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Research article

Results on the solutions of several second order mixed type partial differential difference equations

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Abstract: This article is concerned with the existence of entire solutions for the following complex second order partial differential-difference equation

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^l + f(z_1+c_1,z_2+c_2)^k = 1,$$

where c_1, c_2 are constants in $\mathbb{C}$ and k, l are positive integers. In addition, we also investigate the forms of finite order transcendental entire solutions for several complex second order partial differential-difference equations of Fermat type, and obtain some theorems about the existence and the forms of solutions for the above equations. Meantime, we give some examples to explain the existence of solutions for some theorems in some cases. Our results are some generalizations of the previous theorems given by Qi [23], Xu and Cao [35], Liu, Cao and Cao [17].

Keywords: Nevanlinna theory; Fermat type; entire solution; partial differential difference equation
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1. Introduction and main results

For positive integers m, n, the equation $x^m + y^n = 1$ is called as the Fermat equation. In 1995, A. Wiles and R. Taylor [28, 29] pointed out that this equation does not admit nontrivial solutions in rational numbers for $m = n \geq 3$, and this equation does exist nontrivial rational solution for $m = n = 2$.

The functional equation $f(z)^m + g(z)^n = 1$ can be regarded as the Fermat type equation. It has attracted the attention of many mathematics workers in the studying of Fermat type equation. We know that the Fermat type equation has no transcendental meromorphic solutions when $n = m \geq 4$ (see [5]); and this equation has no transcendental entire solutions when $n = m \geq 3$ (see [21]). In [10], Iyer pointed out that the entire solutions are of the form $f = \cos a(z), g = \sin a(z)$, where $m = n = 2$ and $a(z)$ is an entire function, no other forms exist. In [36], Yang discussed the Fermat type functional equation

$$a(z)f(z)^n + b(z)g(z)^m = 1, \tag{1.1}$$

where $a(z), b(z)$ are small functions with respect to f and obtained

Theorem A. (see [36]). *Let m, n be positive integers satisfying $1/m + 1/n < 1$. Then there are no nonconstant entire solutions f and g that satisfy* (1.1).

When g is replaced by f' or a differential polynomial of f and $m = n = 2$ for Eq (1.1), Yang and Li [37] in 2004 studied the Malmquist type nonlinear differential equational by using Nevanlinna theory of meromorphic functions, and obtained

Theorem B. (see [37]). *Let a_1, a_2 and a_3 be nonzero meromorphic functions. Then a necessary condition for the differential equation*

$$a_1 f^2 + a_2 f'^2 = a_3.$$

to have a transcendental meromorphic solution satisfying $T(r, a_k) = S(r, f), k = 1, 2, 3$, is $\frac{a_1}{a_3} \equiv$constant.

Theorem C. (see [37]). *Let n be a positive integer, $b_0, b_1, \ldots, b_{n-1}$ be constants, b_n be a non-zero constant and let $L(f) = \sum_{k=0}^{n} b_k f^{(k)}$. Then the transcendental meromorphic solution of the following equation*

$$f(z)^2 + L(f)^2 = 1. \tag{1.2}$$

must have the form $f(z) = \frac{1}{2}(Pe^{\lambda z} + \frac{1}{P}e^{-\lambda z})$, where $e^A = P$, P is a non-zero constant and λ satisfies the following equations:

$$\sum_{k=0}^{n} b_k \lambda^k = \frac{1}{i}, \quad \sum_{k-0}^{n} b_k (-\lambda)^k = -\frac{1}{i}.$$

Remark 1.1. *Let $L(f) = f^{(n)}(z)$. From Theorem C, we can see that if n is an odd, then Eq* (1.2) *has transcendental entire solutions. If n is an even, then Eq* (1.2) *has no transcendental entire solutions.*

Over the past two decades, with the help of difference Nevanlinna theory for meromorphic functions (see [4, 6, 7]), the study of the properties of solutions for complex difference equations and complex differential-difference equations has become more and more active, and a series of literatures concerning the existence and forms of solutions for some equations have sprung up (including [15, 16, 18, 19, 23, 30–33]).

When $L(f)$ is replaced by $f(z+c)$ in Eq (1.2), Liu [15] in 2009 investigated the entire solutions of the equation $f(z)^2 + f(z+c)^2 = 1$ by using the difference Nevanlinna theory for meromorphic functions and pointed out that the finite order transcendental entire solutions $f(z)$ of the equation $f(z)^2 + f(z+c)^2 = 1$ must satisfy $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $\frac{h_1(z+c)}{h_1(z)} = i$, $\frac{h_2(z+c)}{h_2(z)} = -i$ and $h_1(z)h_2(z) = 1$. Later, Liu, Cao and Cao [17] in 2012 studied the existence of solutions for some complex difference equations and obtained

Theorem D. (see [17, Theorem 1.1]). *The transcendental entire solutions with finite order of the equation* $f(z)^2 + f(z+c)^2 = 1$ *must satisfy* $f(z) = \sin(Az + B)$*, where* B *is a constant and* $A = \frac{(4k+1)\pi}{2c}$*,* k *is an integer.*

Theorem E. (see [17, Theorem 1.3]). *The transcendental entire solutions with finite order of*

$$f'(z)^2 + f(z+c)^2 = 1,$$

must satisfy $f(z) = \sin(z \pm Bi)$*, where* B *is a constant and* $c = 2k\pi$ *or* $c = (2k+1)\pi$*,* k *is an integer.*

In 2019, Liu and Gao [20] further studied the entire solutions of second order differential and difference equation when $f'(z)$ is replaced by $f''(z)$ in Theorem E and obtained

