}\varepsilon_k^2, ..., \sum_{k=1}^{n}\varepsilon_k^n\right\}}{B(p)} \sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)},$$

and

$$\Delta_1 = \sum_{k=1}^{n}\left[\left|z_k^0\right| + \frac{L_k r}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}\right]. \quad (3.5)$$

Theorem 3.1. *Assume that* (H_1) *holds. If* $\Lambda_1 < 1$, *then the operator* Φ *is completely continuous.*

Proof. First, in view of the continuity of the functions $\mathcal{F}_k$, we notice that the operator Φ is continuous. Define a closed ball

$$\mathcal{B}_r = \{(z_1, z_2, ..., z_n) \in \mathcal{G} : \|(z_1, z_2, ..., z_n)\| \le r\},$$

with $r \ge \frac{\Lambda_2}{1-\Lambda_1}$, where

$$\Lambda_2 = \sum_{k=1}^{n}\left[\left|z_k^0\right| + \frac{\lambda_k}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}\right].$$

Now, for $(z_1, z_2, ..., z_n) \in \mathcal{B}_r, \theta \in \mathcal{J}$, then, by (3.3) and $k = 1, 2, ..., n$, we have

$$\begin{aligned}
&\|\Phi_k(z_1, z_2, ..., z_n)\| \\
\le\ & \left|z_k^0\right| + {}_0^{AB}\mathbf{I}^{p,q,v}\left|\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))\right| \\
\le\ & \left|z_k^0\right| + \frac{\lambda_k + \varepsilon_k^1\|z_1\| + \varepsilon_k^2\|z_2\| + ... + \varepsilon_k^n\|z_n\|}{B(p)} \\
\times\ & \sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}. \qquad (3.6)
\end{aligned}$$

Thus

$$\begin{aligned}
\|\Phi(z_1, z_2, ..., z_n)\| &= \sum_{k=1}^{n}\|\Phi_k(z_1, z_2, ..., z_n)\| \\
&\le \sum_{k=1}^{n}\left[\left|z_k^0\right| + \frac{\lambda_k + \varepsilon_k^1\|z_1\| + \varepsilon_k^2\|z_2\| + ... + \varepsilon_k^n\|z_n\|}{B(p)}\right. \\
&\times \left.\sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}\right] \\
&\le \sum_{k=1}^{n}\left|z_k^0\right| + \frac{\lambda_k}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\max\left\{\sum_{k=1}^{n}\varepsilon_k^1, \sum_{k=1}^{n}\varepsilon_k^2, ..., \sum_{k=1}^{n}\varepsilon_k^n\right\}\|(z_1, z_2, ..., z_n)\|}{B(p)} \\
\times & \sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)} \\
\leq & \Lambda_2 + \Lambda_1 r \leq r.
\end{aligned}
$$

Hence, $\Phi(\mathcal{B}_r)$ is uniformly bounded. Next, for the equicontinuity of the operator Φ, for any $\theta_1, \theta_2 \in \mathcal{J}, \theta_1 < \theta_2$ and $(z_1, z_2, ..., z_n) \in \mathcal{B}_r$, for $k = 1, 2, ..., n$, we have

$$
\begin{aligned}
& \|\Phi_k(z_1, z_2, ..., z_n)(\theta_2) - \Phi_k(z_1, z_2, ..., z_n)(\theta_1)\| \\
\leq & \max_{\theta\in\mathcal{J}}\left|{}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_k(\theta_2, z_1(\theta_2), ..., z_n(\theta_2)) - {}_0^{AB}\mathbf{I}^{p,q,v}\mathcal{F}_k(\theta_1, z_1(\theta_1), ..., z_n(\theta_1))\right| \\
\leq & \frac{\lambda_k + \varepsilon_k^1\|z_1\| + \varepsilon_k^2\|z_2\| + ... + \varepsilon_k^n\|z_n\|}{B(p)} \times \sum_{i=0}^{v}\binom{v}{i}\frac{p^i\left(\theta_2{}^{pi-q+1} - \theta_1{}^{pi-q+1}\right)}{(1-p)^{i-1}\Gamma(pi+2-q)} \\
& \to 0 \text{ as } \theta_2 \to \theta_1.
\end{aligned}
$$

Hence

$$
\|\Phi(z_1, z_2, ..., z_n)(\theta_2) - \Phi(z_1, z_2, ..., z_n)(\theta_1)\| \to 0 \text{ as } \theta_2 - \theta_1.
$$

Thus, Φ is equicontinuous. According to the above analysis together with Arzela´-Ascoli Theorem, we deduce that Φ is relatively compact and so completely continuous. □

