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Research article

Note on error bounds for linear complementarity problems involving B^S-matrices

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Abstract: Using the range for the infinity norm of inverse matrix of a strictly diagonally dominant M-matrix, some new error bounds for the linear complementarity problem are obtained when the involved matrix is a B^S-matrix. Theory analysis and numerical examples show that these upper bounds are more accurate than some existing results.

Keywords: error bounds; linear complementarity problems; diagonally dominant matrices; B^S-matrices; B-matrices
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1. Introduction

The linear complementarity problem (LCP) is to find a vector $x \in R^n$ such that

$$(Mx + z)^T x = 0, \qquad Mx + z \geq 0, \qquad x \geq 0,$$

or to show that no such vector x exists, where $M = (m_{ij}) \in R^{n\times n}$ and $z \in R^n$. Many problems such as the contact problem, Nash equilibrium point of a bimatrix game, and the free boundary problem for journal bearing can be posed in the framework of the LCP, see [1–3].

It is well known that the LCP has a unique solution for any $z \in R^n$ if and only if M is a P-matrix [2]. Here, a matrix $M \in R^{n\times n}$ is called a P-matrix if all its principal minors are positive [4]. In 2006, Chen et al. [5] gave the following result for the LCP when M is a P-matrix:

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty \quad for\ any\ x \in R^n,$$

where x^* is the solution of the LCP, $r(x) = min\{x, Mx + z\}$, $D = diag(d_i)$ with $0 \leq d_i \leq 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. When the matrix M for the LCP

belongs to P-matrices or some subclass of P-matrices, various bounds for $\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ are established [6–14].

In 2012, García-Esnaola et al. [9] gave upper bounds for $\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a B^S-matrix as a subclass of P-matrices. Here a matrix $M = (m_{ij}) \in R^{n\times n}$ is called a B^S-matrix [15] if there exists a subset S of the set $N = \{1, 2, \ldots, n\}$, with $2 \leq card(S) \leq n - 2$, such that for all $i, j \in N$, $t \in T(i) \setminus \{i\}$, and $k \in K(j) \setminus \{j\}$,

$$R_i^S > 0, \qquad R_j^{\overline{S}} > 0, \qquad \left(m_{it} - R_i^S\right)\left(m_{jk} - R_j^{\overline{S}}\right) < R_j^S R_i^{\overline{S}},$$

where $R_i^S = \frac{1}{n}\sum_{k\in S} m_{ik}$, $T(i) = \{t \in S | m_{it} > R_i^S\}$ and $K(j) = \{k \in \overline{S} | m_{jk} > R_j^{\overline{S}}\}$ with $\overline{S} = N \setminus \{S\}$.

A square real matrix $M = (m_{ik})_{1\leq i,k\leq n}$ with positive row sums is a B-matrix [4] if all of its off-diagonal elements are bounded above by the corresponding row means, i.e., for all $i = 1, \ldots, n$,

$$\sum_{k=1}^{n} m_{ik} > 0 \quad and \quad \frac{1}{n}\left(\sum_{k=1}^{n} m_{ik}\right) > m_{ij}, \qquad \forall j \neq i.$$

Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix, and let $X = diag(x_1, \cdots, x_n)$ with

$$x_i = \begin{cases} \gamma, & i \in S, \\ 1, & otherwise, \end{cases}$$

such that $\tilde{M} = MX$ is a B-matrix with the form $\tilde{M} = \tilde{B}^+ + \tilde{C}$, where

$$\tilde{B}^+ = (\tilde{b}_{ij}) = \begin{bmatrix} m_{11}x_1 - \tilde{r}_1^+ & \cdots & m_{1n}x_n - \tilde{r}_1^+ \\ \vdots & & \vdots \\ m_{n1}x_1 - \tilde{r}_n^+ & \cdots & m_{nn}x_n - \tilde{r}_n^+ \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} \tilde{r}_1^+ & \cdots & \tilde{r}_1^+ \\ \vdots & & \vdots \\ \tilde{r}_n^+ & \cdots & \tilde{r}_n^+ \end{bmatrix}, \tag{1.1}$$