Theorem F. (see [20, Theorem 2.1]). *Suppose that* f *is a transcendental entire solution with finite order of the complex differential-difference equation*

$$f''(z)^2 + f(z+c)^2 = Q(z),$$

then $Q(z) = c_1c_2$ *is a constant, and* $f(z)$ *satisfies*

$$f(z) = \frac{c_1e^{az+b} + c_2e^{-az-b}}{2a^2},$$

where $a, b \in \mathbb{C}$*, and* $a^4 = 1, c = \frac{\log(-ia^2)+2k\pi i}{a}$*,* $k \in \mathbb{Z}$*.*

Let us recall some conclusions on the Fermat type equations in several complex variables. Hereinafter, let $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2), w = (w_1, w_2)$. For the equation $f^2 + g^2 = 1$ in $\mathbb{C}^2$, Li [13] showed that meromorphic solutions f, g must be constant if and only if $\frac{\partial f}{\partial z_2}$ and $\frac{\partial g}{\partial z_1}$ have the same zeros. When $f = \frac{\partial u}{\partial z_1}$ and $g = \frac{\partial u}{\partial z_2}$ in $f^2 + g^2 = 1$, then any entire solutions of the equation $(\frac{\partial u}{\partial z_1})^2 + (\frac{\partial u}{\partial z_2})^2 = 1$ in $\mathbb{C}^2$ are necessarily linear ([11]), which was originally investigated by [14, 26].

Recently, Xu and Cao [34] investigated the existence of the entire and meromorphic solutions for some Fermat-type partial differential-difference equations and obtained the following theorems.

Theorem G. (see [34, Theorem 1.1]). *Let* $c = (c_1, c_2)$ *be a constant in* $\mathbb{C}^2$*. Then the Fermat-type partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^m + f(z_1 + c_1, z_2 + c_2)^n = 1.$$

doesn't have any transcendental entire solution with finite order, where m and n are two distinct positive integers.

Remark 1.2. *In fact, the equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + 1, z_2 + 1) = 1. \tag{1.3}$$

admits a finite order transcendental entire solution. For example, let

$$f(z)=\frac{5}{4}-\frac{1}{4}z_1^2+\frac{1}{2}(z_1-1)z_2+(z_1-1)e^{2\pi i z_2}-\left(\frac{1}{2}(z_2-1)+e^{2\pi i z_2}\right)^2,$$

then $f(z)$ is a finite order transcendental entire solution of Eq (1.3).

Theorem H. (see [34, Theorem 1.2]). *Let $c=(c_1,c_2)$ be a constant in $\mathbb{C}^2$. Then any transcendental entire solutions with finite order of the partial differential-difference equation*

$$\left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2+f(z_1+c_1,z_2+c_2)^2=1.$$

has the form of $f(z_1,z_2)=\sin(Az_1+B)$, where A is a constant on $\mathbb{C}$ satisfying $Ae^{iAc_1}=1$, and B is a constant on $\mathbb{C}$; in the special case whenever $c_1=0$, we have $f(z_1,z_2)=\sin(z_1+B)$.

In view of Theorem G and Remark 1.2, one question can be raised as follows.

Question 1.1. *How to deform the equation can guarantee that the conclusion of Theorem G holds under the condition $m\neq n$?*

The forms of equations in Theorem F and Theorem G prompts us to consider the following problems.

Question 1.2. *What can be said about the solution of equation if $\frac{\partial f(z_1,z_2)}{\partial z_1}$ is replaced by*

$$\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}.$$

in Theorem G?

Question 1.3. *What can be said about the existence and the forms of the entire solution of the equation when $\frac{\partial f(z_1,z_2)}{\partial z_1}$ is replaced by*

$$\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\quad or\quad \frac{\partial^2 f(z_1,z_2)}{\partial z_1\partial z_2}$$

in Theorem H?

Motivated by Questions 1.1–1.3, we investigate the existence and the forms of solutions for some second order partial differential-difference equations, by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables (see [3,12]). We give some existence theorems and the forms of entire solutions for some partial differential-difference equations, and also list some examples. Our results are some generalizations of the previous theorems given by Xu and Cao, Liu, Cao and Cao [17,34].

The first theorem is as follows.

Theorem 1.1. *If $c=(c_1,c_2)\in\mathbb{C}^2$, and l,k be two distinct positive integers, then the Fermat-type partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^l+f(z_1+c_1,z_2+c_2)^k=1. \tag{1.4}$$

does not have any transcendental entire solution with finite order.

Remark 1.3. *In fact, on the basis of the proof of Theorem 1.1, it is easily to get that the conclusions of Theorem 1.1 still hold if* $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}$ *or* $\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}$ *is replaced by* $\frac{\partial^2 f(z_1,z_2)}{\partial z_1 \partial z_2}$.

Next, we proceed to study the existence and forms of entire solutions of Eq (1.4) for $l = k = 2$.

Theorem 1.2. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_2 \neq 0$. *If the second order Fermat-type partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1. \tag{1.5}$$

admits a transcendental entire solution with finite order $f(z_1, z_2)$, *then* $f(z_1, z_2)$ *has the following form*

$$f(z_1,z_2) = \eta \frac{e^{i(a_1z_1+a_2z_2+B)} + e^{-i(a_1z_1+a_2z_2+B)}}{2},$$

where $\eta, c_1, c_2, a_1, a_2, B$ *are constants in* $\mathbb{C}$, *and satisfy one of the following cases*

(*i*) $L(c) = 2k\pi + \frac{1}{2}\pi, a_1^2 = 1$, *and* $\eta = -1$, *where* $L(c) := a_1c_1 + a_2c_2$, *here and below* k *is a integer;*

(*ii*) $L(c) = 2k\pi - \frac{1}{2}\pi, a_1^2 = -1$, *and* $\eta = 1$.

Two examples are given to explain the existence of solutions for Eq (1.5).