Theorem 3.2. *Let $\mathcal{F}_k$ be a functions satisfies (H_1) such that $\mathcal{F}_k(0, z_1(0), z_2(0), ..., z_n(0))) = 0$ in the case $q = 1$. If $\Lambda_1 < 1$, then the model (1.1) has at least one solution $(z_1, z_2, ..., z_n) \in \mathcal{H} := \underbrace{C^{p,q,v}(\mathcal{J}, \mathbb{R}^+) \times .... \times C^{p,q,v}(\mathcal{J}, \mathbb{R}^+)}_{n-time}$.*

Proof. From Theorem (3.1), we have Φ is completely continuous. Now, by means of Schaefer's fixed point approaches, we need only prove that the set

$$
\xi(\Phi) = \{(z_1, z_2, ..., z_n) \in \mathcal{G} : (z_1, z_2, ..., z_n) = \delta\Phi(z_1, z_2, ..., z_n), \delta \in [0, 1]\},
$$

is bounded. Let $(z_1, z_2, ..., z_n) \in \xi(\Phi)$. Then, $z_k = \delta\Phi_k(z_1, z_2, .., z_n), k = 1, 2, ..., n$. For $\theta \in \mathcal{J}$, by (3.3), we get

$$
\begin{aligned}
\|z_k\| &= \|\delta\Phi_k(z_1, z_2, ..., z_n)\| \\
&\leq \|\Phi_k(z_1, z_2, ..., z_n)\| \\
&\leq \left|z_k^0\right| + \frac{\lambda_k + \varepsilon_k^1\|z_1\| + \varepsilon_k^2\|z_2\| + ... + \varepsilon_k^n\|z_n\|}{B(p)} \\
&\times \sum_{i=0}^{v}\binom{v}{i}\frac{p^i T^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}.
\end{aligned}
$$

Thus

$$
\|(z_1, z_2, ..., z_n)\| \leq \sum_{k=1}^{n}\left|z_k^0\right| + \sum_{k=1}^{n}\frac{\lambda_k + \varepsilon_k^1\|z_1\| + \varepsilon_k^2\|z_2\| + ... + \varepsilon_k^n\|z_n\|}{B(p)}
$$

$$
\begin{aligned}
&\times \quad \sum_{i=0}^{v} \binom{v}{i} \frac{p^{i} T^{pi-q+1}}{(1-p)^{i-1} \Gamma(pi+2-q)} \Bigg] \\
&\leq \quad \Lambda_2 + \Lambda_1 r \leq r.
\end{aligned}
$$

Hence $\xi(\Phi)$ is bounded. So, by Theorem (2.7), we deduce that Φ has one fixed point in $\mathcal{X}$. Consequently, the model (1.1) has at least one solution $(z_1, z_2, ..., z_n) \in \mathcal{G}$. In addition, by the definition of Φ and $(z_1, z_2, ..., z_n)$ possesses the form $z_k(\theta) = z_k^0 +_0^{AB} \mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))$, we get

$$
\lim_{m \to \infty} \left\| \Phi^m z_k^0 - z_k \right\| = 0. \tag{3.7}
$$

By Lemmas 2.4 and 2.5, the identity ${}_0^{ABR}\mathbf{D}^{p,q,v}\, {}_0^{AB}\mathbf{I}^{p,q,v}\, z_k(\theta) = z_k(\theta)$, and taking into account that $\mathcal{F}_k(0, z_1(0), z_2(0), ..., z_n(0))) = 0$ for $q = 1$. So, we can shown that $(z_1, z_2, ..., z_n)(\theta)$ satisfies the model (1.1) if and only if it satisfies (3.1). Finally, we have the estimate

$$
\left\| {}_0^{ABC}\mathbf{D}^{p,q,v} \Phi^m z_k^0 - {}_0^{ABC}\mathbf{D}^{p,q,v} z_k \right\| \leq L \left\| \Phi^m z_k^0 - z_k \right\|.
$$

From (3.7), we conclude that

$$
\lim_{m \to \infty} \left\| {}_0^{ABC}\mathbf{D}^{p,q,v} \Phi^m z_k^0 - {}_0^{ABC}\mathbf{D}^{p,q,v} z_k \right\| = 0.
$$

That is ${}_0^{ABC}\mathbf{D}^{p,q,v} z_k \in C^{p,q,v}(\mathcal{J}, \mathbb{R}^+)$ and hence

$$
(z_1, z_2, ..., z_n) \in \mathcal{H} := \underbrace{C^{p,q,v}(\mathcal{J}, \mathbb{R}^+) \times .... \times C^{p,q,v}(\mathcal{J}, \mathbb{R}^+)}_{n-time}.
$$