and $\tilde{r}_i^+ = \max\{0, m_{ij}x_j | j \neq i\}$. Then

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{(n-1)\max\{\gamma, 1\}}{\min\{\tilde{\beta}, \gamma, 1\}}, \tag{1.2}$$

where $\tilde{\beta}_i = \tilde{b}_{ii} - \sum_{j\neq i} |\tilde{b}_{ij}|$, $\tilde{\beta} = \min_{i\in N}\{\tilde{\beta}_i\}$, and

$$(0 <)\gamma \in \left(\max_{j\in N, k\in K(j)\setminus\{j\}} \frac{m_{jk} - R_j^{\overline{S}}}{R_j^S}, \quad \max_{i\in N, t\in T(i)\setminus\{i\}} \frac{R_i^{\overline{S}}}{m_{it} - R_i^S} \right),$$

assuming that if $K(j) \setminus \{j\} = \emptyset (T(i) \setminus \{i\} = \emptyset)$, then max (min) is set to be $-\infty(\infty)$ [9].

In 2018, Gao [14] presented a new bound: Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix. Then

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \sum_{i=1}^{n} \frac{(n-1)\max\{\gamma, 1\}}{\min\{\hat{\beta}_i, x_i\}} \prod_{j=1}^{i-1} \frac{\tilde{b}_{jj}}{\hat{\beta}_j}, \tag{1.3}$$

where $\hat{\beta}_i = \tilde{b}_{ii} - \sum\limits_{j=i+1}^{n} |\tilde{b}_{ij}| l_i(\tilde{B}^+)$, $l_k(\tilde{B}^+) = \max\limits_{k\le i\le n} \left\{ \frac{1}{|\tilde{b}_{ii}|} \sum\limits_{j=k,\neq i}^{n} |\tilde{b}_{ij}| \right\}$, and

$$\prod_{j=1}^{i-1} \frac{\tilde{b}_{jj}}{\hat{\beta}_j} = 1, \quad if\ i = 1.$$

In order to improving the above results, in this paper, we establish some new upper bounds for the condition constant $\max\limits_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a B^S-matrix.

Next, we recall the following definition and lemmas for an $n \times n$ matrix.

Definition 1. [13] *A matrix $M = (m_{ij}) \in C^{n\times n}$ is called a row strictly diagonally dominant matrix if for each $i \in N$, $|m_{ii}| > \sum\limits_{j=1,\neq i}^{n} |m_{ij}|$. A matrix $M = (m_{ij})$ is called a Z-matrix if $m_{ij} \le 0$ for any $i \neq j$, and an M-matrix if M is a Z-matrix with M^{-1} being nonnegative.*

Lemma 1. [9] *Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix. Then there exists a positive diagonal matrix $X = diag(x_1, \cdots, x_n)$ with*

$$x_i = \begin{cases} \gamma, & i \in S, \\ 1, & otherwise, \end{cases}$$

such that $\tilde{M} = MX$ is a B-matrix, where $\gamma > 0$,

$$\max_{j\in N, k\in K(j)\setminus\{j\}} \frac{m_{jk} - R_j^{\overline{S}}}{R_j^S} < \gamma < \max_{i\in N, t\in T(i)\setminus\{i\}} \frac{R_i^{\overline{S}}}{m_{it} - R_i^S}, \tag{1.4}$$

and max (min) is set to be $-\infty(\infty)$ if $K(j) \setminus \{j\} = \emptyset(T(i) \setminus \{i\} = \emptyset)$.

Remark 1. *From the definitions of B-matrix and B^S-matrix, if M is a B^S-matrix such that $T(i) = \{i\}$ for all $i \in S$ and $K(j) = \{j\}$ for all $j \in \overline{S}$, then M is a B-matrix. Moreover, each 3×3 B-matrix is not a B^S-matrix and there exists a B^S-matrix that is not a B-matrix* [9]. *Thus the notions of B-matrix and B^S-matrix are only related in the sense of Lemma* 1.