Example 1.1. *Let* $a = (a_1, a_2) = (1, -1)$, $c = (c_1, c_2) = (\pi, \frac{1}{2}\pi)$, *and* $B \in \mathbb{C}$. *Thus, the function*

$$f(z_1,z_2) = -\frac{e^{i(z_1-z_2+B)} + e^{-i(z_1-z_2+B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1 + \pi, z_2 + \frac{1}{2}\pi)^2 = 1.$$

Example 1.2. *Let* $a = (a_1, a_2) = (i, 1)$, $c = (c_1, c_2) = (\pi i, \frac{1}{2}\pi)$, *and* $B \in \mathbb{C}$. *Thus, the function*

$$f(z_1,z_2) = \frac{e^{i(iz_1+z_2+B)} + e^{-i(iz_1+z_2+B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\right)^2 + f(z_1 + 2\pi i, z_2 - \frac{1}{2}\pi)^2 = 1.$$

Corollary 1.1. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_1 \neq 0$. *If the second order Fermat-type partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1. \tag{1.6}$$

admits a transcendental entire solution with finite order $f(z_1, z_2)$, *then* $f(z_1, z_2)$ *has the following form*

$$f(z_1,z_2) = \eta \frac{e^{i(a_1z_1+a_2z_2+B)} + e^{-i(a_1z_1+a_2z_2+B)}}{2},$$

where $\eta, c_1, c_2, a_1, a_2, B$ *are constants in* $\mathbb{C}$*, and satisfy one of the following cases*

(*i*) $L(c) = 2k\pi + \frac{1}{2}\pi, a_2^2 = 1$, *and* $\eta = -1$, *where* $L(c) := a_1c_1 + a_2c_2$*;*

(*ii*) $L(c) = 2k\pi - \frac{1}{2}\pi, a_2^2 = -1$, *and* $\eta = 1$.

Theorem 1.3. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_1c_2 \neq 0$*. If the second order Fermat-type partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1. \tag{1.7}$$

admits a transcendental entire solution with finite order $f(z_1, z_2)$*, then* $f(z_1, z_2)$ *has the following form*

$$f(z_1, z_2) = \eta \frac{e^{i(a_1z_1+a_2z_2+B)} + e^{-i(a_1z_1+a_2z_2+B)}}{2},$$

where $\eta, c_1, c_2, a_1, a_2, B$ *are constants in* $\mathbb{C}$*, and satisfy one of the following cases*

(*i*) $L(c) = 2k\pi + \frac{1}{2}\pi, a_1a_2 = 1$, *and* $\eta = -1$*;*

(*ii*) $L(c) = 2k\pi - \frac{1}{2}\pi, a_1a_2 = -1$, *and* $\eta = 1$.

Theorem 1.4. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_1 \neq 0$, $c_2 \neq 0$*. If the second order Fermat-type partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1. \tag{1.8}$$

admits a transcendental entire solution with finite order $f(z_1, z_2)$*, then* $f(z_1, z_2)$ *has the following form*

$$f(z_1, z_2) = \eta \frac{e^{i(a_1z_1+a_2z_2+B)} + e^{-i(a_1z_1+a_2z_2+B)}}{2},$$

where $\eta, c_1, c_2, a_1, a_2, B$ *are constants in* $\mathbb{C}$*, and satisfy one of the following cases*

(*i*) $L(c) = 2k\pi + \frac{1}{2}\pi, a_1^2 + a_2^2 = 1$, *and* $\eta = -1$*;*

(*ii*) $L(c) = 2k\pi - \frac{1}{2}\pi, a_1^2 + a_2^2 = -1$, *and* $\eta = 1$.

Theorem 1.5. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$ *and* $c_1 \neq 0$, $c_2 \neq 0$*. If the second order Fermat-type partial differential-difference equation*

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1. \tag{1.9}$$

admits a transcendental entire solution with finite order $f(z_1, z_2)$*, then* $f(z_1, z_2)$ *has the following form*

$$f(z_1, z_2) = \eta \frac{e^{i(a_1z_1+a_2z_2+B)} + e^{-i(a_1z_1+a_2z_2+B)}}{2},$$

where $\eta, c_1, c_2, a_1, a_2, B$ *are constants in* $\mathbb{C}$*, and satisfy one of the following cases*

(*i*) $L(c) = 2k\pi + \frac{1}{2}\pi, a_1(a_1 + a_2) = 1$, *and* $\eta = -1$*;*

(*ii*) $L(c) = 2k\pi - \frac{1}{2}\pi, a_1(a_1 + a_2) = -1$, *and* $\eta = 1$.

Remark 1.4. *From Theorems 1.1–1.5, the Eq* (1.4) *has no nonconstant entire solution for the case* $l = k > 2$*, and has no nonconstant meromorphic solutions for the case* $l = k > 3$*. Hence based on Theorems 1.1–1.5, an open question is: What will happen for the meromorphic solutions of the Fermat type partial difference-differential equation*

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right)^3 + f(z_1 + c_1, z_2 + c_2)^3 = 1.$$

in $\mathbb{C}^2$*?*

Similar to Examples 1.1 and 1.2, it is easy to give some solutions for Eqs (1.7), (1.8) and (1.9).