□

Theorem 3.3. *Let $\mathcal{F}_k$ be continuous functions satisfies (H_2). Then, the model (1.1) has a unique solution in the space $\mathcal{H} := \underbrace{C^{p,q,v}(\mathcal{J}, \mathbb{R}^+) \times .... \times C^{p,q,v}(\mathcal{J}, \mathbb{R}^+)}_{n-time}$, provided that $\Delta_1 < 1$. Moreover, the case $q = 1$ requires that $\mathcal{F}_k(0, z_1(0), z_2(0), ..., z_n(0))) = 0$.*

Proof. Define a closed ball set $\mathcal{A}_r = \{(z_1, z_2, ..., z_n) \in \mathcal{G} : \|(z_1, z_2, ..., z_n)\| \leq r\}$ with $r \geq \frac{\Delta_2}{1-\Delta_1}$, where

$$
\Delta_2 = \sum_{k=1}^{n} \frac{\mathcal{K}_k}{B(p)} \sum_{i=0}^{v} \binom{v}{i} \frac{p^{i} T^{pi-q+1}}{(1-p)^{i-1} \Gamma(pi+2-q)},
$$

and $\mathcal{K}_k = \max_{\theta \in \mathcal{J}} |\mathcal{F}_k(\theta, 0, 0, ..., 0)|$. In order to prove $\Phi(\mathcal{A}_r) \subset \mathcal{A}_r$, let $(z_1, z_2, ..., z_n) \in \mathcal{A}_r$. Then, for $\theta \in \mathcal{J}$, by (3.4), we have

$$
\begin{aligned}
& \|\Phi_k(z_1, z_2, ..., z_n)\| \\
\leq \quad & \left|z_k^0\right| +_0^{AB} \mathbf{I}^{p,q,v} \left(|\mathcal{F}_1(\theta, z_1(\theta), .., z_n(\theta)) - \mathcal{F}_1(\theta, 0, 0, ..., 0)| + |\mathcal{F}_1(\theta, 0, 0, ..., 0)|\right) \\
\leq \quad & \left|z_k^0\right| +_0^{AB} \mathbf{I}^{p,q,v} L_k \sum_{j=1}^{n} \left\|z_j\right\| +_0^{AB} \mathbf{I}^{p,q,v} \mathcal{K}_k \\
\leq \quad & \left|z_k^0\right| + \frac{L_k \sum_{j=1}^{n} \left\|z_j\right\|}{B(p)} \sum_{i=0}^{v} \binom{v}{i} \frac{p^{i} T^{pi-q+1}}{(1-p)^{i-1} \Gamma(pi+2-q)}
\end{aligned}
$$

$$+\frac{\mathcal{K}_k}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^iT^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}.$$

Thus

$$\begin{aligned}\|\Phi(z_1,z_2,..,z_n)\| &= \sum_{k=1}^{n}\|\Phi_k(z_1,z_2,....,z_n)\|\\ &\le \sum_{k=1}^{n}\left[\left|z_k^0\right|+\frac{L_kr}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^iT^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}\right]\\ &\quad+\sum_{k=1}^{n}\frac{\mathcal{K}_k}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^iT^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}\\ &\le \Delta_1r+\Delta_2\le r.\end{aligned}$$

Hence $\Phi(\mathcal{A}_r)\subset\mathcal{A}_r$. For any $(z_1,z_2,....,z_n),\left(z_1^*,z_2^*,.....,z_n^*\right)\in\mathcal{A}_r$ and $\theta\in\mathcal{J}$, we have

$$\begin{aligned}&\left\|\Phi_k(z_1,z_2,....,z_n)-\Phi_k(z_1^*,z_2^*,.....,z_n^*)\right\|\\ \le&\left[\frac{L_k}{B(p)}\sum_{i=0}^{v}\binom{v}{i}\frac{p^iT^{pi-q+1}}{(1-p)^{i-1}\Gamma(pi+2-q)}\right]\sum_{j=1}^{n}\left\|z_j-z_j^*\right\|.\end{aligned}$$

Thus

$$\left\|\Phi(z_1,z_2,....,z_n)-\Phi(z_1^*,z_2^*,.....,z_n^*)\right\|\le\Delta_1\sum_{j=1}^{n}\left\|z_j-z_j^*\right\|.$$

Due to $\Delta_1<1$, we conclude that Φ a contraction on $C(\mathcal{J},\mathbb{R}^+)$. Therefore, due to Banach fixed point Theorem, the model (1.1) has a unique fixed point $(z_1,z_2,....,z_n)\in\mathcal{G}$. By the same way in Theorem 3.2, one can prove that $(z_1,z_2,....,z_n)\in\mathcal{H}$. In case $q=1$, the condition $\mathcal{F}_k(0,z_1(0),...,z_n(0))=0$ is needed in order to guarantee that solution given by (2.3) will satisfy $z_k(0)=z_k^0, k=1,2,.....,n$. However, in case $q\neq1$, one may note that $z_k(0)=z_k^0$ without any restrictions. □