Lemma 2. [9] *Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix and let X be the diagonal matrix in Lemma* 1 *such that $\tilde{M} = MX$ is a B-matrix with the form $\tilde{M} = \tilde{B}^+ + \tilde{C}$, where $\tilde{B}^+ = (\tilde{b}_{ij})$ is the matrix in* (1.1). *Then $\tilde{B}^+$ is strict diagonal dominant by rows with positive diagonal entries.*

Lemma 3. [9] *If $M = (m_{ij}) \in R^{n\times n}$ is a B^S-matrix that is not a B-matrix, then there exist $k, i \in N$ with $k \neq i$ such that*

$$\frac{1}{n} \sum_{j=1}^{n} m_{ij} \le m_{ik}.$$

Furthermore, if $k \in S$ (resp., $k \in \overline{S}$), then $\gamma < 1$ (resp., $\gamma > 1$), where γ is the parameter γ satisfying (1.4).

Some notations are given, which will be used in the sequel. Let $A=(a_{ij})\in R^{n\times n}$. For $i,j,k\in N$, $i\neq j$, denote

$$u_i(A)=\frac{1}{|a_{ii}|}\sum_{j=i+1}^{n}|a_{ij}|,\qquad l_k(A)=\max_{k\le i\le n}\left\{\frac{1}{|a_{ii}|}\sum_{j=k,\neq i}^{n}|a_{ij}|\right\},$$
$$v_k(A)=\max_{k+1\le i\le n}\left\{\frac{|a_{ik}|}{|a_{ii}|-\sum\limits_{j=k+1,\neq i}^{n}|a_{ij}|}\right\},\qquad w_k(A)=\max_{k+1\le i\le n}\left\{\frac{|a_{ik}|+\sum\limits_{j=k+1,\neq i}^{n}|a_{ij}|v_k(A)}{|a_{ii}|}\right\}.$$

Lemma 4. [16] *Let $A=(a_{ij})\in R^{n\times n}$ be a row strictly diagonally dominant M-matrix. Then*

$$\begin{aligned}\|A^{-1}\|_\infty \ \le\ \max\Bigg\{&\sum_{i=1}^{n}\left(\frac{1}{a_{ii}(1-u_i(A)w_i(A))}\prod_{j=1}^{i-1}\frac{u_j(A)}{1-u_j(A)w_j(A)}\right),\\ &\sum_{i=1}^{n}\left(\frac{w_i(A)}{a_{ii}(1-u_i(A)w_i(A))}\prod_{j=1}^{i-1}\frac{1}{1-u_j(A)w_j(A)}\right)\Bigg\},\end{aligned}$$

where

$$\prod_{j=1}^{i-1}\frac{u_j(A)}{1-u_j(A)w_j(A)}=1,\quad \prod_{j=1}^{i-1}\frac{1}{1-u_j(A)w_j(A)}=1,\quad if\ i=1.$$

Lemma 5. [13] *Let $\gamma>0$ and $\eta\ge 0$. Then for any $x\in[0,1]$,*

$$\frac{1}{1-x+\gamma x}\le\frac{1}{\min\{\gamma,1\}},\qquad \frac{\eta x}{1-x+\gamma x}\le\frac{\eta}{\gamma}.$$

Lemma 6. [12] *Let $A=(a_{ij})\in R^{n\times n}$ with*

$$a_{ii}>\sum_{j=i+1}^{n}|a_{ij}|,\quad \forall i\in N.$$

Then for any $x_i\in[0,1]$, $i\in N$,

$$\frac{1-x_i+a_{ii}x_i}{1-x_i+a_{ii}x_i-\sum\limits_{j=i+1}^{n}|a_{ij}|x_i}\le\frac{a_{ii}}{a_{ii}-\sum\limits_{j=i+1}^{n}|a_{ij}|}.$$

The rest of this paper is organized as follows: In Section 2, we present some new bounds for $\max\limits_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty$ when M is a B^S-matrix, and new perturbation bounds of B^S-matrices linear complementarity problems are also considered. In Section 3, a numerical example is given to show that our proposed bounds are respectively better than those in [6,11] in some cases.

2. Error bounds for LCPs of B^S-matrices

In this section, we propose some new error bounds for linear complementarity problems involved with B^S-matrices.