Example 1.3. *Let* $a = (a_1, a_2) = (i, -i)$*,* $c = (c_1, c_2) = (-\pi i, -\frac{1}{2}\pi i)$*, and* $B \in \mathbb{C}$*. Thus, the function*

$$f(z_1, z_2) = -\frac{e^{-(z_1 - z_2 + B)} + e^{(z_1 - z_2 + B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 - \pi i, z_2 - \frac{1}{2}\pi i)^2 = 1.$$

Example 1.4. *Let* $a = (a_1, a_2) = (1, -1)$*,* $c = (c_1, c_2) = (\frac{1}{2}\pi, \pi)$*, and* $B \in \mathbb{C}$*. Thus, the function*

$$f(z_1, z_2) = \frac{e^{i(z_1 - z_2 + B)} + e^{-i(z_1 - z_2 + B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 + \frac{1}{2}\pi, z_2 + \pi)^2 = 1.$$

Example 1.5. *Let* $a = (a_1, a_2) = (\sqrt{2}, i)$*,* $c = (c_1, c_2) = (\frac{\pi}{\sqrt{2}}, \frac{\pi i}{2})$*, and* $B \in \mathbb{C}$*. Thus, the function*

$$f(z_1, z_2) = -\frac{e^{i(\sqrt{2}z_1 + iz_2 + B)} + e^{-i(\sqrt{2}z_1 + iz_2 + B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right)^2 + f(z_1 + \frac{\pi}{\sqrt{2}}, z_2 + \frac{\pi i}{2})^2 = 1.$$

Example 1.6. *Let* $a = (a_1, a_2) = (\sqrt{2}i, 1)$*,* $c = (c_1, c_2) = (\frac{\sqrt{2}}{2}\pi i, \frac{\pi}{2})$*, and* $B \in \mathbb{C}$*. Thus, the function*

$$f(z_1, z_2) = \frac{e^{i(\sqrt{2}iz_1 + z_2 + B)} + e^{-i(\sqrt{2}iz_1 + z_2 + B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right)^2 + f(z_1 + \frac{\sqrt{2}}{2}\pi i, z_2 + \frac{\pi}{2})^2 = 1.$$

Example 1.7. *Let* $a = (a_1, a_2) = (i, -2i)$*,* $c = (c_1, c_2) = (-\pi i, -\frac{\pi i}{4})$*, and* $B \in \mathbb{C}$*. Thus, the function*

$$f(z_1, z_2) = -\frac{e^{-(z_1-2z_2)+B} + e^{z_1-2z_2-B}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 - \pi i, z_2 - \frac{\pi i}{4})^2 = 1.$$

Example 1.8. *Let* $a = (a_1, a_2) = (1, -2)$*,* $c = (c_1, c_2) = (\frac{\pi}{2}, \frac{\pi}{2})$*, and* $B \in \mathbb{C}$*. Thus, the function*

$$f(z_1, z_2) = \frac{e^{i(z_1-2z_2+B)} + e^{-i(z_1-2z_2+B)}}{2}.$$

satisfies the following equation

$$\left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}\right)^2 + f(z_1 + \frac{\pi}{2}, z_2 + \frac{\pi}{2})^2 = 1.$$

2. Proof of Theorem 1.1

To prove Theorem 1.1, the following lemmas should be required.

Lemma 2.1. *([2]). Let* f *be a nonconstant meromorphic function on* $\mathbb{C}^n$ *and let* $I = (i_1, \ldots, i_n)$ *be a multi-index with length* $|I| = \sum_{j=1}^n i_j$*. Assume that* $T(r_0, f) \geq e$ *for some* r_0*. Then*

$$m\left(r, \frac{\partial^I f}{f}\right) = S(r, f),$$

holds for all $r \geq r_0$ *outside a set* $E \subset (0, +\infty)$ *of finite logarithmic measure* $\int_E \frac{dt}{t} < \infty$*, where* $\partial^I f = \frac{\partial^{|I|} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}$*.*

Lemma 2.2. *([3, 12]). Let* f *be a nonconstant meromorphic function on* $\mathbb{C}^n$ *such that* $f(0) \neq 0, \infty$*, and let* $\varepsilon > 0$*. If* $\sigma_2(f) := \sigma < 1$*, then*

$$m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T_f(r)}{r^{1-\sigma-\varepsilon}}\right),$$

holds for all $r \geq r_0$ *outside a set* $E \subset (0, +\infty)$ *of finite logarithmic measure* $\int_E \frac{dt}{t} < \infty$*, where*

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ T_f(r)}{\log r}.$$

Remark 2.1. *In view of Lemma 2.2, one can get that if* f *is a nonconstant meromorphic function with finite order on* $\mathbb{C}^n$ *such that* $f(0) \neq 0, \infty$*, for* $c \in \mathbb{C}^n$*, then*

$$m\left(r, \frac{f(z)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f),$$

where $S(r, f)$ *denotes any quantity satisfying* $S(r, f) = o(T(r, f))$ *as* r *sufficiently large outside possibly a set* E *of* r *with finite Lebesgue measure.*

The proof of Theorem 1.1: The proof of Theorem 1.1 is very similar to the argument as in Ref. [35]. Assume that f is a finite order transcendental entire solution of Eq (1.4), then $f(z_1 + c_1, z_2 + c_2)$ is transcendental. Thus, in view of (1.4), $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}$ is also transcendental. Here, two cases will be considered below.

Case 1. $k > l$. Thus, it follows from Lemma 2.2 that

$$m\left(r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2)}\right) = S(r, f). \tag{2.1}$$

holds for all $r > 0$ outside of a possible exceptional set $E \subset [1, +\infty)$ with finite logarithmic measure $\int_E \frac{dt}{t} < \infty$. Thus, by (2.1) and combining with the properties of $m(r, f)$, we can deduce that

$$\begin{aligned} T(r, f(z_1, z_2)) = m(r, f(z_1, z_2)) &= m\left(r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2)} f(z_1 + c_1, z_2 + c_2)\right) \\ &\leq m\left(r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2)}\right) + m(r, f(z_1 + c_1, z_2 + c_2)) + \log 2 \\ &= m(r, f(z_1 + c_1, z_2 + c_2)) + \log 2 + S(r, f) \\ &= T(r, f(z_1 + c_1, z_2 + c_2)) + \log 2 + S(r, f), \end{aligned} \tag{2.2}$$