4. Ulam-Hyers stability

Definition 4.1. *[30] The model (1.1) is UH stable if there exists a real number* $\mathfrak{R}=\max\{\mathfrak{R}_1,\mathfrak{R}_2,......,\mathfrak{R}_n\}>0$ *such that for each* $\epsilon=\max\{\epsilon_1,\epsilon_2,.....,\epsilon_n\}>0$ *there exists a solution* $(\widetilde{z}_1,\widetilde{z}_2,.....,\widetilde{z}_n)\in\mathcal{G}$ *of the inequality*

$$\left|{}_0^{ABC}\mathbf{D}^{p,q,v}\widetilde{z}_k(\theta)-\mathcal{F}_k(\theta,\widetilde{z}_1(\theta),\widetilde{z}_2(\theta),.....\widetilde{z}_n(\theta))\right|\le\epsilon_k, \tag{4.1}$$

corresponding to a solution $(z_1,z_2,....,z_n)\in\mathcal{G}$ *of model (1.1) such that*

$$\|(\widetilde{z}_1,\widetilde{z}_2,.....,\widetilde{z}_n)-(z_1,z_2,....,z_n)\|\le\mathfrak{R}\epsilon,\qquad \theta\in\mathcal{J}.$$

Remark 4.2. *A function* $(\widetilde{z}_1,\widetilde{z}_2,.....,\widetilde{z}_n)\in\mathcal{G}$ *satisfies the inequality (4.1) if and only if there exist a small perturbation* $(z_1,z_2,...,z_n)\in\mathcal{G}$ *such that for* $k=1,2,....,n$, *we have*

(i) $\{|z_k(\theta)|\le\epsilon_k,\ \theta\in\mathcal{J}$.

(ii) ${}_0^{ABC}\mathbf{D}^{p,q,v}\widetilde{z}_k(\theta)=\mathcal{F}_k(\theta,\widetilde{z}_1(\theta),..,\widetilde{z}_n(\theta))+z_k(\theta),\theta\in\mathcal{J}$.

Lemma 4.3. *Let $0 \leq p < 1, q, v > 0$. If a function $(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n) \in \mathcal{G}$ satisfies the inequality (4.1), then $(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n)$ satisfies the following integral inequalities*

$$\left| \widetilde{z}_k(\theta) - \widetilde{z}_k^0 - {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, \widetilde{z}_1(\theta), ...\widetilde{z}_n(\theta)) \right| \leq \epsilon_k \mathbf{K},$$

where

$$\mathbf{K} = \frac{1}{B(p)} \sum_{i=0}^{v} \begin{pmatrix} v \\ i \end{pmatrix} \frac{p^i T^{pi-q+1}}{(1-p)^{i-1} \Gamma(pi + 2 - q)}.$$

Proof. Indeed by Remark 4.2, we have the following model

$${}_0^{ABC}\mathbf{D}^{p,q,v} \widetilde{z}_k(\theta) = \mathcal{F}_k(\theta, \widetilde{z}_1(\theta), ..., \widetilde{z}_n(\theta)) + z_k(\theta), \theta \in \mathcal{J}.$$

Then, the solution of the above model is given as

$$\widetilde{z}_k(\theta) = \widetilde{z}_k^0 + {}_0^{AB}\mathbf{I}^{p,q,v} \left[\mathcal{F}_k(\theta, \widetilde{z}_1(\theta), \widetilde{z}_2(\theta), ..., \widetilde{z}_n(\theta)) + z_k(\theta) \right].$$

It follows that

$$\left| \widetilde{z}_k(\theta) - \widetilde{z}_k^0 - {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, \widetilde{z}_1(\theta), ..., \widetilde{z}_n(\theta)) \right| \leq {}_0^{AB}\mathbf{I}^{p,q,v} \left| z_k(\theta) \right|.$$

Hence

$$\left| \widetilde{z}_k(\theta) - \widetilde{z}_k^0 - {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, \widetilde{z}_1(\theta), ..., \widetilde{z}_n(\theta)) \right| \leq \epsilon_1 \mathbf{K}.$$

□

Theorem 4.4. *Assume that the preconditions of Theorem 3.3 are satisfied. Then the model (1.1) is UH stable in $C(\mathcal{J}, \mathbb{R}^+)$.*