Theorem 1. *Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix and let X be the diagonal matrix given by Lemma* 1 *such that $\tilde{M} = MX$ is a B-matrix with the form $\tilde{M} = \tilde{B}^+ + \tilde{C}$, where $\tilde{B}^+ = (\tilde{b}_{ij})$ is the matrix in* (1.1). *Then*

$$\begin{aligned}&\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \\ \leq\ & \max\left\{\sum_{i=1}^{n} \frac{(n-1)\max\{\gamma, 1\}}{\min\{\bar{\beta}_i, x_i\}} \prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right),\ \sum_{i=1}^{n} \frac{(n-1)\max\{\gamma, 1\}w_i(\tilde{B}^+)}{\min\{\bar{\beta}_i, x_i\}} \prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j}\right\},\end{aligned} \quad (2.1)$$

where $\bar{\beta}_i = \tilde{b}_{ii} - \sum\limits_{j=i+1}^{n} |\tilde{b}_{ij}| w_i(\tilde{B}^+)$, and

$$\prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right) = 1, \quad \prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j} = 1, \quad if\ i = 1.$$

Proof. Let $\tilde{M}_D = X - DX + D\tilde{M}$. From Lemma 1, we deduce that

$$\tilde{M}_D = X - DX + D\tilde{M} = X - DX + D(\tilde{B}^+ + \tilde{C}) = \tilde{B}^+_D + \tilde{C}_D,$$

where $\tilde{B}^+_D = X - DX + D\tilde{B}^+$, $\tilde{C}_D = D\tilde{C}$. By Lemma 2, $\tilde{B}^+$ is a strictly diagonal dominant matrix. Let $M_D = I - D + DM$. Note that $M = \tilde{M}X^{-1}$. Then, similarly to the proof of Theorem 2.2 in [8], we can obtain that $\tilde{B}^+_D$ is a strictly diagonally dominant M-matrix with positive diagonal elements and that

$$\|M_D^{-1}\|_\infty \leq \|X^{-1}\|_\infty \|(I + (\tilde{B}^+_D)^{-1}\tilde{C}_D)^{-1}\|_\infty \|(\tilde{B}^+_D)^{-1}\|_\infty \leq (n-1)\max\{\gamma, 1\}\|(\tilde{B}^+_D)^{-1}\|_\infty. \quad (2.2)$$

Next, by Lemma 4, we have

$$\begin{aligned}&\|(\tilde{B}^+_D)^{-1}\|_\infty \\ \leq\ & \max\left\{\sum_{i=1}^{n} \frac{1}{(x_i - d_ix_i + \tilde{b}_{ii}d_i)(1 - u_i(\tilde{B}^+_D)w_i(\tilde{B}^+_D))} \prod_{j=1}^{i-1} \frac{u_j(\tilde{B}^+_D)}{1 - u_j((\tilde{B}^+_D))w_j(\tilde{B}^+_D)},\right.\\ &\left.\sum_{i=1}^{n} \frac{w_i(\tilde{B}^+_D)}{(x_i - d_ix_i + \tilde{b}_{ii}d_i)(1 - u_i((\tilde{B}^+_D))w_i(\tilde{B}^+_D))} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(\tilde{B}^+_D)w_j(\tilde{B}^+_D)}\right\}.\end{aligned}$$

From Lemma 5, we can easily get the following results: For each $i, j, k \in N$,

$$\begin{aligned}
v_k(\tilde{B}_D^+) &= \max_{k+1\le i\le n}\left\{\frac{|\tilde{b}_{ik}|d_i}{x_i - d_i x_i + \tilde{b}_{ii}d_i - \sum\limits_{j=k+1,\ne i}^{n}|\tilde{b}_{ij}|d_i}\right\}\\
&= \max_{k+1\le i\le n}\left\{\frac{\frac{|\tilde{b}_{ik}|}{x_i}d_i}{1 - d_i + \frac{\tilde{b}_{ii}}{x_i}d_i - \frac{1}{x_i}\sum\limits_{j=k+1,\ne i}^{n}|\tilde{b}_{ij}|d_i}\right\}\\
&\le \max_{k+1\le i\le n}\left\{\frac{|\tilde{b}_{ik}|}{\tilde{b}_{ii} - \sum\limits_{j=k+1,\ne i}^{n}|\tilde{b}_{ij}|}\right\}\\
&= v_k(\tilde{B}^+),
\end{aligned}$$