for all $r \notin E$. In view of (2.2), by applying Lemma 2.1 and the Mokhon'ko theorem in several complex variables [8, Theorem 3.4], we have

$$\begin{aligned} kT(r, f(z_1, z_2)) &\leq kT(r, f(z_1 + c_1, z_2 + c_2)) + S(r, f) \\ &= T(r, f(z_1 + c_1, z_2 + c_2)^k) + S(r, f) \\ &= T\left(r, \left(\frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right)^l - 1\right) + S(r, f) \\ &= lT\left(r, \frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right) + S(r, f) \\ &= lm\left(r, \frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right) + S(r, f) \\ &\leq l\left(m\left(r, \frac{\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}}{f_1(z_1, z_2)}\right) + m(r, f(z_1, z_2))\right) + \log 2 + S(r, f) \\ &= lT(r, f(z_1, z_2)) + \log 2 + S(r, f), \end{aligned} \tag{2.3}$$

for all $r \notin E$. This means

$$(k - l)T(r, f(z_1, z_2)) \leq \log 2 + S(r, f), \quad r \notin E. \tag{2.4}$$

Since f is transcendental, so this is a contradiction.

Case 2. $l > k \geq 2$. Then $\frac{1}{l} + \frac{1}{k} \leq \frac{2}{k} < 1$. Thus, it follows that $l > \frac{k}{k-1}$. In view of the Nevanlinna second fundamental theorem, Lemma 2.2, and Eq (1.4), we have

$$(l - 1)T\left(r, \frac{\partial^2 f(z_1, z_2)}{\partial z_1^2} + \frac{\partial^2 f(z_1, z_2)}{\partial z_2^2}\right)$$

$$
\begin{aligned}
\leq&\overline{N}\left(r, \frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)+\sum_{q=1}^{l}\overline{N}\left(r, \frac{1}{\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}-w_q}\right)\\
&+S\left(r, \frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)\\
\leq&\overline{N}\left(r, \frac{1}{\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^l-1}\right)+S\left(r, \frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)\\
\leq&\overline{N}\left(r, \frac{1}{f(z_1+c_1,z_2+c_2)}\right)+S(r,f)\\
\leq&T(r, f(z_1+c_1,z_2+c_2))+S(r,f), \qquad (2.5)
\end{aligned}
$$

where w_q is a root of $w^l-1=0$.

On the other hand, in view of Eq (1.4), and by applying the Mokhon'ko theorem in several complex variables [8, Theorem 3.4], it yields that

$$
\begin{aligned}
lT(r, f(z_1+c_1,z_2+c_2)) &= T(r, f(z_1+c_1,z_2+c_2)^l)+S(r,f)\\
&= T\left(r, \left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^k-1\right)+S(r,f)\\
&= kT\left(r, \frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)+S(r,f). \qquad (2.6)
\end{aligned}
$$

In view of (2.5)–(2.6) and $l>\frac{k}{k-1}$, it follows

$$
(l-\frac{k}{k-1})T(r, f(z_1+c_1,z_2+c_2))\leq S(r,f).
$$

This is impossible since f is transcendental.

Case 3. $l>k=1$. Then it follows

$$
\left(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}\right)^l+f(z_1+c_1,z_2+c_2)=1. \qquad (2.7)
$$

Differentiating this equation for z_1, z_2, respectively, we have

$$
l\left(\frac{\partial^2 f}{\partial z_1^2}+\frac{\partial^2 f}{\partial z_2^2}\right)^{l-1}Q(f)=-\left(\frac{\partial f(z+c)}{\partial z_1}+\frac{\partial f(z+c)}{\partial z_2}\right),
$$

where $Q(f)$ is a differential polynomial in $\frac{\partial f}{\partial z_1}$ and $\frac{\partial f}{\partial z_2}$ of the form

$$
Q(f)=\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\left(\frac{\partial^2 f}{\partial z_1^2}+\frac{\partial^2 f}{\partial z_2^2}\right).
$$

Divide both sides of the above equation by f^l, it follows that

$$l\left(\frac{\frac{\partial^2 f}{\partial z_1^2}+\frac{\partial^2 f}{\partial z_2^2}}{f}\right)^{l-1}\frac{Q(f)}{f}=-\frac{\frac{\partial f(z+c)}{\partial z_1}+\frac{\partial f(z+c)}{\partial z_2}}{f}\frac{1}{f^{l-1}}. \tag{2.8}$$

By Lemmas 3.1 and 2.2, we have

$$m\left(r,\frac{\frac{\partial^2 f}{\partial z_1^2}+\frac{\partial^2 f}{\partial z_2^2}}{f}\right)=S(r,f),\ \ m\left(r,\frac{Q(f)}{f}\right)=S(r,f), \tag{2.9}$$

and

$$m\left(r,\frac{\frac{\partial f(z+c)}{\partial z_1}+\frac{\partial f(z+c)}{\partial z_2}}{f}\right)=S(r,f). \tag{2.10}$$

Thus, in view of (2.8)–(2.10), it follows

$$m\left(r,\frac{1}{f^{l-1}}\right)=S(r,f),$$

which is a contradiction with the assumption of f being transcendental and $l>1$.

Therefore, this completes the proof of Theorem 1.1.

3. Proofs of Theorems 1.2–1.5

The following lemmas play the key roles in proving Theorems 1.2–1.5.

Lemma 3.1. *([9, Lemma 3.1]). Let $f_j(\not\equiv 0)$, $j=1,2,3$, be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, and $f_1+f_2+f_3=1$, and such that*

$$\sum_{j=1}^{3}\left\{N_2(r,\frac{1}{f_j})+2\overline{N}(r,f_j)\right\}<\lambda T(r,f_1)+O(\log^+ T(r,f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda<1$ is a positive number. Then either $f_2=1$ or $f_3=1$.