Proof. Let $\epsilon = \max\{\epsilon_1, \epsilon_2, ..., \epsilon_n\} > 0$ and

$$\Omega_k = \frac{L_k}{B(p)} \sum_{i=0}^{v} \begin{pmatrix} v \\ i \end{pmatrix} \frac{p^i T^{pi-q+1}}{(1-p)^{i-1} \Gamma(pi + 2 - q)}. \tag{4.2}$$

Let $(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n) \in \mathcal{G}$ be a functions satisfying the inequalities

$$\left| {}_0^{ABC}\mathbf{D}^{p,q,v} \widetilde{z}_k(\theta) - \mathcal{F}_k(\theta, \widetilde{z}_1(\theta), \widetilde{z}_2(\theta), ..., \widetilde{z}_n(\theta)) \right| \leq \epsilon_k,$$

and let $(z_1, z_2, ..., z_n) \in \mathcal{G}$ be the unique solution of the following model

$$\begin{cases} {}_0^{ABC}\mathbf{D}^{p,q,v} z_k(\theta) = \mathcal{F}_k(\theta, z_1(\theta), ..., z_n(\theta)), \\ z_k(0) = \widetilde{z}_k(0), k = 1, 2, ..., n. \end{cases}$$

Now, in the light of Lemma 2.8, we have

$$z_k(\theta) = z_k^0 + {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)),$$

Since $z_k(0) = \widetilde{z}_k(0), (k = 1, 2, ..., n)$, we get

$$z_k(\theta) = \widetilde{z}_k^0 + {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)),$$

Hence, from (3.4) and Lemma 4.3, then for each $\theta \in \mathcal{J}$, we have

$$\|\widetilde{z}_k - z_k\| \leq \mathbf{K}\epsilon_k + \Omega_k \sum_{j=1}^{n} \left\|\widetilde{z}_j - z_j\right\|,$$

which implies

$$\begin{aligned} & \|(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n) - (z_1, z_2, ..., z_n)\| \\ \leq & \sum_{k=1}^{n} \mathbf{K}\epsilon_k + \sum_{k=1}^{n} \Omega_k \|(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n) - (z_1, z_2, ..., z_n)\|. \end{aligned}$$

For $\epsilon = \max\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ and

$$\mathfrak{R} = \frac{n\mathbf{K}}{1 - \sum_{k=1}^{n} \Omega_k},$$

we get

$$\|(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n) - (z_1, z_2, ..., z_n)\| \leq \mathfrak{R}\epsilon.$$

Hence the model (1.1) is U-H stable. □

5. Continuous dependence and δ-approximate solutions

In this section, we shall discuss the results of continuous dependence of solutions of the proposed model (1.1) on initial conditions.

Definition 5.1. *[29] A function $(z_1, z_2, ..., z_n) \in \mathcal{G}$ satisfying the following fractional differential inequality*

$$\left\|{}_0^{ABC}\mathbf{D}^{p,q,v} z_k(\theta) - \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))\right\| \leq \delta, \theta \in \mathcal{J}, \tag{5.1}$$

and

$$z_k(0) = z_k^0,$$

is called δ-approximate solutions of model (1.1).

Theorem 5.2. *For $p \in (0,1), q > 0$, $v \in \mathbb{R}$ and $\lambda = -\frac{p}{1-p}$. Let $\mathcal{F}_k : \mathcal{J} \times \mathcal{G} \to \mathbb{R}$ be a continuous function and satisfies Lipschitz condition 3.4, let $(z_1, z_2, ..., z_n)$ and $\left(z_1^*, z_2^*, ..., z_n^*\right)$ be a δ-approximation solutions of the model (1.1). If $\triangle \neq 0$ and*

$$\begin{pmatrix} m_{11} & m_{12} & . & . & m_{1n} \\ m_{21} & m_{22} & . & . & m_{2n} \\ . & . & & & . \\ . & . & & & . \\ m_{n1} & m_{n2} & . & . & m_{nn} \end{pmatrix}^{-1} = \frac{1}{\triangle} \begin{pmatrix} l_{11} & l_{12} & . & . & l_{1n} \\ l_{21} & l_{22} & . & . & l_{2n} \\ . & . & & & . \\ . & . & & & . \\ l_{n1} & l_{n2} & . & . & l_{nn} \end{pmatrix},$$

where $l_{ij} \in \mathbb{R}, i, j = 1, 2, ...n$ and

$$m_{ij} = \begin{cases} 1 - \Omega_i; & i = j \\ -\Omega_i; & i \neq j, \end{cases}$$

where

$$\Omega_k = \frac{L_k}{B(p)} \Theta_{q=1}^{q\neq 1}(T), k = 1, 2, ..., n,$$

then

$$\begin{aligned} & \left\|(z_1, z_2, .., z_n) - (z_1^*, z_2^*, ..., z_n^*)\right\| \\ \leq \ & \frac{1}{\triangle} \sum_{k=1}^{n} \sum_{r=1}^{n} l_{kr} \left({}_0^{AB}\mathbf{I}^{p,q,v} |\delta_1 + \delta_2| + \left|z_0^r - z_0^{*r}\right| \right), \end{aligned}$$