$$\begin{aligned}
w_k(\tilde{B}_D^+) &= \max_{k+1\le i\le n}\left\{\frac{|\tilde{b}_{ik}|d_i + \sum\limits_{j=k+1,\ne i}^{n}|\tilde{b}_{ij}|d_i v_k(\tilde{B}_D^+)}{x_i - d_i x_i + \tilde{b}_{ii}d_i}\right\}\\
&= \max_{k+1\le i\le n}\left\{\frac{\frac{1}{x_i}|\tilde{b}_{ik}|d_i + \frac{1}{x_i}\sum\limits_{j=k+1,\ne i}^{n}|\tilde{b}_{ij}|d_i v_k(\tilde{B}_D^+)}{1 - d_i + \frac{1}{x_i}\tilde{b}_{ii}d_i}\right\}\\
&\le \max_{k+1\le i\le n}\left\{\frac{|\tilde{b}_{ik}| + \sum\limits_{j=k+1,\ne i}^{n}|\tilde{b}_{ij}|v_k(\tilde{B}^+)}{\tilde{b}_{ii}}\right\}\\
&= w_k(\tilde{B}^+),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{(x_i - d_i x_i + \tilde{b}_{ii}d_i)(1 - u_i(\tilde{B}_D^+)w_i(\tilde{B}_D^+))} &= \frac{1}{x_i - d_i x_i + \tilde{b}_{ii}d_i - \sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|d_i w_i(\tilde{B}_D^+)}\\
&\le \frac{1}{\min\left\{\tilde{b}_{ii} - \sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|w_i(\tilde{B}^+), x_i\right\}}\\
&= \frac{1}{\min\left\{\bar{\beta}_i, x_i\right\}}.
\end{aligned} \tag{2.3}$$

Furthermore, by Lemma 5 and Lemma 6, we have

$$\begin{aligned}
\frac{u_i(\tilde{B}_D^+)}{1-u_i(\tilde{B}_D^+)w_i(\tilde{B}_D^+)} &= \frac{\sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|d_i}{x_i-d_ix_i+\tilde{b}_{ii}d_i-\sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|d_iw_i(\tilde{B}_D^+)} \\
&\leq \frac{\sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|}{\tilde{b}_{ii}-\sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|w_i(\tilde{B}^+)} \\
&= \frac{1}{\bar{\beta}_i}\sum_{j=i+1}^{n}|\tilde{b}_{ij}|, \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{1-u_i(\tilde{B}_D^+)w_i(\tilde{B}_D^+)} &= \frac{x_i-d_ix_i+\tilde{b}_{ii}d_i}{x_i-d_ix_i+\tilde{b}_{ii}d_i-\sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|d_iw_i(\tilde{B}_D^+)} \\
&\leq \frac{\tilde{b}_{ii}}{\tilde{b}_{ii}-\sum\limits_{j=i+1}^{n}|\tilde{b}_{ij}|w_i(\tilde{B}^+)} = \frac{\tilde{b}_{ii}}{\bar{\beta}_i}. \qquad (2.5)
\end{aligned}$$

Finally, by (2.3)–(2.5), we obtain

$$\|(B_D^+)^{-1}\|_\infty \leq \max\left\{\sum_{i=1}^{n}\frac{1}{\min\{\bar{\beta}_i,x_i\}}\prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right),\ \sum_{i=1}^{n}\frac{w_i(\tilde{B}^+)}{\min\{\bar{\beta}_i,x_i\}}\prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j}\right\}. \qquad (2.6)$$

Therefore, the result in (2.1) holds from (2.2) and (2.6). □

Based on Theorem 1 and Lemma 3, the following Corollary can be proved easily.