Remark 3.1. *Here, $N_2(r,\frac{1}{f})$ is the counting function of the zeros of f in $|z|\le r$, where the simple zero is counted once, and the multiple zero is counted twice.*

Lemma 3.2. *([25,27]). For an entire function F on $\mathbb{C}^n$, $F(0)\neq 0$ and put $\rho(n_F)=\rho<\infty$. Then there exist a canonical function f_F and a function $g_F\in\mathbb{C}^n$ such that $F(z)=f_F(z)e^{g_F(z)}$. For the special case $n=1$, f_F is the canonical product of Weierstrass.*

Remark 3.2. *Here, denote $\rho(n_F)$ to be the order of the counting function of zeros of F.*

Lemma 3.3. *([22]). If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(a) the internal function h is a polynomial and the external function g is of finite order; or else

(b) the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.

3.1. The Proof of Theorem 1.2

Proof. Suppose that f is a finite order transcendental entire solutions of Eq (1.5), then it follows that $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}$ is transcendental. Otherwise, $f(z_1+c_1, z_2+c_2)$ is not transcendental, this is a contradiction with the condition. Firstly, Eq (1.5) can be represented as the following form

$$\left[\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + if(z_1+c_1,z_2+c_2)\right]\left[\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2} - if(z_1+c_1,z_2+c_2)\right] = 1. \tag{3.1}$$

Since $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}$ and $f(z_1+c_1, z_2+c_2)$ are transcendental, then by Lemma 3.2 and Lemma 3.3, from (3.1), there exists a nonconstant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\begin{cases} \dfrac{\partial^2 f(z_1,z_2)}{\partial z_1^2} + if(z_1+c_1,z_2+c_2) = e^{ip(z_1,z_2)}, \\ \dfrac{\partial^2 f(z_1,z_2)}{\partial z_1^2} - if(z_1+c_1,z_2+c_2) = e^{-ip(z_1,z_2)}. \end{cases} \tag{3.2}$$

Thus, in view of (3.2), it yields

$$\begin{cases} \dfrac{\partial^2 f(z_1,z_2)}{\partial z_1^2} = \dfrac{e^{ip(z_1,z_2)} + e^{-ip(z_1,z_2)}}{2}, \\ f(z_1+c_1,z_2+c_2) = \dfrac{e^{ip(z_1,z_2)} - e^{-ip(z_1,z_2)}}{2i}. \end{cases} \tag{3.3}$$

In view of (3.3), we have

$$\begin{aligned} & e^{ip(z_1+c_1,z_2+c_2)} + e^{-ip(z_1+c_1,z_2+c_2)} \\ =& \left(\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2} + i\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2\right)e^{ip(z_1,z_2)} \\ &+ \left(\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2} - i\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2\right)e^{-ip(z_1,z_2)}. \end{aligned} \tag{3.4}$$

Now, we claim that $\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2} - i\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2 \not\equiv 0$. If $\frac{\partial p(z_1,z_2)}{\partial z_1} \equiv 0$, then Eq (3.4) becomes $e^{ip(z_1+c_1,z_2+c_2)} + e^{-ip(z_1+c_1,z_2+c_2)} \equiv 0$, this is impossible since $p(z)$ is a nonconstant polynomial. If $\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2} - i\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2 \equiv 0$ and $\frac{\partial p(z_1,z_2)}{\partial z_1} \not\equiv 0$, then $\frac{\partial u}{\partial z_1} = iu^2$, where $u = \frac{\partial p(z_1,z_2)}{\partial z_1}$. Solving this equation, we have $-\frac{1}{u} = iz_1 + \varphi_1(z_2)$, that is, $\frac{\partial p(z_1,z_2)}{\partial z_1} = u = -\frac{1}{iz_1+\varphi_1(z_2)}$, where $\varphi_1(z_2)$ is a polynomial in z_2. Thus, it follows that $p(z_1,z_2) = i\log[iz_1+\varphi_1(z_2)]+\varphi_2(z_2)$, where $\varphi_2(z_2)$ is a polynomial in z_2. This is a contradiction with the assumption of $p(z_1,z_2)$ being a nonconstant polynomial. Hence, $\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2} - i\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2 \not\equiv 0$. Similarly, we have $\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2} + i\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2 \not\equiv 0$. Thus, (3.4) becomes

$$\left(\frac{\partial^2 p}{\partial z_1^2} + i\left(\frac{\partial p}{\partial z_1}\right)^2\right)e^{i(p(z+c)+p(z))} + \left(\frac{\partial^2 p}{\partial z_1^2} - i\left(\frac{\partial p}{\partial z_1}\right)^2\right)e^{i(p(z+c)-p(z))} - e^{2ip(z+c)} \equiv 1. \tag{3.5}$$

Since $p(z)$ is a nonconstant polynomial, we have that $e^{2ip(z+c)}$ is not a constant, and

$$N\left(r, (\frac{\partial^2 p}{\partial z_1^2} + i(\frac{\partial p}{\partial z_1})^2)e^{i(p(z+c)+p(z))}\right) = N\left(r, (\frac{\partial^2 p}{\partial z_1^2} - i(\frac{\partial p}{\partial z_1})^2)e^{i(p(z+c)-p(z))}\right)$$
$$= N(r, -e^{2ip(z+c)}) = 0,$$

and

$$N\left(r, \frac{1}{(\frac{\partial^2 p}{\partial z_1^2} + i(\frac{\partial p}{\partial z_1})^2)e^{i(p(z+c)+p(z))}}\right) = O(\log r) = S(r, e^{2ip(z+c)}),$$
$$N\left(r, \frac{1}{(\frac{\partial^2 p}{\partial z_1^2} - i(\frac{\partial p}{\partial z_1})^2)e^{i(p(z+c)-p(z))}}\right) = O(\log r) = S(r, e^{2ip(z+c)}).$$