Proof. Let $(z_1, z_2, ..., z_n), \left(z_1^*, z_2^*, ..., z_n^*\right) \in \mathcal{G}$ be an δ-approximation solutions of the model (1.1). Then, we have

$$\begin{cases} \left\|{}_0^{ABC}\mathbf{D}^{p,q,v} z_k(\theta) - \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))\right\| \leq \delta_1, \theta \in \mathcal{J}, \\ \left\|{}_0^{ABC}\mathbf{D}^{p,q,v} z_k^*(\theta) - \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right\| \leq \delta_2, \theta \in \mathcal{J}, \end{cases} \tag{5.2}$$

and

$$\begin{cases} z_k(0) = z_0^k \\ z_k^*(0) = z_0^{*k}, \\ k = 1, 2, ..., n. \end{cases}$$

Applying ${}_0^{AB}\mathbf{I}^{p,q,v}$ on both sides of the above inequalities, and using Lemma 2.8, we get

$$\begin{cases} \left|z_k(\theta) - z_0^k - {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))\right| \leq {}_0^{AB}\mathbf{I}^{p,q,v} \delta_1, \\ \left|z_k^*(\theta) - z_0^{*k} - {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right| \leq {}_0^{AB}\mathbf{I}^{p,q,v} \delta_2. \end{cases}$$

Using the fact $|z| - |y| \leq |z - y| \leq |z| + |y|$, we get

$$\begin{aligned} & {}_0^{AB}\mathbf{I}^{p,q,v} |\delta_1 + \delta_2| \geq \\ & \left|z_k(\theta) - z_0^k + {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))\right| \\ & + \left|z_k^*(\theta) - z_0^{*k} + {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right| \\ \geq \ & \left|\left[z_k(\theta) - z_0^k + {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))\right]\right. \\ & \left. - \left[z_k^*(\theta) - z_0^{*k} + {}_0^{AB}\mathbf{I}^{p,q,v} \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right]\right| \\ \geq \ & \left|(z_k(\theta) - z_k^*(\theta)) - \left(z_0^k - z_0^{*k}\right)\right. \\ & \left. + {}_0^{AB}\mathbf{I}^{p,q,v} \left[\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)) - \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right]\right| \\ \geq \ & \left|(z_k(\theta) - z_k^*(\theta))\right| - \left|\left(z_0^k - z_0^{*k}\right)\right| \\ & + \left|{}_0^{AB}\mathbf{I}^{p,q,v} \left[\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)) - \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right]\right|. \end{aligned}$$

Then, we have

$$\begin{aligned} \left\|z_k - z_k^*\right\| &= \sup_{\theta\in\mathcal{J}} \left|z_k(\theta) - z_k^*(\theta)\right| \\ &\leq {}_0^{AB}\mathbf{I}^{p,q,v} |\delta_1 + \delta_2| + \left|z_0^k - z_0^{*k}\right| \\ &\quad + {}_0^{AB}\mathbf{I}^{p,q,v} \left|\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ...z_n(\theta)) - \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ...z_n^*(\theta))\right| \\ &\leq {}_0^{AB}\mathbf{I}^{p,q,v} |\delta_1 + \delta_1| + \left|z_0^k - z_0^{*k}\right| \end{aligned}$$

$$+\Omega_k \sum_{j=1}^{n} \left\| z_j - z_j^* \right\|.$$

Consequently, we have the following inequalities

$$\begin{aligned}
(1-\Omega_1)\left\| z_1 - z_1^* \right\| - \Omega_1 \sum_{j=2}^{n} \left\| z_j - z_j^* \right\| &\leq {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_1 \right| + \left| z_0^1 - z_0^{*1} \right|, \\
(1-\Omega_2)\left\| z_2 - z_2^* \right\| - \Omega_2 \sum_{\substack{j=1 \\ j\neq 2}}^{n} \left\| z_j - z_j^* \right\| &\leq {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^2 - z_0^{*2} \right|, \\
(1-\Omega_3)\left\| z_3 - z_3^* \right\| - \Omega_3 \sum_{\substack{j=1 \\ j\neq 3}}^{n} \left\| z_j - z_j^* \right\| &\leq {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^3 - z_0^{*3} \right|, \\
&\;\;. \\
&\;\;. \\
(1-\Omega_n)\left\| z_n - z_n^* \right\| - \Omega_n \sum_{j=1}^{n-1} \left\| z_j - z_j^* \right\| &\leq {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^n - z_0^{*n} \right|.
\end{aligned} \tag{5.3}$$