Corollary 1. *Let $M=(m_{ij})\in R^{n\times n}$ be a B^S-matrix that is not a B-matrix and let $k_0,i_0\in N$ with $k_0\neq i_0$ such that $m_{i_0k_0}\geq\frac{1}{n}\sum\limits_{j\in N}m_{i_0j}$. If $k_0\in\overline{S}$, then*

$$\begin{aligned}
&\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \\
&\leq \max\left\{\sum_{i=1}^{n}\frac{(n-1)\gamma}{\min\{\bar{\beta}_i,1\}}\prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right),\ \sum_{i=1}^{n}\frac{(n-1)\gamma w_i(\tilde{B}^+)}{\min\{\bar{\beta}_i,1\}}\prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j}\right\}.
\end{aligned}$$

If $k_0\in S$, then

$$\begin{aligned}
&\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \\
&\leq \max\left\{\sum_{i=1}^{n}\frac{(n-1)}{\min\{\bar{\beta}_i,\gamma\}}\prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right),\ \sum_{i=1}^{n}\frac{(n-1)w_i(\tilde{B}^+)}{\min\{\bar{\beta}_i,\gamma\}}\prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j}\right\}.
\end{aligned}$$

Similarly to the proof of Theorem 2.4 in [6], we can also obtain new perturbation bounds for linear complementarity problems of B^S-matrices based on Theorem 1.

Theorem 2. *Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix and let $\tilde{B}^+ = (\tilde{b}_{ij})$ be the matrix in* (1.1). *Then*

$$\beta_\infty(M) \leq \max\left\{\sum_{i=1}^{n}\frac{(n-1)\max\{\gamma,1\}}{\min\{\bar{\beta}_i, x_i\}}\prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right),\quad \sum_{i=1}^{n}\frac{(n-1)\max\{\gamma,1\}w_i(\tilde{B}^+)}{\min\{\bar{\beta}_i, x_i\}}\prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j}\right\},$$

where $\beta_\infty(M) = \max\limits_{d\in[0,1]^n}\|(I-D+DM)^{-1}D\|_\infty$, $D = diag(d_i)$ with $0 \leq d_i \leq 1$ for each $i \in N$, and

$$\bar{\beta}_i = \tilde{b}_{ii} - \sum_{j=i+1}^{n}|\tilde{b}_{ij}|w_i(\tilde{B}^+),\qquad \prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right) = 1\ if\ i = 1,\qquad \prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j} = 1\ if\ i = 1.$$

Finally, we give a comparison of the bounds in (1.3) and (2.1) as follows.

Theorem 3. *Let $M = (m_{ij}) \in R^{n\times n}$ be a B^S-matrix and let X be the diagonal matrix given by Lemma* 1 *such that $\tilde{M} = MX$ is a B-matrix with the form $\tilde{M} = \tilde{B}^+ + \tilde{C}$, where $\tilde{B}^+ = (\tilde{b}_{ij})$ is the matrix in* (1.1). *Let $\hat{\beta}_i$ and $\bar{\beta}_i$ be defined as in* (1.3) *and* (2.1)*, respectively. Then*

$$\begin{aligned}&\max\left\{\sum_{i=1}^{n}\frac{(n-1)\max\{\gamma,1\}}{\min\{\bar{\beta}_i, x_i\}}\prod_{j=1}^{i-1}\left(\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}|\right),\ \sum_{i=1}^{n}\frac{(n-1)\max\{\gamma,1\}w_i(\tilde{B}^+)}{\min\{\bar{\beta}_i, x_i\}}\prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\bar{\beta}_j}\right\}\\ \leq\ & \sum_{i=1}^{n}\frac{(n-1)\max\{\gamma,1\}}{\min\{\hat{\beta}_i, x_i\}}\prod_{j=1}^{i-1}\frac{\tilde{b}_{jj}}{\hat{\beta}_j}. \end{aligned} \tag{2.7}$$

Proof. For any $i \in N$, based on $\tilde{B}^+$ is strict diagonal dominant, we have

$$0 < w_i(\tilde{B}^+) \leq l_i(\tilde{B}^+) < 1, \tag{2.8}$$

and by (2.8), we get

$$\hat{\beta}_i = \tilde{b}_{ii} - \sum_{j=i+1}^{n}|\tilde{b}_{ij}|l_i(\tilde{B}^+) \leq \tilde{b}_{ii} - \sum_{j=i+1}^{n}|\tilde{b}_{ij}|w_i(\tilde{B}^+) = \bar{\beta}_i. \tag{2.9}$$