Thus, by Lemma 3.1, it yields

$$\left(\frac{\partial^2 p}{\partial z_1^2} - i\left(\frac{\partial p}{\partial z_1}\right)^2\right)e^{i(p(z+c)-p(z))} \equiv 1. \tag{3.6}$$

In view of (3.5) and (3.6), it follows

$$\left(\frac{\partial^2 p}{\partial z_1^2} + i\left(\frac{\partial p}{\partial z_1}\right)^2\right)e^{i(p(z)-p(z+c))} \equiv 1. \tag{3.7}$$

Here, we claim that $p(z) = L(z) + B$, where $L(z)$ is a linear function as the form $L(z) = a_1z_1 + a_2z_2$, B is a constant in $\mathbb{C}$. In fact, since $p(z)$ is a nonconstant polynomial, and in view of (3.6) and (3.7), we conclude that $p(z) = L(z) + H(s) + B$, where $H(s)$ is a polynomial in s, $s := c_2z_1 - c_1z_2$. Thus, it follows from (3.6) that $H''c_2 - i(H'c_2 + a_1)^2$ must be a constant in $\mathbb{C}$. By combining with $c_2 \neq 0$, then we have $\deg_s H \leq 1$. Thus, $L(z) + H(s) + B$ is still a linear form of z_1, z_2. Hence, we have $p(z) = L(z) + B$ and $L(z) = a_1z_1 + a_2z_2$. Thus, it follows

$$\frac{\partial^2 p}{\partial z_1^2} \equiv 0, \quad -ia_1^2e^{iL(c)} = 1, \;\; ia_1^2e^{-iL(c)} = 1. \tag{3.8}$$

Thus, it follows from (3.8) that

$$a_1^2 = 1, \;\; L(c) = 2k\pi + \frac{1}{2}\pi, \quad or \quad a_1^2 = -1, \;\; L(c) = 2k\pi - \frac{1}{2}\pi. \tag{3.9}$$

By observing the second equation in (3.3), we can define the form of $f(z_1, z_2)$ as

$$f(z_1, z_2) = \frac{e^{i(L(z)-L(c)+B)} - e^{-i(L(z)-L(c)+B)}}{2i}. \tag{3.10}$$

By combining with (3.9) and (3.10), it yields

$$f(z_1, z_2) = \eta\frac{e^{i(L(z)+B)} + e^{-i(L(z)+B)}}{2},$$

where η, a_1, a_2, B are constants in $\mathbb{C}$ satisfying one of the following conditions

(i) $\eta = -1$, $a_1^2 = 1$, and $L(c) = 2k\pi + \frac{1}{2}\pi$;

(ii) $\eta = 1$, $a_1^2 = -1$, and $L(c) = 2k\pi - \frac{1}{2}\pi$.

Therefore, this completes the proof of Theorem 1.2. □

3.2. The Proof of Theorem 1.4

Proof. Suppose that f is a finite order transcendental entire solutions of Eq (1.8), then it follows that $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}$ is transcendental. Otherwise, $f(z_1+c_1,z_2+c_2)$ is not transcendental, this is a contradiction with the condition. Firstly, Eq (1.8) can be rewritten as the following form

$$\left[\frac{\partial^2 f(z)}{\partial z_1^2}+\frac{\partial^2 f(z)}{\partial z_2^2}+if(z+c)\right]\left[\frac{\partial^2 f(z)}{\partial z_1^2}+\frac{\partial^2 f(z)}{\partial z_2^2}-if(z+c)\right]=1. \tag{3.11}$$

Since $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 f(z_1,z_2)}{\partial z_2^2}$ and $f(z_1+c_1,z_2+c_2)$ are transcendental, then by Lemma 3.2 and 3.3, it follows from (3.11) that

$$\begin{cases}\dfrac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f(z_1,z_2)}{\partial z_2^2}+if(z_1+c_1,z_2+c_2)=e^{ip(z_1,z_2)},\\ \dfrac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f(z_1,z_2)}{\partial z_2^2}-if(z_1+c_1,z_2+c_2)=e^{-ip(z_1,z_2)},\end{cases} \tag{3.12}$$

where $p(z)$ is a nonconstant polynomial in $\mathbb{C}^2$. Thus, in view of (3.12), it yields

$$\begin{cases}\dfrac{\partial^2 f(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f(z_1,z_2)}{\partial z_2^2}=\dfrac{e^{ip(z_1,z_2)}+e^{-ip(z_1,z_2)}}{2},\\ f(z_1+c_1,z_2+c_2)=\dfrac{e^{ip(z_1,z_2)}-e^{-ip(z_1,z_2)}}{2i}.\end{cases} \tag{3.13}$$

In view of (3.13), we have

$$\begin{aligned}&e^{ip(z_1+c_1,z_2+c_2)}+e^{-ip(z_1+c_1,z_2+c_2)}\\ =&(A_1+iA_2)\,e^{ip(z_1,z_2)}+(A_1-iA_2)\,e^{-ip(z_1,z_2)}.\end{aligned} \tag{3.14}$$

where $A_1=\frac{\partial^2 p(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 p(z_1,z_2)}{\partial z_2^2}$ and $A_2=\left(\frac{\partial p(z_1,z_2)}{\partial z_1}\right)^2+\left(\frac{\partial p(z_1,z_2)}{\partial z_2}\right)^2$.