Inequalities (5.3) can be writting as matrices as followes

$$\begin{pmatrix} m_{11} & m_{12} & . & . & m_{1n} \\ m_{21} & m_{22} & . & . & m_{2n} \\ . & . & & & . \\ . & . & & & . \\ m_{n1} & m_{n2} & . & . & m_{nn} \end{pmatrix} \begin{pmatrix} \left\| z_1 - z_1^* \right\| \\ \left\| z_2 - z_2^* \right\| \\ . \\ . \\ \left\| z_n - z_n^* \right\| \end{pmatrix} \leq \begin{pmatrix} {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^1 - z_0^{*1} \right| \\ {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^2 - z_0^{*2} \right| \\ . \\ . \\ {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^n - z_0^{*n} \right| \end{pmatrix}.$$

By simple computations, the above inequality becomes

$$\begin{pmatrix} \left\| z_1 - z_1^* \right\| \\ \left\| z_2 - z_2^* \right\| \\ . \\ . \\ \left\| z_n - z_n^* \right\| \end{pmatrix} \leq \frac{1}{\Delta} \begin{pmatrix} l_{11} & l_{12} & . & . & l_{1n} \\ l_{21} & l_{22} & . & . & l_{2n} \\ . & . & & & . \\ . & . & & & . \\ l_{n1} & l_{n2} & . & . & l_{nn} \end{pmatrix} \times \begin{pmatrix} {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^1 - z_0^{*1} \right| \\ {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^2 - z_0^{*2} \right| \\ . \\ . \\ {}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^n - z_0^{*n} \right| \end{pmatrix}.$$

Since $\Delta \neq 0$. This leads to

$$\begin{aligned}
\left\| z_1 - z_1^* \right\| &\leq \sum_{r=1}^{n} \frac{l_{1r}}{\triangle} \left({}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^r - z_0^{*r} \right| \right) \\
\left\| z_2 - z_2^* \right\| &\leq \sum_{r=1}^{n} \frac{l_{2r}}{\triangle} \left({}_0^{AB}\mathbf{I}^{p,q,v} \left| \delta_1 + \delta_2 \right| + \left| z_0^r - z_0^{*r} \right| \right) \\
&\;\;\vdots \\
&\;\;\vdots
\end{aligned}$$

$$\left\|z_n - z_n^*\right\| \leq \sum_{r=1}^{n} \frac{l_{nr}}{\triangle}\left({}_{0}^{AB}\mathbf{I}^{p,q,v}\left|\delta_1+\delta_2\right| + \left|z_0^r - z_0^{*r}\right|\right).$$

From the fact

$$\left\|(z_1, z_2, ..., z_n) - (z_1^*, z_2^*, ..., z_n^*)\right\| = \sum_{k=1}^{n}\left\|z_k - z_k^*\right\|.$$

It follows that

$$\begin{aligned} &\left\|(z_1, z_2, ..., z_n) - (z_1^*, z_2^*, ..., z_n^*)\right\| \\ \leq\ & \sum_{k=1}^{n}\sum_{r=1}^{n} \frac{l_{kr}}{\triangle}\left({}_{0}^{AB}\mathbf{I}^{p,q,v}\left|\delta_1+\delta_2\right| + \left|z_0^r - z_0^{*r}\right|\right) \\ \leq\ & \frac{1}{\triangle}\sum_{k=1}^{n}\sum_{r=1}^{n} l_{kr}\left({}_{0}^{AB}\mathbf{I}^{p,q,v}\left|\delta_1+\delta_2\right| + \left|z_0^r - z_0^{*r}\right|\right). \end{aligned} \tag{5.4}$$

□

Remark 5.3. *If $\delta_1 = \delta_2 = 0$ in the inequality (5.4), then $(z_1, z_2, ..., z_n), \left(z_1^*, z_2^*, ..., z_n^*\right)$ are solutions of the model (1.1) and the inequality (5.4) reduces to*

$$\left\|(z_1, z_2, ..., z_n) - (z_1^*, z_2^*, ..., z_n^*)\right\| \leq \frac{1}{\triangle}\sum_{k=1}^{n}\sum_{r=1}^{n} l_{kr}\left(\left|z_0^r - z_0^{*r}\right|\right),$$

which provides the continuous dependence of the model (1.1). Also if $z_0^r = z_0^{*r}$ for all $r = 1, 2, ..., n$, then

$$\left\|(z_1, z_2, ..., z_n) - (z_1^*, z_2^*, ..., z_n^*)\right\| = 0,$$

which provides the uniqueness of a solution of model (1.1).