Furthermore, by (2.9), for all $i \in N$, we have

$$\frac{1}{\min\{\bar{\beta}_i, x_i\}} \leq \frac{1}{\min\{\hat{\beta}_i, x_i\}}, \tag{2.10}$$

and for each $j = 1, 2, \ldots, n-1$,

$$\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}| \leq \frac{1}{\hat{\beta}_j}\sum_{k=j+1}^{n}|\tilde{b}_{jk}| \leq \frac{\tilde{b}_{jj}}{\hat{\beta}_j}. \tag{2.11}$$

The result in (2.7) follows by (2.10) and (2.11). □

3. A numerical example

In this section, we give a numerical example to illustrate the advantages of new bound.

Example 1. Consider the family of B^S-matrices for $S = \{1, 2\}$ in [14]:

$$M_k = \begin{bmatrix} 2 & 1 & 1 & 1.5 \\ -\frac{2k}{k+1} & 2 & \frac{1}{k+1} & \frac{1}{k+1} \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix},$$

where $k \geq 1$. We choose $X = diag\{\gamma, \gamma, 1, 1\}$ with $\gamma \in (\frac{3.5}{3}, 1.5)$. So $\tilde{M}_k = M_k X$ can be written $\tilde{M} = \tilde{B}_k^+ + \tilde{C}_k$ as in (1.1), where

$$\tilde{B}_k^+ = \begin{bmatrix} 2\gamma - 1.5 & \gamma - 1.5 & -0.5 & 0 \\ -\frac{2k}{k+1}\gamma - \frac{1}{k+1} & 2\gamma - \frac{1}{k+1} & 0 & 0 \\ 0 & 0 & 2-\gamma & 1-\gamma \\ 0 & 0 & 1-\gamma & 2-\gamma \end{bmatrix}.$$

In fact, the bound (1.2), with the hypotheses that $k \geq 1$, is

$$\frac{(4-1)\max\{\gamma, 1\}}{\min\{\tilde{\beta}, \gamma, 1\}} = \frac{3\gamma}{2\gamma - 1}(k+1),$$

and it can be arbitrarily large when $k \to +\infty$.

In particular, let $\gamma = 1.3$, then we can use the bound (1.2), the bound (1.3) and the bound (2.1) for $k = 2, 20, 30, 60, 100 \ldots, +\infty$ to estimate $\max\limits_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$, see Table 1.

Table 1. The bound (1.2), the bound (1.3) and the bound (2.1).

k	2	20	30	60	100	$\cdots$	$+\infty$
bound (1.2)	7.3125	51.1875	75.5625	148.6875	246.1875	$\cdots$	$+\infty$
bound (1.3)	48.1089	54.4704	54.8144	55.1699	55.3155	$\cdots$	55.5375
bound (2.1)	29.8235	31.4335	33.1377	33.4355	33.7785	$\cdots$	33.9556

Remark 2. *From Example 1, it is easy to see that each bound* (1.2) *or* (2.1) *is better than the other one. Thus it is difficult to say in advance which one is better. However, for a B^S-matrix M with $\tilde{M} = \tilde{B}^+ + \tilde{C}$, where the diagonal dominance of $\tilde{B}^+$ is weak (e.g., for a matrix M_k with a large number of k here), the bound* (2.1) *is more effective than the bound* (1.2).

4. Conclusions

We present a new error bound for linear complementarity problems associated with B^S-matrices, which improves some existing results. A numerical example shows the feasibility and effectiveness of the results which are obtained in this paper. Besides B^S-matrices, some similar assertions for linear complementarity problems of other classes of matrices are provided, such as DB-matrices, SB-matrices and MB-matrices. So we conjecture here that by the technique above, new sharper bounds for linear complementarity problems of these classes of matrices could be given.

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Conflict of interest

The author declare that they have no competing interests.

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