If $A_1+iA_2\equiv 0$, then it follows that

$$e^{2ip(z_1+c_1,z_2+c_2)}=(A_1-iA_2)\,e^{i(p(z_1+c_1,z_2+c_2)-p(z_1,z_2))}-1. \tag{3.15}$$

By making use of the Nevanlinna second fundamental theorem and (3.15), it follows that

$$\begin{aligned}&T\left(r,e^{2ip(z_1+c_1,z_2+c_2)}\right)\\ \le&N\left(r,\frac{1}{e^{2ip(z_1+c_1,z_2+c_2)}}\right)+N\left(r,\frac{1}{e^{2ip(z_1+c_1,z_2+c_2)}+1}\right)+S\left(r,e^{2ip(z_1+c_1,z_2+c_2)}\right)\\ \le&N\left(r,\frac{1}{(A_1-iA_2)\,e^{i(p(z_1+c_1,z_2+c_2)-p(z_1,z_2))}}\right)+S\left(r,e^{2ip(z_1+c_1,z_2+c_2)}\right)\\ \le&N\left(r,\frac{1}{A_1-iA_2}\right)+S\left(r,e^{2ip(z_1+c_1,z_2+c_2)}\right).\end{aligned} \tag{3.16}$$

If $A_1-iA_2\equiv 0$, in view of (3.15), it yields that $e^{2ip(z_1+c_1,z_2+c_2)}=-1$, a contradiction. If $A_1-iA_2\not\equiv 0$, then from (3.16), it leads to

$$T\left(r,e^{2ip(z_1+c_1,z_2+c_2)}\right)\le T(r,A_1-iA_2)+S\left(r,e^{2ip(z_1+c_1,z_2+c_2)}\right)$$

$$\leq O(T(r,p)) + S\left(r, e^{2ip(z_1+c_1,z_2+c_2)}\right),$$

outside possibly a set of finite Lebesgue measure. This is a contradiction with the fact

$$\lim_{r\to+\infty} \frac{T(r,e^{2ip})}{T(r,p)} = +\infty,$$

for $p(z)$ is a nonconstant polynomial. Thus, it follows that $A_1+iA_2 \not\equiv 0$. Similarly, we have $A_1-iA_2 \not\equiv 0$. Thus, (3.14) becomes

$$(A_1 + iA_2)e^{i(p(z+c)+p(z))} + (A_1 - iA_2)e^{i(p(z+c)-p(z))} - e^{2ip(z+c)} \equiv 1. \tag{3.17}$$

Since $p(z)$ is a nonconstant polynomial, we have that $e^{2ip(z+c)}$ is not a constant, and

$$N\left(r, (A_1 + iA_2)e^{i(p(z+c)+p(z))}\right) = N\left(r, (A_1 - iA_2)e^{i(p(z+c)-p(z))}\right)$$
$$= N(r, -e^{2ip(z+c)}) = 0,$$

and

$$N\left(r, \frac{1}{(A_1 + iA_2)e^{i(p(z+c)+p(z))}}\right) = O(\log r) = S(r, e^{2ip(z+c)}),$$
$$N\left(r, \frac{1}{(A_1 - iA_2)e^{i(p(z+c)-p(z))}}\right) = O(\log r) = S(r, e^{2ip(z+c)}).$$

Thus, by Lemma 3.1, it yields

$$(A_1 - iA_2)\, e^{i(p(z+c)-p(z))} \equiv 1. \tag{3.18}$$

In view of (3.17) and (3.18), it follows

$$(A_1 + iA_2)\, e^{i(p(z)-p(z+c))} \equiv 1. \tag{3.19}$$

Since $p(z)$ is a nonconstant polynomial, in view of (3.18) and (3.19), similar to the argument as in the proof of Theorem 1.2, we conclude that $p(z) = L(z) + B$, where $L(z)$ is a linear function as the form $L(z) = a_1z_1 + a_2z_2$, B is a constant in $\mathbb{C}$. Thus, it follows

$$\frac{\partial^2 p}{\partial z_1^2} \equiv 0, \quad \frac{\partial^2 p}{\partial z_2^2} \equiv 0 \quad -i(a_1^2 + a_2^2)e^{iL(c)} = 1, \quad i(a_1^2 + a_2^2)e^{-iL(c)} = 1. \tag{3.20}$$

Thus, it follows from (3.20) that

$$(a_1^2 + a_2^2) = 1, \quad L(c) = 2k\pi + \frac{1}{2}\pi, \quad or \quad (a_1^2 + a_2^2) = -1, \quad L(c) = 2k\pi - \frac{1}{2}\pi. \tag{3.21}$$

By observing the second equation in (3.13), we can define the form of $f(z_1, z_2)$ as

$$f(z_1, z_2) = \frac{e^{i(L(z)-L(c)+B)} - e^{-i(L(z)-L(c)+B)}}{2i}. \tag{3.22}$$

By combining with (3.21) and (3.22), it yields

$$f(z_1, z_2) = \eta \frac{e^{i(L(z)+B)} + e^{-i(L(z)+B)}}{2},$$

where η, a_1, a_2, B are constants in $\mathbb{C}$ satisfying one of the following conditions

(i) $\eta - 1$, $(a_1^2 + a_2^2) = 1$, and $L(c) = 2k\pi + \frac{1}{2}\pi$;

(ii) $\eta = 1$, $(a_1^2 + a_2^2) = -1$, and $L(c) = 2k\pi - \frac{1}{2}\pi$.

Therefore, this completes the proof of Theorem 1.4. □

3.3. Proofs of Theorems 1.3 and 1.5

By using the same argument as in the proof of Theorem 1.4, we can easily prove the conclusions of Theorems 1.3 and 1.5.

4. Conclusions

We can see that Theorem 1.1 is an extension of Theorem G. Meantime, it is also a positive answer to Question 1.1. Moreover, Theorems 1.2–1.5 are the answer to Questions 1.2–1.3. More important, a series of examples show that our results are accurate.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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