6. An example

Consider the following model

$$\begin{cases} {}_{0}^{ABC}\mathbf{D}^{\frac{1}{2},\frac{1}{2},1} z_1(\theta) = \mathcal{F}_1\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right), \\ {}_{0}^{ABC}\mathbf{D}^{\frac{1}{2},\frac{1}{2},1} z_2(\theta) = \mathcal{F}_2\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right), \\ \vdots \\ \vdots \\ {}_{0}^{ABC}\mathbf{D}^{\frac{1}{2},\frac{1}{2},1} z_n(\theta) = \mathcal{F}_n\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right) \\ z_k(0) = z_k^0 \in \mathbb{R}, k = 1, 2, ..., n. \end{cases}$$

Here $p = \frac{1}{2}, q = \frac{1}{2}, v = 1, a = 0, \theta = 1,$

$$\mathcal{F}_k\left(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)\right) = \frac{\theta\left(1 + \sum_{j=1}^{n}\left|z_j(\theta)\right|\right)}{3^{\theta}\left[1 + \sum_{j=1}^{n}\left|z_j(\theta)\right|\right]}, \theta \in [0, 1], (z_1, z_2, ..., z_n) \in \mathcal{G}.$$

Clearly, $\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))$ are continuous, $\mathcal{F}_k(0, z_1(0), z_2(0), ..., z_n(0)) = 0$ and

$$\left|\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta)) - \mathcal{F}_k(\theta, z_1^*(\theta), z_2^*(\theta), ..., z_n^*(\theta))\right| \leq \frac{1}{3}\sum_{j=1}^{n}\left|z_j - z_j^*\right|,$$

for all $(z_1, z_2, ..., z_n), \left(z_1^*, z_2^*, ..., z_n^*\right) \in \mathcal{G}$. Also

$$|\mathcal{F}_k(\theta, z_1(\theta), z_2(\theta), ..., z_n(\theta))| \leq \frac{1}{3}\left(1 + \sum_{j=1}^{n}\left|z_j(\theta)\right|\right).$$

Here $L_k = \lambda_k = \varepsilon_k^j = \frac{1}{3}$ for all $k = 1, 2, ..., n$ and $j = 1, 2, ..., n$. Now, by simple calculation, we get

$$\Lambda_1 \simeq \begin{cases} 0.35, & \text{for } q \neq 1, \\ 0.45, & \text{for } q = 1. \end{cases}$$

Thus all conditions of Theorem 3.3 are satisfied. Hence, model (1.1) has a unique solution on $(0, 1]$. Finally, for $\epsilon = \max\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$, we find that

$$\left|{}_{0}^{ABC}\mathbf{D}^{p,q,v}\widetilde{z}_k(\theta) - \mathcal{F}_k(\theta, \widetilde{z}_1(\theta), \widetilde{z}_2(\theta), ...\widetilde{z}_n(\theta))\right| \leq \epsilon_k.$$

is satisfied. Hence the model (1.1) is U-H stable with

$$\|(\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_n) - (z_1, z_2, ..., z_n)\| \leq \mathfrak{R}\epsilon,$$

where

$$\mathfrak{R} = \frac{n\mathbf{K}}{1 - \sum_{k=1}^{n}\Omega_k} > 0.$$

7. Conclusions

ABC fractional operators with multi-parameters ML kernels on certain time scales and the integral equations expressed by them are some of the keys in developing fractional calculus. In this work, we have obtained some existence, uniqueness, UH stability results for the fractional system (1.1) in the frame of generalized FD in AB settings containing a multi-parameter ML kernel. As well, the data dependence analysis and δ-approximate solutions of the proposed system are discussed. Our approach is based on some fixed point theorems and mathematical techniques. As an application, one example has been provided in order to illustrate the validity of our results. We realized that, if $q \neq 1$, then the condition $\mathcal{F}_k(0, z_1(0), ..., z_n(0)) = 0, (k = 1, 2, ..., n)$ not necessary to guarantee a unique solution. The considered system has been investigated under the generalized ML law. Observed that in the case of the classical ABC fractional operator, the requirement of the vanishing status condition of the right-hand side of the dynamic system to full the initial data needs recuperation on the modeled population. However, the nature of the generalized ML kernel managed to get rid of any restrictions on the initial data. Due to the fractional operators used in this work have semigroup property and recently proposed, the results obtained here are new and open the door for the researchers to study more real-world problems in different fields. Besides, the results obtained in this work are very significant in developing the theory of fractional analytical dynamics of different biological models.

So in the future, the same analysis can be extended to the system of delay equations under the generalized fractional operator.

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Conflicts of interest

The authors declare that they have no competing interests.

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